

Common Belief in Rationality in Games with Unawareness*

Andrés Perea[†]
Maastricht University

August 27, 2018

Abstract

This paper investigates static games with unawareness, where players may be unaware of some of the choices that can be made. That is, different players may have different *views* on the game. We propose an epistemic model that encodes players' belief hierarchies on choices and views, and use it to formulate the basic reasoning concept of *common belief in rationality*. We do so for two scenarios: one in which we only limit the possible views that may enter the players' belief hierarchies, and one in which we fix the players' belief hierarchies on views. For both scenarios we design a recursive elimination procedure that yields for every possible view the choices that can rationally be made under common belief in rationality.

Keywords: Unawareness, common belief in rationality, epistemic game theory, elimination procedure

JEL Classification: C72, C73

1 Introduction

A standard assumption in game theory is that all ingredients of the game – the players, their choices and their utility functions – are perfectly transparent to everybody involved. However, there are many situations of interest in which players may not be fully informed about some of these ingredients. For instance, a player may be uncertain about the precise utility functions of his opponents. Such situations may be modelled as *games with incomplete information*, and Harsanyi (1967–1968) opened the door towards a formal analysis of this class of games. In some cases the lack of information may even be more basic, as a player may be unaware of

*I would like to thank Joseph Halpern, Aviad Heifetz, Niels Mourmans, Burkhard Schipper, Marciano Siniscalchi, the participants at LOFT 2018 in Milano, an editor and three referees for useful comments on the paper.

[†]*Address:* Epicenter and Dept. of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. *E-mail:* a.perea@maastrichtuniversity.nl *Web:* <http://www.epicenter.name/Perea/>

certain choices that can be made, or may even be unaware of the presence of certain players in the game. Harsanyi (1967–1968, pp.167–168) argued that unawareness of choices can also be modelled within the framework of incomplete information, by assigning a very low utility to the choices that players are unaware of. But conceptually this still seems very different from being truly unaware of these choices.¹ This type of situations, where players are unaware of certain choices or the presence of certain players in the game, has recently given rise to the study of *games with unawareness*. For an overview of the relatively young literature in this field, see Schipper (2014).

In terms of reasoning there is a crucial difference between these two classes of games. In a game with incomplete information, a player may not be informed about the true utility function of an opponent, yet at the same time may reason about all the possible utility functions that this opponent may have. And he may reason about an opponent reasoning about all the possible utility functions that some third player may have, and so on. That is, if we list all the possible utility functions that the players may have, then there is no limit to the players' reasoning about these utility functions.

The same is not true for games with unawareness, however. If a player is unaware of an opponent's choice c , then he cannot reason about other players who are aware of c . In a sense, the choice c is not part of his language, or state space, and hence this choice c cannot enter at any level of his reasoning. These endogenous constraints on the players' reasoning constitute the key factor that distinguishes games with unawareness from other classes of games.

At the same time, this reasoning about the level of unawareness of other players is at the central stage of games with unawareness. Indeed, if a player in a game with unawareness must decide what to do, then he must base his choice not only on his own (possibly partial) view of the game, but also on what he believes about the opponents' views of the game, what he believes that his opponents believe about the views of other players, and so on. In other words, a player holds a belief hierarchy on the players' views of the game, and bases his choice upon this belief hierarchy.

In that light, the reasoning of players in games with unawareness is considerably more complex than in standard games, as a player must form beliefs about the opponents' choices *and* the opponents' views, where his beliefs about the opponents' choices will depend on his belief about their views. In the literature, the reasoning about views has typically been disentangled from the reasoning about choices, as most models for games with unawareness *exogenously* specify a belief hierarchy on views for every player. The strategic reasoning is then modelled by using an equilibrium or rationalizability concept that assumes these fixed belief hierarchies on views.

In this paper we take a different approach by *combining* the players' reasoning about views and choices into *one* belief hierarchy that models both. More precisely, we propose a model of static games with unawareness that no longer fixes the players' belief hierarchies on views, but where we only limit the possible views that may enter the players' belief hierarchies. We impose

¹See Hu and Stuart (2001) and Meier and Schipper (2014, p.227) for a discussion of this issue.

no restrictions, however, on how these views enter the belief hierarchies, or what probabilities these views receive at the various levels of a belief hierarchy. Subsequently, we encode the players' belief hierarchies on *choices and views* by an appropriately designed epistemic model with types. Types in this epistemic model thus simultaneously describe the players' reasoning about views and their strategic reasoning – something that proves to be very convenient for an epistemic analysis. Another difference with most of the existing literature is that we allow for *probabilistic* beliefs about the opponents' views, and not only deterministic beliefs. We find this important, as a player who is truly uncertain about the level of unawareness of his opponent may well ascribe positive probability to *various* possible views for this opponent. Such probabilistic beliefs on views can naturally be captured by our choice of an epistemic model.

We then use this epistemic model to investigate the strategic reasoning of players in games with unawareness, which is the main purpose of this paper. To do so, we focus on the central yet basic reasoning concept of *common belief in rationality* (Spohn (1982), Brandenburger and Dekel (1987) and Tan and Werlang (1988)) which in standard two-player static games characterizes *rationalizability* (Bernheim (1984), Pearce (1984)), while characterizing *correlated rationalizability* (Brandenburger and Dekel (1987)) and the iterated strict dominance procedure in standard static games with two players or more.² In the context of games with unawareness, this concept states that a player believes that his opponents choose optimally *given their views of the game*, that a player believes that his opponents believe that the other players choose optimally *given their views of the game*, and so on. It turns out that this concept can naturally be formulated within the language of our epistemic model which, as we saw, encodes belief hierarchies on choices and views.

A natural question is whether we can find a recursive elimination procedure à la *iterated strict dominance* that characterizes precisely those choices that can rationally be made under common belief in rationality. We indeed propose such a procedure, and call it *iterated strict dominance for unawareness*. The main difference with the standard strict dominance procedure is that, in every round and for every player, we eliminate choices for *every possible view* that this player can hold in the game. More precisely, at a given view v_i for player i we first eliminate those choices for opponent j that have not survived the previous round for *any possible view of player j* that player i can reason about when holding the view v_i . Subsequently, at view v_i we eliminate for player i those choices that are strictly dominated, given the current set of opponents' choices.

We show in Theorem 4.2 that this procedure selects, for every player and every view, precisely those choices that this player can make with this particular view under common belief in rationality. Since the procedure always yields a non-empty output, it immediately follows that for every static game with unawareness there is for every player and every view at least one

²The difference between rationalizability (Bernheim (1984), Pearce (1984)) and correlated rationalizability (Brandenburger and Dekel (1987)) is that the former, in games with three players or more, requires player i 's belief about opponent j 's choice to be independent from his belief about another opponent k 's choice. The latter concept does not impose this independence condition.

belief hierarchy on choices and views that expresses common belief in rationality. The procedure is very similar to the *generalized iterated strict dominance procedure* (Bach and Perea (2017)) which has been designed for static games with *incomplete information*. The main difference is that in the latter procedure, choices are being eliminated at every possible *utility function* that a player can have in the game, instead of at every possible *view* that a player can hold.

As a second step, we reconcile the concept of common belief in rationality with the common assumption that the players' belief hierarchies on views are fixed. The new concept then selects, for every view and every fixed belief hierarchy on views, those choices that a player can rationally make under common belief in rationality if he holds this particular view and belief hierarchy on views. Also for this concept we design a recursive elimination procedure, called *iterated strict dominance with fixed beliefs on views*, that yields precisely these choices. See our Theorem 5.2.

To define this procedure we first encode the given belief hierarchy on views by means of an epistemic model with types, similar to the one mentioned above. The difference is that there is no reference to choices in this epistemic model, only to views. Types in this epistemic model are called *view-types*, as they encode belief hierarchies on views only. More precisely, every view-type in the model can be identified with a probability distribution on the opponents' views and view-types. The new procedure is more refined as above, as it now eliminates, in every round and for every player, choices at every possible view *and every possible view-type* for that player. Moreover, at a given view v_i and view-type r_i for player i , the opponents' choices that can be eliminated at (v_i, r_i) are based on the probability distribution that r_i induces on the opponents' views and view-types. In that sense, the procedure is closely related to the *interim correlated rationalizability procedure* (Dekel, Fudenberg and Morris (2007)) for static games with *incomplete information*. The key difference is that in the latter procedure, choices are being eliminated at pairs of *utility functions* and belief hierarchies on *utility functions*, whereas in this paper choices are eliminated at pairs of *views* and belief hierarchies on *views*.

With these two procedures we thus characterize the behavioral consequences of common belief in rationality in games with unawareness in two scenarios: the basic scenario where we only limit the possible *views* that may enter the players' belief hierarchies, but where no other restrictions are imposed, and a scenario where the belief hierarchies on views are fixed. Moreover, if the belief hierarchies on views are fixed and *deterministic*, then our procedure becomes equivalent to the *extensive-form rationalizability procedure* in Heifetz, Meier and Schipper (2013b) when applied to *static* games with unawareness. Our analysis is also closely related to Feinberg (2012) who investigates the concept of *rationalizability* for static games with unawareness. Most other papers on games with unawareness investigate *equilibrium* concepts instead of rationalizability concepts.

The rest of the paper is organized as follows. In Section 2 we provide our definition of static games with unawareness. In Section 3 we encode belief hierarchies on choices and views by means of an epistemic model with types, and use it to formally define common belief in rationality for static games with unawareness. In Section 4 we present the *iterated strict dominance procedure for unawareness* and show that it characterizes the behavioral consequences of common belief in

rationality. In Section 5 we impose a fixed belief hierarchy on views for every player, present the *iterated strict dominance procedure with fixed beliefs on views*, and show that it characterizes the behavioral consequences of common belief in rationality with fixed belief hierarchies on views. In Section 6 we relate our work to other papers on unawareness in the literature. We conclude in Section 7. The appendix (Section 8) contains all proofs, and shows how to formally derive belief hierarchies on views from types in an epistemic model.

2 Static Games with Unawareness

In this paper we restrict to *static games*, and focus on unawareness about the possible *choices* that the players can make. That is, a player may be unaware of certain choices that he, or his opponents, can make in the game. Feinberg (2012) allows players, in addition, to be unaware of some of the other *players* in the game. Such unawareness, however, will not be part of our framework.

Before we can analyze games with unawareness, we must first establish how we *describe* the possible unawareness of players about some of the choices in the game. We will do so by defining, for every player, a collection of *partial descriptions* of the full game, which contain some – but not necessarily all – possible choices that can be made. These partial descriptions will be called the possible *views* that the player can hold. Every view can thus be interpreted as a personal, and possibly incomplete, perception of the full game.

Formally, a *static game* is a tuple $G = (C_i, u_i)_{i \in I}$ where I is a finite set of players, C_i is a finite set of choices, and $u_i : \times_{j \in I} C_j \rightarrow \mathbf{R}$ is a utility function for every player i . A *view* of the game G is a tuple $v = (D_i)_{i \in I}$ where $D_i \subseteq C_i$ is a possibly reduced set of choices for every player i . The interpretation is that a player with view $v = (D_i)_{i \in I}$ is only aware of the choices in D_i for every player i . We implicitly assume that a player with view $v = (D_i)_{i \in I}$ believes that the utilities induced by the choice combinations in v coincide with those of the game G . For that reason, it is not necessary to specify new utility functions for a view. For any two views $v = (D_i)_{i \in I}$ and $v' = (D'_i)_{i \in I}$ we say that v is *contained* in v' if $D_i \subseteq D'_i$ for all players i . That is, all choices considered possible in v are also considered possible in v' . An important principle in this – and any other – paper on unawareness is that a player with view v can only reason about views that are contained in v .

We can now define a static game with unawareness as a tuple consisting of a full static game, containing all choices that the players can possibly make, and for every player a finite collection of possible views of the full game.

Definition 2.1 (Static game with unawareness) *A static game with unawareness is a tuple $G^u = (G^{base}, (V_i)_{i \in I})$ where G^{base} is a static game, and V_i is a finite collection of views for player i of the game G^{base} . Moreover, for every player i , every view v_i in V_i , and every opponent $j \neq i$ there must be a view in V_j that is contained in v_i .*

Here, we refer to G^{base} as the *base game*. The condition above thus guarantees that for every possible view $v_i \in V_i$ that player i can have, there is for every opponent j at least one view $v_j \in V_j$ that player i can reason about.

Unlike most other definitions in game theory, not all ingredients in a static game with unawareness are commonly known among the players. In particular, if player i holds a certain view v_i , he will not be aware of – and hence, not know of – the existence of views in the model that are not contained in v_i . This will be illustrated below in Example 1.

Note that, for every player i , the collection of views V_i need not contain *all* possible views of the game G^{base} . By considering *limited* collections of views, we put restrictions on the possible belief hierarchies on views that we allow for. Indeed, for every player i we restrict to belief hierarchies in which i only deems possible views in V_j for every opponent j , believes that every opponent j only deems possible views in V_k for every other player k , and so on.

The reasons for imposing such restrictions are two-fold. First, for a specific game-theoretic context, some views just make more sense than other views, and it thus seems reasonable to restrict to the more plausible views. Second, the concept of common belief in rationality, which is the main object of study in this paper, would hardly have any bite if we were to allow for all possible views. In that case, every choice that would be optimal for at least *some* belief about the opponents’ choices could be rationalized under the concept of common belief in rationality. To see this, consider some choice c_i for player i that is optimal for some belief b_i about the opponents’ choices. If all views are allowed, then player i is free to believe that every opponent only has one available choice, that every opponent believes that every other player only has one available choice, and so on. In that way, we can trivially embed the belief b_i in a belief hierarchy on choices and views that expresses common belief in rationality, thus rationalizing the choice c_i under common belief in rationality. However, by imposing some restrictions on the possible belief hierarchies on views, we may be able to derive some non-trivial behavioral consequences from common belief in rationality. In Section 5 we will impose further restrictions on the players’ belief hierarchies on views by assuming a *unique* belief hierarchy on views for every player.

As another special case of our model, one may select the “full view” $(C_i)_{i \in I}$ as the only possible view for every player. In that case, the game with unawareness would reduce to a traditional static game in which all players agree that the game being played is G^{base} and no other.

The main difference between this model and other definitions for games with unawareness, such as Feinberg (2012), Rêgo and Halpern (2012) and Heifetz, Meier and Schipper (2013b)³, is that the latter fix for every player a view and a *belief hierarchy on views*, whereas we do not. That is, these papers exogenously describe, for every player, the view he holds on the game, what the player believes about the opponents’ views, what he believes about the opponents’ beliefs about the views by the other players, and so on. In contrast, we allow players to hold any view and any belief hierarchy on views they wish, as long as these only use views from

³Other papers that model games with unawareness can be found in Section 6.

the collections $(V_i)_{i \in I}$. As we already said, the case of fixed belief hierarchies on views will be explored in Section 5.

Moreover, in our model we allow such belief hierarchies on views to be *probabilistic*, whereas Feinberg (2012) and Heifetz, Meier and Schipper (2013b) restrict to deterministic belief hierarchies on views. Rêgo and Halpern (2012), in turn, do allow for probabilistic belief hierarchies on views through the introduction of chance moves.

A last difference we wish to outline is that the models by Rêgo and Halpern (2012) and Heifetz, Meier and Schipper (2013b) were specifically designed for *dynamic* games with unawareness. But their definitions capture static games as a special case.

We now illustrate the definition of a static game with unawareness by means of an example.

Example 1. A day at the beach.

Barbara and you spend a holiday on an island with four small beaches, the *Nextdoor Beach*, the *Closeby Beach*, the *Faraway Beach* and the *Distant Beach*. The first two beaches are close to the hotel, whereas the latter two are more remote and hard to find. You happen to know about the two remote beaches because a local person told you about it. However, you are unsure whether Barbara is aware of the two remote beaches or not. Today, both you and Barbara would like to go to the beach. The question is: To which beach do you go?

As to the utilities, suppose you had a fierce argument with Barbara yesterday, and therefore you would both prefer to avoid each other by going to different beaches today. Assume, moreover, that you prefer the *Faraway Beach* to the *Distant Beach*, the *Distant Beach* to the *Nextdoor Beach*, and the *Nextdoor Beach* to the *Closeby Beach*. You know that Barbara prefers the *Nextdoor Beach* to the *Closeby Beach*, and suspect that she prefers the *Closeby Beach* to the *Faraway Beach*, and the *Faraway Beach* to the *Distant Beach* in case she is aware of the two remote beaches.

This situation can be represented by the game with unawareness in Table 1, where G^{base} is the base game, $V_1 = \{v_1, v'_1\}$ contains the views for you that are relevant for the story, and $V_2 = \{v_2, v'_2\}$ contains the views for Barbara that are relevant for the situation at hand. In the base game, your choices are in the rows and Barbara's choices are in the columns. In both of your possible views v_1 and v'_1 , your choices are in the rows and Barbara's choices in the columns. In the corresponding cells we have put your utilities. In Barbara's views v_2 and v'_2 we have put her choices in the rows and your choices in the columns, and have written the induced utilities for her in the cells. This is a general convention we adopt for depicting views of a player i : we always put i 's choices in the rows, the opponents' choice combinations in the columns, and the induced utilities for player i in the corresponding cells.

The interpretation of the views is as follows: The view v_1 represents your actual view, in which you are aware of all four beaches. Since you are unsure whether Barbara is aware of the two remote beaches or not, you believe that Barbara's view is either v_2 , in which she is aware of all four beaches, or v'_2 , in which she is not aware of the two remote beaches. If you believe

	G^{base}	Faraway	Distant	Nextdoor	Closeby		
Base game	Faraway	0, 0	4, 1	4, 4	4, 3		
	Distant	3, 2	0, 0	3, 4	3, 3		
	Nextdoor	2, 2	2, 1	0, 0	2, 3		
	Closeby	1, 2	1, 1	1, 4	0, 0		
	v_1	Faraway	Distant	Nextdoor	Closeby		
Your views	Faraway	0	4	4	4		
	Distant	3	0	3	3		
	Nextdoor	2	2	0	2		
	Closeby	1	1	1	0		
	v_2	Faraway	Distant	Nextdoor	Closeby		
Barbara's views	Faraway	0	2	2	2		
	Distant	1	0	1	1		
	Nextdoor	4	4	0	4		
	Closeby	3	3	3	0		
	v'_1	Nextdoor	Closeby				
	Nextdoor	0	2				
	Closeby	1	0				
	v'_2	Nextdoor	Closeby				
	Nextdoor	0	4				
	Closeby	3	0				

Table 1: “A day at the beach”, modelled as a game with unawareness

that Barbara is not aware of the two remote beaches, and hence that her view is v'_2 , you must necessarily believe that Barbara believes that your view is v'_1 . Indeed, if Barbara is not aware of the two remote beaches, she cannot even reason about the possibility that you are aware of these remote beaches, and hence she must believe that you are only aware of the two local beaches. That is, she must believe that your view is v'_1 , different from your actual view v_1 .

It may be verified that Table 1 yields a well-defined static game with unawareness, meeting the condition on views as specified in Definition 2.1. Indeed, for both of your views v_1 and v'_1 there is the view v'_2 for Barbara that is contained in it, and for Barbara's views v_2 and v'_2 there is the view v'_1 for you that is contained in it.

Note that this scenario allows for multiple belief hierarchies on views for you, provided your view is v_1 , and similarly for Barbara if her view is v_2 . Indeed, if your view is v_1 , then one possible belief hierarchy on views is that you believe that Barbara holds the view v_2 , that you believe that Barbara believes that you hold the view v_1 , that you believe that Barbara believes that you believe that Barbara holds the view v_2 , and so on. Another belief hierarchy could be that you believe that Barbara's view is v'_2 , that you believe that Barbara believes that you hold the view v'_1 , that you believe that Barbara believes that you believe that Barbara holds the view v'_2 , and so on. Or you could also hold a probabilistic belief hierarchy in which, at the first layer, you assign probability 0.3 to Barbara holding the view v_2 and probability 0.7 to Barbara holding the view v'_2 . Many other belief hierarchies on views would be possible as well.

However, if your view is v'_1 , then you can only reason about the view v'_2 for Barbara and the view v'_1 for yourself. Hence, the only possible belief hierarchy on views would be the one where you believe that Barbara's view is v'_2 , believe that Barbara believes that your view is v'_1 , and so on.

Note that not all the ingredients of this game with unawareness are common knowledge among the two players. Consider, for instance, the case where Barbara's view is v'_2 . Then, she only has mental access to the views v'_1 and v'_2 in the model, and is not even aware of the existence of the other views v_1 and v_2 .

Finally, we wish to mention that a specific view v_i for player i only specifies the choices – for himself, but also for the opponents – that he is aware of himself, but does not specify what player i believes about the awareness of other players. For instance, the view v_1 above only tells us that you are aware of the four beaches yourself. In particular, you are aware of four possible choices for Barbara. This does not mean, however, that you believe that Barbara is aware of these four choices also. Indeed, if you hold the view v_1 , then you may very well believe that Barbara has the view v'_2 , and hence that she is unaware of two of her potential choices, the *Faraway Beach* and the *Distant Beach*. Of course, if you have the view v_1 and believe that Barbara has the view v'_2 , you must necessarily believe that Barbara will only choose between the two local beaches.

3 Common Belief in Rationality

The idea of *common belief in rationality* (Spohn (1982), Brandenburger and Dekel (1987) and Tan and Werlang (1988)) is that a player believes that every opponent chooses optimally given his view, that he believes that every opponent believes that every other player chooses optimally given his view, and so on. In order to formally define this idea for static games with unawareness, we must specify (i) what a player believes about the possible choices and views of his opponents, (ii) what he believes about the opponents' beliefs about their opponents' choices and views, and so on. Such belief hierarchies can be encoded by means of epistemic models with types, where every type holds a probabilistic belief about the opponents' choices, views and types.

Definition 3.1 (Epistemic model) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. An epistemic model for G^u is a tuple $M = (T_i, b_i)_{i \in I}$ where T_i is the finite set of types for player i , and b_i is a belief mapping that assigns to every type $t_i \in T_i$ some probabilistic belief $b_i(t_i) \in \Delta(C_{-i} \times V_{-i} \times T_{-i})$. Moreover, for every type $t_i \in T_i$ the belief $b_i(t_i)$ only assigns positive probability to opponents' choice-view pairs (c_j, v_j) where c_j is part of the view v_j .

Here, C_{-i} is a short-hand for $\times_{j \neq i} C_j$, and denotes the set of opponents' choice combinations. Similarly for V_{-i} and T_{-i} . For every finite set X , we denote by $\Delta(X)$ the set of probability distributions on X . Hence, $\Delta(C_{-i} \times V_{-i} \times T_{-i})$ denotes the set of probability distributions on $C_{-i} \times V_{-i} \times T_{-i}$. The last condition states that if a player believes that the opponent holds the view v_j , then he must believe that the opponent will make a choice c_j that the opponent is actually aware of given the view v_j .

For every type in the epistemic model, we can then derive a full infinite belief hierarchy about the choices and views of the players. Indeed, if we take a type t_i , then the first-order belief about the opponents' choices and views would simply be the marginal of the probability distribution $b_i(t_i)$ on $C_{-i} \times V_{-i}$. As $b_i(t_i)$ induces a probability distribution on T_{-i} , and every opponent's type holds a first-order belief about the other players' choices and views, we can also derive the belief that t_i has about the first-order belief that every opponent j has about the other players' choices and views. This yields the second-order belief that type t_i has. By continuing in this fashion we can also derive the third-order belief, and all higher-order beliefs, for the type t_i , representing t_i 's belief hierarchy about the players' choices and views.

Without any further restrictions, this could result in unreasonable belief hierarchies, however. In the game of Table 1, for instance, we could construct an epistemic model with a type t_1 for you that believes that Barbara has view v'_2 , and at the same time believes that Barbara believes that you have view v_1 . Such beliefs are unreasonable, because Barbara cannot reason about the larger view v_1 if she holds the smaller view v'_2 . To exclude such unreasonable beliefs we impose a condition that is called *common belief in smaller views*. To formalize this notion, we first define what it means to *believe in smaller views*.

Definition 3.2 (Belief in smaller views) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M = (T_i, b_i)_{i \in I}$ for G^u . A view-type pair $(v_i, t_i) \in V_i \times T_i$ believes in smaller views if $b_i(t_i)$ only assigns positive probability to opponents' views that are contained in v_i .

This is a cognitive constraint that seems indispensable for modelling unawareness. It states that if a player has a certain view, then he can only reason about opponents' views that are contained in his own view. Similar conditions can be found in other papers on games with unawareness. Indeed, this condition corresponds to Condition 2 in Feinberg (2012), condition C2 in Rêgo and Halpern (2012), condition I4 in Heifetz, Meier and Schipper (2013b), condition (iii)(a) from Definition 1 in Heinsalu (2014), and the “confinement” condition in Heifetz, Meier and Schipper (2013a) and Meier and Schipper (2014).

By iterating this condition we finally arrive at *common belief in smaller views*.

Definition 3.3 (Common belief in smaller views) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M = (T_i, b_i)_{i \in I}$ for G^u . A view type-pair (v_i, t_i) expresses 1-fold belief in smaller views if (v_i, t_i) believes in smaller views. For $k > 1$, we inductively say that (v_i, t_i) expresses k -fold belief in smaller views if t_i only assigns positive probability to opponents' view-type pairs (v_j, t_j) that express $(k - 1)$ -fold belief in smaller views. Finally, we say that (v_i, t_i) expresses common belief in smaller views if it expresses k -fold belief in smaller views for all $k \geq 1$.

That is, you believe in smaller views, believe that all opponents believe in smaller views, and so on. In particular, if (v_i, t_i) expresses common belief in smaller views, then type t_i , at any layer of its belief hierarchy, will only reason about views that are contained in v_i – as it should be. If (v_i, t_i) expresses common belief in smaller views then, in a sense, the belief hierarchy on views induced by t_i is both reasonable and cognitively feasible for the view v_i .

The condition of common belief in smaller views is the main ingredient that distinguishes epistemic models for unawareness from epistemic models for incomplete information. In the latter scenario no condition of this kind is needed, as a player with a certain utility function has in principle mental access to *all* other utility functions in the model, and hence no restrictions need to be imposed on belief hierarchies on utility functions. One could say that, *a priori*, all belief hierarchies on utility functions are equally plausible. This is not the case for games with unawareness.

We have seen in the previous section that a player with a certain view may not have mental access to all possible views in a game with unawareness. A similar property holds for types in an epistemic model. More precisely, a player i with view v_i may not have mental access to *all* types in the epistemic model, since he is only able to reason about views that are contained in v_i . Indeed, a type with view v_i only has mental access to opponents' types – and types of his own – that reason exclusively about views contained in v_i . In other words, a type with view

Types	$T_1 = \{t_1, t'_1, t''_1\}, T_2 = \{t_2, t'_2, t''_2\}$
Beliefs for you	$b_1(t_1) = (\text{Nextdoor}, v'_2, t_2)$
	$b_1(t'_1) = (\text{Faraway}, v_2, t'_2)$
	$b_1(t''_1) = (\text{Closeby}, v'_2, t''_2)$
Beliefs for Barbara	$b_2(t_2) = (\text{Closeby}, v'_1, t_1)$
	$b_2(t'_2) = (0.6) \cdot (\text{Nextdoor}, v'_1, t''_1) + (0.4) \cdot (\text{Closeby}, v'_1, t_1)$
	$b_2(t''_2) = (\text{Nextdoor}, v'_1, t''_1)$

Table 2: An epistemic model for “A day at the beach”

v_i has a “subjective state space” that only includes types which reason exclusively about views contained in v_i .

Our choice of an epistemic model is closely related to that in Meier and Schipper (2014), which also encodes belief hierarchies about choices and views in the game. The crucial difference is that the model in Meier and Schipper (2014) assigns a choice and a view to every type, whereas we do not. Types in our model hold *beliefs* about the opponents’ choices and views, but are not attached to a specific choice or view themselves.

Additionally, there is a resemblance with Heifetz, Meier and Schipper (2013a) and Heinsalu (2014) who also use type structures to model probabilistic belief hierarchies on views. The key difference is that the latter two papers exogenously partition the types (or the states at which the types are defined) into different awareness levels, whereas we do not. However, within our epistemic model we can *derive* the possible awareness levels for every type t_i , which would correspond to all the views $v_i \in V_i$ such that (v_i, t_i) expresses common belief in smaller views.

As an illustration of an epistemic model, consider the one in Table 2 for the game “A day at the beach”. The beliefs for the types should be read as follows: Type t_1 for you assigns probability 1 to the event that Barbara chooses *Nextdoor Beach*, holds view v'_2 and has type t_2 . Type t'_2 for Barbara assigns probability 0.6 to the event that you choose *Nextdoor Beach*, hold view v'_1 and have type t''_1 , and assigns probability 0.4 to the event that you choose *Closeby Beach*, hold view v'_1 and have type t_1 . Similarly for the other types.

It may be verified that the view-type pairs

$$(v_1, t_1), (v_1, t'_1), (v_1, t''_1), (v'_1, t_1), (v'_1, t''_1), \\ (v_2, t_2), (v_2, t'_2), (v_2, t''_2), (v'_2, t_2), (v'_2, t'_2) \text{ and } (v'_2, t''_2)$$

all believe in smaller views. As all types in the model only assign positive probability to the view-type pairs above, we may conclude that each of the view-type pairs above expresses *common* belief in smaller views.

For every type, we can now derive the full belief hierarchy about choices and views it encodes. Consider, for instance, type t_1 for you which believes that Barbara chooses *Nextdoor Beach* while having view v'_2 , believes that Barbara believes that you choose *Closeby Beach* while having view v'_1 , believes that Barbara believes that you believe that Barbara chooses *Nextdoor Beach* while having view v'_2 , and so on. Similarly for the other types.

Note that in this scenario, you have mental access to *all* types in the model if your view is v_1 , but you do not have mental access to type t'_1 if your view is v'_1 .

Now that we know how to encode belief hierarchies on choices and views, the next step towards a formal definition of common belief in rationality is to define optimal choice for a particular view, and belief in the opponents' rationality. For a given type t_i in an epistemic model, and a choice c_i , we denote by

$$u_i(c_i, t_i) := \sum_{(c_{-i}, v_{-i}, t_{-i}) \in C_{-i} \times V_{-i} \times T_{-i}} b_i(t_i)(c_{-i}, v_{-i}, t_{-i}) \cdot u_i(c_i, c_{-i})$$

the *expected utility* induced by choice c_i under t_i 's first-order belief about the opponents' choice combinations. We now define what it means for a choice to be optimal for a view-type pair (v_i, t_i) .

Definition 3.4 (Optimal choice) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$, an epistemic model $M = (T_i, b_i)_{i \in I}$ for G^u , a type $t_i \in T_i$ and a view $v_i \in V_i$. Let $C_i(v_i)$ be the set of choices that player i has available at view v_i . A choice $c_i \in C_i(v_i)$ is optimal for the view-type pair (v_i, t_i) if

$$u_i(c_i, t_i) \geq u_i(c'_i, t_i) \text{ for all } c'_i \in C_i(v_i).$$

We next define what it means to believe in the opponents' rationality. In words, it means that you only deem possible combinations of choices, views and types for the opponent where the choice is optimal for the view and the type.

Definition 3.5 (Belief in the opponents' rationality) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$, an epistemic model $M = (T_i, b_i)_{i \in I}$ for G^u , and a type $t_i \in T_i$. We say that type t_i believes in the opponents' rationality if $b_i(t_i)$ only assigns positive probability to opponents' choice-view-type combinations $(c_j, v_j, t_j) \in C_j \times V_j \times T_j$ where the choice c_j is optimal for (v_j, t_j) .

In the epistemic model of Table 2, it may be verified that all types believe in the opponent's rationality. With this definition at hand, we can now define common belief in rationality in an iterative fashion.

Definition 3.6 (Common belief in rationality) Consider a static game with unawareness G^u and an epistemic model $M = (T_i, b_i)_{i \in I}$ for G^u . A type $t_i \in T_i$ expresses 1-fold belief in rationality if it believes in the opponents' rationality. For $k > 1$, we recursively say that a type t_i expresses k -fold belief in rationality if $b_i(t_i)$ only assigns positive probability to opponents' types that express $(k - 1)$ -fold belief in rationality. A type t_i expresses common belief in rationality if it expresses k -fold belief in rationality for every $k \geq 1$.

Hence, type t_i believes in the opponents' rationality, believes that the opponents believe in the other players' rationality, and so on. Now, consider a choice c_i for player i and a view v_i for player i that contains c_i . Rational choice under common belief in rationality with a particular view can be defined as follows.

Definition 3.7 (Rational choice under common belief in rationality) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$, a view $v_i \in V_i$, and a choice $c_i \in C_i(v_i)$ available at that view. Choice c_i can rationally be chosen under common belief in rationality with the view v_i if there is an epistemic model $M = (T_j, b_j)_{j \in I}$ and a type $t_i \in T_i$ such that (v_i, t_i) expresses common belief in smaller views, t_i expresses common belief in rationality, and c_i is optimal for (v_i, t_i) .

To illustrate these notions, consider again the epistemic model from Table 2. As all types believe in the opponent's rationality, it follows that all types in the epistemic model express *common* belief in rationality as well. Note that *Faraway Beach* is optimal for your type t_1 and the view v_1 , and *Distant Beach* is optimal for your type t'_1 and the view v_1 . As (v_1, t_1) and (v_1, t'_1) express common belief in smaller views, with the view v_1 you can rationally choose *Faraway Beach* and *Distant Beach* under common belief in rationality. In the next section we will see that these are also the *only* choices you can rationally make under common belief in rationality while holding the view v_1 .

4 Recursive Procedure

In this section we wish to characterize the choices a player can rationally make under common belief in rationality while holding a particular view. To that purpose we introduce a recursive elimination procedure, called *iterated strict dominance for unawareness*, which iteratively eliminates choices from every possible view in the game. We show that the procedure delivers, for every view, exactly those choices that can rationally be made under common belief in rationality with that particular view.

4.1 Definition

To formally define the procedure, we need some additional terminology. A *decision problem* for player i is a pair (D_i, D_{-i}) where $D_i \subseteq C_i$ and $D_{-i} \subseteq C_{-i}$. We say that $c_i \in D_i$ is *strictly*

dominated within the decision problem (D_i, D_{-i}) if there is some randomized choice $\rho_i \in \Delta(D_i)$ such that

$$u_i(c_i, c_{-i}) < \sum_{c'_i \in D_i} \rho_i(c'_i) \cdot u_i(c'_i, c_{-i}) \text{ for all } c_{-i} \in D_{-i}.$$

In the procedure below we start by defining, for every player i and every possible view $v_i \in V_i$, the full decision problem $(C_i(v_i), C_{-i}(v_i))$ that corresponds to the view v_i . Here $C_i(v_i)$ is the set of player i 's choices and $C_{-i}(v_i)$ the set of opponents' choice combinations that player i is aware of with the view v_i . At every round we then recursively reduce these decision problems at the various views by eliminating choices and opponents' choice combinations.

Definition 4.1 (Iterated strict dominance for unawareness) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$.

(Initial step) For every player i and every view $v_i \in V_i$, define the full decision problem $(C_i^0(v_i), C_{-i}^0(v_i)) := (C_i(v_i), C_{-i}(v_i))$.

(Inductive step) For $k \geq 1$, every player i , and every view $v_i \in V_i$, define

$$C_{-i}^k(v_i) := \{(c_j)_{j \neq i} \in C_{-i}^{k-1}(v_i) \mid \text{for all } j \neq i \text{ choice } c_j \text{ is in } C_j^{k-1}(v_j) \text{ for some view } v_j \in V_j \text{ that is contained in } v_i\},$$

and

$$C_i^k(v_i) := \{c_i \in C_i^{k-1}(v_i) \mid c_i \text{ not strictly dominated within the decision problem } (C_i^{k-1}(v_i), C_{-i}^k(v_i))\}.$$

A choice-view pair (c_i, v_i) is said to survive the procedure if $c_i \in C_i^k(v_i)$ for every $k \geq 0$.

Hence, in this procedure we recursively restrict, for every view v_i , the possible beliefs that player i can hold about his opponents' choices, through the sets $C_{-i}^k(v_i)$, and the possible choices that player i can make himself, through the sets $C_i^k(v_i)$. In that sense, it is very similar to the *generalized iterated strict dominance procedure* (Bach and Perea (2017)) for static games with *incomplete information*. The latter procedure recursively restricts such beliefs and choices for every possible *utility function* that player i can have in the game with incomplete information, instead of for every possible *view* in the game, as we do here.

An important difference is that in the case of incomplete information, a player with a certain utility function is able to reason about *all* other utility functions in the model – in a sense, the collection of all utility functions is common knowledge among the players – whereas the same is not true for views in a game with unawareness. Indeed, a player with a certain view can only reason about views in the model that are *contained* in his own view. This fact is reflected in the procedure above, by the way the sets $C_{-i}^k(v_i)$ are defined. Note that in $C_{-i}^k(v_i)$ we only keep those opponents' choices c_j that are in $C_j^{k-1}(v_j)$ for some view v_j that is *contained in* v_i . A

similar condition is not present in the generalized iterated strict dominance procedure for games with incomplete information.

Observe that in the special case where V_i only contains the “full view” $(C_j)_{j \in I}$ for every player i , the procedure above reduces to the well-known iterated strict dominance procedure for standard static games without unawareness.

In the following subsection we will show that this procedure always delivers a non-empty set of choices for every possible view, and indeed characterizes precisely those choice-view pairs where the choice is possible for the view under common belief in rationality.

4.2 Non-Empty Output and Characterization Result

We first show that the iterated strict dominance procedure for unawareness always yields a non-empty output. More precisely, we show that for every possible view in the game, there is always at least one choice for the respective player that survives the procedure.

Theorem 4.1 (Non-empty output) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i and every view $v_i \in V_i$ there is some choice $c_i \in C_i$ such that (c_i, v_i) survives the iterated strict dominance procedure for unawareness.*

We next present the main result in this section, showing that the iterated strict dominance procedure for unawareness selects for every view precisely those choices that can rationally be made under common belief in rationality.

Theorem 4.2 (Characterization of common belief in rationality) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i , every view $v_i \in V_i$ and every choice $c_i \in C_i(v_i)$, choice c_i can rationally be made under common belief in rationality with the view v_i , if and only if, (c_i, v_i) survives the procedure of iterated strict dominance for unawareness.*

Consider now the special case where V_i contains all possible views for every player i . That is, we do not impose any restrictions on the players’ belief hierarchies on views. Then, for every $k \geq 1$ we have that $C_{-i}^k(v_i) = C_{-i}^0(v_i)$ for every player i and view v_i , because every opponent’s choice c_j in v_i is optimal for a view of player j that is contained in v_i and in which c_j appears as the unique choice for player j . Consequently, the procedure terminates already at round 1, and every choice $c_i \in C_i^1(v_i)$ survives the procedure at v_i . In view of Theorem 4.2 we thus see that in this case, every choice that is optimal for *some* belief at a certain view can automatically be chosen rationally under *common belief in rationality* with that particular view. Hence, the concept of common belief in rationality is very permissive if we allow for all possible views in the game.

One direction of Theorem 4.2 states that if (c_i, v_i) survives the procedure, then we can always find an epistemic model, and a type $t_i \in T_i$ for player i within that epistemic model such that (v_i, t_i) expresses common belief in smaller views, t_i expresses common belief in rationality,

and the choice c_i is optimal for (v_i, t_i) . For the construction of this epistemic model we rely on Theorem 4.1, which guarantees that for every player j , and every view v_j , there is at least one choice c_j that survives the procedure at v_j .

In particular, this direction implies that for every finite static game with unawareness, we can always construct for every player i , and every view v_i , a type t_i such that (v_i, t_i) expresses common belief in smaller views and t_i expresses common belief in rationality.

Corollary 4.1 (Common belief in rationality is always possible) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. Then, for every player i and every view $v_i \in V_i$, there is an epistemic model $M = (T_j, b_j)_{j \in I}$ and a type $t_i \in T_i$ such that (v_i, t_i) expresses common belief in smaller views and t_i expresses common belief in rationality.*

In other words, for every view it is always possible to reason in accordance with common belief in rationality, while respecting the bounds set by the view.

4.3 Example

In this subsection we will illustrate the iterated strict dominance procedure for unawareness by means of the example we discussed above. To save space, we use the abbreviations f, d, n and c for the four beaches.

Example 2. Procedure for “A day at the beach”.

Consider the game with unawareness as depicted in Table 1. At the beginning of the procedure we have the full decision problems at the different views, given by

$$\begin{aligned} C_1^0(v_1) &= \{f, d, n, c\}, & C_{-1}^0(v_1) &= \{f, d, n, c\}, \\ C_1^0(v'_1) &= \{n, c\}, & C_{-1}^0(v'_1) &= \{n, c\}, \\ \\ C_2^0(v_2) &= \{f, d, n, c\}, & C_{-2}^0(v_2) &= \{f, d, n, c\} \\ C_2^0(v'_2) &= \{n, c\}, & C_{-2}^0(v'_2) &= \{n, c\}. \end{aligned}$$

Round 1. By definition we have that $C_{-1}^1(v_1) = C_{-1}^0(v_1)$, $C_{-1}^1(v'_1) = C_{-1}^0(v'_1)$, $C_{-2}^1(v_2) = C_{-2}^0(v_2)$ and $C_{-2}^1(v'_2) = C_{-2}^0(v'_2)$. Note that c is strictly dominated for you by the randomized choice $(0.5) \cdot f + (0.5) \cdot d$ within the decision problem $(C_1^0(v_1), C_{-1}^1(v_1))$, and that d is strictly dominated for Barbara by $(0.5) \cdot n + (0.5) \cdot c$ within her decision problem $(C_2^0(v_2), C_{-2}^1(v_2))$. No other choices are strictly dominated in this round. We can therefore eliminate your choice c from $C_1^0(v_1)$ and Barbara’s choice d from $C_2^0(v_2)$, yielding the reduced decision problems

$$\begin{aligned} C_1^1(v_1) &= \{f, d, n\}, & C_{-1}^1(v_1) &= \{f, d, n, c\}, \\ C_1^1(v'_1) &= \{n, c\}, & C_{-1}^1(v'_1) &= \{n, c\}, \\ \\ C_2^1(v_2) &= \{f, n, c\}, & C_{-2}^1(v_2) &= \{f, d, n, c\}, \\ C_2^1(v'_2) &= \{n, c\}, & C_{-2}^1(v'_2) &= \{n, c\}. \end{aligned}$$

Round 2. As Barbara's choice d is not in her decision problems at v_2 and v'_2 anymore, we can eliminate Barbara's choice d from your current decision problem at v_1 . That is, $C_{-1}^2(v_1) = \{f, n, c\}$ and $C_{-1}^2(v'_1) = \{n, c\}$. Note that we cannot eliminate your choice c from Barbara's decision problems at v_2 and v'_2 , since your choice c is still present in $C_1^1(v'_1)$, and your view v'_1 is contained in both v_2 and v'_2 . We thus have that $C_{-2}^2(v_2) = \{f, d, n, c\}$ and $C_{-2}^2(v'_2) = \{n, c\}$.

In your decision problem $(C_1^1(v_1), C_{-1}^2(v_1)) = (\{f, d, n\}, \{f, n, c\})$ at v_1 , your choice n is strictly dominated by d , and can thus be eliminated from $C_1^1(v_1)$. No other choices can be eliminated in this round. We thus obtain the reduced decision problems

$$\begin{aligned} C_1^2(v_1) &= \{f, d\}, & C_{-1}^2(v_1) &= \{f, n, c\}, \\ C_1^2(v'_1) &= \{n, c\}, & C_{-1}^2(v'_1) &= \{n, c\}, \\ \\ C_2^2(v_2) &= \{f, n, c\}, & C_{-2}^2(v_2) &= \{f, d, n, c\}, \\ C_2^2(v'_2) &= \{n, c\}, & C_{-2}^2(v'_2) &= \{n, c\}. \end{aligned}$$

After this round no further choices can be eliminated at any of the possible views, and hence the procedure terminates at the end of round 2. The choice-view pairs that survive for you are (f, v_1) , (d, v_1) , (n, v'_1) and (c, v'_1) , whereas the choice-view pairs surviving for Barbara are (f, v_2) , (n, v_2) , (c, v_2) , (n, v'_2) and (c, v'_2) .

Hence, in view of Theorem 4.2, these are exactly the choice-view pairs that are possible under common belief in rationality. That is, under common belief in rationality, you can rationally choose *Faraway Beach* and *Distant Beach* with the view v_1 , you can rationally choose *Nextdoor Beach* and *Closeby Beach* with the view v'_1 , Barbara can rationally choose *Faraway Beach*, *Nextdoor Beach* and *Closeby Beach* with the view v_2 , and can rationally choose *Nextdoor Beach* and *Closeby Beach* with the view v'_2 .

5 Fixed Beliefs on Views

In the literature on games with unawareness, it is typically assumed that every player holds some *exogenously given belief hierarchy on views*. See, for instance, Feinberg (2012), Rêgo and Halpern (2012) and Heifetz, Meier and Schipper (2013b). Following this approach, we reconcile in this section the concept of common belief in rationality with the assumption that the *belief hierarchy on views is fixed*. One important difference with Feinberg (2012) and Heifetz, Meier and Schipper (2013b) is that we allow for *truly probabilistic* belief hierarchies on views, and not only belief hierarchies consisting of probability 1 beliefs on views. The reason is that we wish to allow for situations in which a player is *uncertain* about the precise view adopted by his opponent, and therefore assigns positive probability to various possible views for this opponent.

5.1 Common Belief in Rationality with Fixed Beliefs on Views

Different from Feinberg (2012), Rêgo and Halpern (2012) and Heifetz, Meier and Schipper (2013b), but in accordance with, for instance, Heinsalu (2014), Heifetz, Meier and Schipper (2013a) and Meier and Schipper (2014), we decide to encode belief hierarchies on views by means of epistemic models with *types*. The reason is that such encodings are easy to work with, and turn out to be convenient for designing proofs and an associated elimination procedure as well. Such an epistemic model may be seen as a reduced version of the one used in Section 3, since now a type only holds a belief about the opponents' *views* and types, instead of the opponents' *choices*, *views* and types.

Definition 5.1 (Epistemic model for views) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$. An epistemic model for views is a tuple $M^{view} = (R_i, p_i)_{i \in I}$ where R_i is the finite set of types for player i , and p_i is a belief mapping that assigns to every type $r_i \in R_i$ some probabilistic belief $p_i(r_i) \in \Delta(V_{-i} \times R_{-i})$.

We call the types in this model *view-types*, since they generate belief hierarchies on views. Similarly as before, we can define *common belief in smaller views* in order to rule out unreasonable belief hierarchies on views.

Definition 5.2 (Common belief in smaller views) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model for views $M^{view} = (R_i, p_i)_{i \in I}$. A view-type pair $(v_i, r_i) \in V_i \times R_i$ believes in smaller views if $p_i(r_i)$ only assigns positive probability to opponents' views v_j that are contained in v_i . The view-type pair (v_i, r_i) is said to express 1-fold belief in smaller views if it believes in smaller views. For $k > 1$, we recursively say that (v_i, r_i) expresses k -fold belief in smaller views if $p_i(r_i)$ only assigns positive probability to opponents' view-type pairs (v_j, r_j) that express $(k-1)$ -fold belief in smaller views. The pair (v_i, r_i) expresses common belief in smaller views if it expresses k -fold belief in smaller views for all $k \geq 1$.

For every view-type $r_i \in R_i$, let $h_i^{view}(r_i)$ be the belief hierarchy on views induced by r_i . The precise construction of this belief hierarchy can be found in Section 8.2.1 of the appendix. Note that if (v_i, r_i) expresses common belief in smaller views, then the belief hierarchy $h_i^{view}(r_i)$ on views only considers, at each of its layers, views that are contained in v_i , as it should be.

Compare this to the epistemic models we considered in Definition 3.1, used to encode belief hierarchies on *choices and views*. In such an epistemic model $M = (T_i, b_i)_{i \in I}$, every type t_i induces a belief hierarchy on choices and views, and hence also on views alone. Let $h_i^{view}(t_i)$ be the induced belief hierarchy on views. The precise construction of $h_i^{view}(t_i)$ can be found in Section 8.2.2 of the appendix.

With these definitions at hand, we can now formally define what we mean by common belief in rationality with fixed beliefs on views.

Definition 5.3 (Common belief in rationality with fixed beliefs on views) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$, an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views, and a view-type pair $(v_i, r_i) \in V_i \times R_i$ that expresses common belief in smaller views. A choice $c_i \in C_i(v_i)$ can rationally be made under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i , if there is an epistemic model $M = (T_j, b_j)_{j \in I}$ for choices and views, and a type $t_i \in T_i$ such that $h_i^{view}(t_i) = h_i^{view}(r_i)$, type t_i expresses common belief in rationality, and c_i is optimal for (v_i, t_i)

Note that if (v_i, r_i) expresses common belief in smaller views and $h_i^{view}(t_i) = h_i^{view}(r_i)$, then also (v_i, t_i) expresses common belief in smaller views. In the following sections we will design a procedure that yields precisely the choices that can rationally be made under this concept, and show that it is always possible to reason in accordance with this concept.

5.2 Recursive Procedure

We will now present a recursive elimination procedure, called *iterated strict dominance with fixed beliefs on views*, that characterizes precisely those choices that can rationally be made, with every possible view, under common belief in rationality with a fixed belief hierarchy on views. Not surprisingly, the procedure is quite similar to *iterated strict dominance for unawareness* (without fixed belief hierarchies on views). There are two important differences. The first is that decision problems will now be defined for every view v_i and every view-type $r_i \in R_i$ such that (v_i, r_i) expresses common belief in smaller views. Moreover, the sets $C_{-i}^k(v_i)$ of opponents' choice combinations as defined in iterated strict dominance with unawareness, restricting the possible beliefs that player i can hold at round k , will now be replaced by sets of possible *probabilistic* beliefs $B_i^k(v_i, r_i)$, representing the possible probabilistic beliefs that player i can hold at round k if he holds view v_i and has the belief hierarchy on views induced by r_i .

To define the procedure formally, we need some additional notation. Consider some Euclidean space \mathbf{R}^n , some subsets A_1, \dots, A_K of \mathbf{R}^n , and some numbers $x_1, \dots, x_K \in \mathbf{R}$. Then, by

$$\sum_{k \in \{1, \dots, K\}} x_k \cdot A_k := \left\{ \sum_{k \in \{1, \dots, K\}} x_k \cdot a_k \mid a_k \in A_k \text{ for all } k \in \{1, \dots, K\} \right\}$$

we define the corresponding ‘‘linear combination’’ of these sets A_1, \dots, A_K .

Definition 5.4 (Iterated strict dominance with fixed beliefs on views) Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views.

(Initial step) For every player i , every view $v_i \in V_i$, and every view-type $r_i \in R_i$ such that (v_i, r_i) expresses common belief in smaller views, define

$$B_i^0(v_i, r_i) := \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \Delta(\times_{j \neq i} C_j(v_j)),$$

and $C_i^0(v_i, r_i) := C_i(v_i)$.

(Inductive step) For $k \geq 1$, every player i , every view $v_i \in V_i$ and every view-type $r_i \in R_i$ such that (v_i, r_i) expresses common belief in smaller views, define

$$B_i^k(v_i, r_i) := \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \Delta(\times_{j \neq i} C_j^{k-1}(v_j, r_j)),$$

and

$$C_i^k(v_i, r_i) := \{c_i \in C_i^{k-1}(v_i, r_i) \mid c_i \text{ is optimal for some belief } \beta_i \in B_i^k(v_i, r_i) \text{ among choices in } C_i^{k-1}(v_i, r_i)\}.$$

A triple (c_i, v_i, r_i) , consisting of a choice, view and view-type, is said to survive the procedure if $c_i \in C_i^k(v_i, r_i)$ for every $k \geq 0$.

More precisely, this procedure is the iterated strict dominance procedure with fixed beliefs on views as given by M^{view} . As a short-hand, we will refer to this procedure as the *iterated strict dominance procedure for M^{view}* .

Consider now the special case where every view-type in M^{view} assigns probability 1 to one specific view for every opponent. Then, it may be verified that the procedure above is equivalent to the *extensive-form rationalizability procedure* in Heifetz, Meier and Schipper (2013b), when applied to the special case of static games. The procedure in Heifetz, Meier and Schipper (2013b) is designed for *dynamic* games with unawareness, and hence can also be applied to *static* games.

Our procedure above is quite similar to the *interim correlated rationalizability* procedure (Dekel, Fudenberg and Morris (2007)) for games with *incomplete information*, which in turn is analogous to the concept of *interim (independent) rationalizability* in Ely and Pęski (2006). Also interim correlated rationalizability assumes a fixed belief hierarchy, not on views but on *utility functions*. The interim correlated rationalizability procedure then recursively restricts, for every possible *utility function* and every belief hierarchy on *utilities*, the set of choices for the respective player. In turn, we recursively restrict the player's set of choices for every possible *view* and belief hierarchy on *views* (as encoded by a view-type r_i).

Similarly to the case without fixed belief hierarchies on views, there is still an important difference between the two procedures. In the case of unawareness, not every belief hierarchy on views can be chosen, because this belief hierarchy must express common belief in smaller views for an appropriately chosen view of the respective player. A similar condition is not present in the case of incomplete information, as in principle every possible belief hierarchy on utility functions may be regarded as reasonable. The reason, again, is that in the context of incomplete information, a player with a certain utility function has mental access to *all* utility functions in the model – something that is not true for views in games with unawareness.

5.3 Non-Empty Output and Characterization Result

Like in Section 4, we first show that the procedure always delivers a non-empty output, and subsequently prove that the procedure yields, for every view and view-type, exactly those choices that can rationally be made under common belief in rationality with this particular view and view-type.

Theorem 5.1 (Non-empty output) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views. Then, for every player i , every view $v_i \in V_i$ and every view-type $r_i \in R_i$ such that (v_i, r_i) expresses common belief in smaller views, there is some choice $c_i \in C_i$ such that (c_i, v_i, r_i) survives the iterated strict dominance procedure for M^{view} .*

The reader will note that the proof for this result is very similar to the one we gave for Theorem 4.1. We thus conclude that, no matter which belief hierarchy on views we impose, it is always possible for a player to reason in accordance with this particular belief hierarchy on views, while respecting common belief in rationality.

We next show that the procedure selects, for every view and every belief hierarchy on views encoded by M^{view} , exactly those choices that can rationally be made under common belief in rationality for this specific view and belief hierarchy on views.

Theorem 5.2 (Characterization of common belief in rationality) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views. Then, for every player i , every choice $c_i \in C_i$, every view $v_i \in V_i$ and every view-type $r_i \in R_i$ such that (v_i, r_i) expresses common belief in smaller views, player i can rationally choose c_i under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i , if and only if, (c_i, v_i, r_i) survives the iterated strict dominance procedure for M^{view} .*

Also here, the proof follows a similar structure as the one for Theorem 4.2. From Theorem 5.1 we know that the procedure always delivers a non-empty set of choices for every possible view and view-type in the game. The “if” direction of Theorem 5.2 therefore implies that for every view v_i and view-type r_i such that (v_i, r_i) expresses common belief in smaller views, we can always construct an epistemic model and a type t_i within it that expresses common belief in rationality, and which holds the belief hierarchy on views induced by r_i . The following result thus obtains.

Corollary 5.1 (Common belief in rationality is always possible) *Consider a static game with unawareness $G^u = (G^{base}, (V_i)_{i \in I})$ and an epistemic model $M^{view} = (R_i, p_i)_{i \in I}$ for views. Then, for every player i , every view $v_i \in V_i$ and view-type $r_i \in R_i$ such that (v_i, r_i) expresses common belief in smaller views, there is an epistemic model $M = (T_j, b_j)_{j \in I}$ and a type $t_i \in T_i$, such that t_i has the belief hierarchy on views induced by r_i , and expresses common belief in rationality.*

In other words, it is always possible to reason in accordance with common belief in rationality, while respecting the bounds set by a fixed view and a fixed belief hierarchy on views.

5.4 Example

To see how the procedure of *iterated strict dominance with fixed beliefs on views* works, consider the example “A day at the beach”.

Example 3. Procedure for “A day at the beach”

Recall that you are unsure whether Barbara is aware of the two remote beaches or not. Assume now that you deem the event that she is aware of these two beaches equally likely as the event that she is not. In case Barbara is aware of the two remote beaches, you believe that Barbara believes that you are also aware of these two beaches. Indeed, you know by experience that Barbara believes that you are aware of everything that she is aware of herself. In case Barbara is not aware of these two beaches, she must of course believe that you are also not aware of these. This situation can be summarized by Table 3, with the fixed belief hierarchy on views induced by your view-type r_1 at the bottom of the table. This belief hierarchy on views is also graphically represented by the arrows between the various views. Indeed, if you have view v_1 and view-type r_1 , then the induced belief hierarchy on views matches exactly the story above. Note that in M^{view} you have mental access to *all* view-types in the model if your view is v_1 , whereas you only have mental access to the types r'_1 and r'_2 if your view is v'_1 . Similarly for Barbara.

The *iterated strict dominance procedure for M^{view}* proceeds as follows.

Initial step. Note that, given the epistemic model for views M^{view} , the only relevant pairs of views and view-types are (v_1, r_1) , (v'_1, r'_1) , (v_2, r_2) and (v'_2, r'_2) . The initial sets of beliefs are given by

$$\begin{aligned}
B_1^0(v_1, r_1) &= (0.5) \cdot \Delta(C_2(v_2)) + (0.5) \cdot \Delta(C_2(v'_2)) \\
&= (0.5) \cdot \Delta(\{f, d, n, c\}) + (0.5) \cdot \Delta(\{n, c\}) \\
&= \{\beta_1 \in \Delta(\{f, d, n, c\}) \mid \beta_1(f) \leq 0.5, \beta_1(d) \leq 0.5\} \\
B_1^0(v'_1, r'_1) &= \Delta(C_2(v'_2)) = \Delta(\{n, c\}), \\
B_2^0(v_2, r_2) &= \Delta(C_1(v_1)) = \Delta(\{f, d, n, c\}), \\
B_2^0(v'_2, r'_2) &= \Delta(C_1(v'_1)) = \Delta(\{n, c\}),
\end{aligned}$$

whereas the initial sets of choices are

$$\begin{aligned}
C_1^0(v_1, r_1) &= \{f, d, n, c\}, \quad C_1^0(v'_1, r'_1) = \{n, c\}, \\
C_2^0(v_2, r_2) &= \{f, d, n, c\}, \quad C_2^0(v'_2, r'_2) = \{n, c\}.
\end{aligned}$$

		G^{base}						
Base game		Faraway	Distant	Nextdoor	Closeby			
	Faraway	0, 0	4, 1	4, 4	4, 3			
	Distant	3, 2	0, 0	3, 4	3, 3			
	Nextdoor	2, 2	2, 1	0, 0	2, 3			
		Closeby	1, 2	1, 1	1, 4	0, 0		
Your views		v_1	Faraway	Distant	Nextdoor	Closeby		
	Faraway	0	4	4	4	4		
	Distant	3	0	3	3	3		
	Nextdoor	2	2	0	2	2		
	Closeby	1	1	1	0	0		
			↓ (0.5)		(0.5) ↘			
			↑					
Barbara's views		v_2	Faraway	Distant	Nextdoor	Closeby		
	Faraway	0	2	2	2	2		
	Distant	1	0	1	1	1		
	Nextdoor	4	4	0	4	4		
	Closeby	3	3	3	3	0		

		v'_1					
		Nextdoor	Closeby				
Nextdoor	0	2					
Closeby	1	0					

↓

↑

		v'_2					
		Nextdoor	Closeby				
Nextdoor	0	4					
Closeby	3	0					

Epistemic model for views M^{view}

$$R_1 = \{r_1, r'_1\}, R_2 = \{r_2, r'_2\}$$

$$p_1(r_1) = (0.5) \cdot (v_2, r_2) + (0.5) \cdot (v'_2, r'_2)$$

$$p_1(r'_1) = (v'_2, r'_2)$$

$$p_2(r_2) = (v_1, r_1)$$

$$p_2(r'_2) = (v'_1, r'_1)$$

Table 3: “A day at the beach” with fixed beliefs on views

Round 1. By definition, the sets of beliefs remain the same as in the initial step. Note that choice c is not optimal for you at view v_1 for any belief in $B_1^1(v_1, r_1)$, and that Barbara's choice d is not optimal for her at view v_2 for any belief in $B_2^1(v_2, r_2)$. Hence, we obtain

$$\begin{aligned} C_1^1(v_1, r_1) &= \{f, d, n\}, & C_1^1(v'_1, r'_1) &= \{n, c\}, \\ C_2^1(v_2, r_2) &= \{f, n, c\}, & C_2^1(v'_2, r'_2) &= \{n, c\}. \end{aligned}$$

Round 2. The new sets of beliefs are

$$\begin{aligned} B_1^2(v_1, r_1) &= (0.5) \cdot \Delta(C_2^1(v_2, r_2)) + (0.5) \cdot \Delta(C_2^1(v'_2, r'_2)) \\ &= (0.5) \cdot \Delta\{f, n, c\} + (0.5) \cdot \Delta\{n, c\} \\ &= \{\beta_1 \in \Delta(\{f, n, c\} \mid \beta_1(f) \leq 0.5)\} \\ B_1^2(v'_1, r'_1) &= \Delta(C_2^1(v'_2, r'_2)) = \Delta(\{n, c\}), \\ B_2^2(v_2, r_2) &= \Delta(C_1^1(v_1, r_1)) = \Delta(\{f, d, n\}), \\ B_2^2(v'_2, r'_2) &= \Delta(C_1^1(v'_1, r'_1)) = \Delta(\{n, c\}). \end{aligned}$$

Then, your choice n is not optimal at your view v_1 for any belief in $B_1^2(v_1, r_1)$. Moreover, Barbara's choice f is not optimal at her view v_2 for any belief in $B_2^2(v_2, r_2)$. The new sets of choices are thus given by

$$\begin{aligned} C_1^2(v_1, r_1) &= \{f, d\}, & C_1^2(v'_1, r'_1) &= \{n, c\}, \\ C_2^2(v_2, r_2) &= \{n, c\}, & C_2^2(v'_2, r'_2) &= \{n, c\}. \end{aligned}$$

Round 3. The new sets of beliefs are

$$\begin{aligned} B_1^3(v_1, r_1) &= (0.5) \cdot \Delta(C_2^2(v_2, r_2)) + (0.5) \cdot \Delta(C_2^2(v'_2, r'_2)) \\ &= \Delta(\{n, c\}), \\ B_1^3(v'_1, r'_1) &= \Delta(C_2^2(v'_2, r'_2)) = \Delta(\{n, c\}), \\ B_2^3(v_2, r_2) &= \Delta(C_1^2(v_1, r_1)) = \Delta(\{f, d\}), \\ B_2^3(v'_2, r'_2) &= \Delta(C_1^2(v'_1, r'_1)) = \Delta(\{n, c\}). \end{aligned}$$

Note that at your view v_1 , your choice d is not optimal for any belief in $B_1^3(v_1, r_1)$. Moreover, Barbara's choice c is not optimal at her view v_2 for any belief in $B_2^3(v_2, r_2)$. Hence, the new sets of choices are

$$\begin{aligned} C_1^3(v_1, r_1) &= \{f\}, & C_1^3(v'_1, r'_1) &= \{n, c\}, \\ C_2^3(v_2, r_2) &= \{n\}, & C_2^3(v'_2, r'_2) &= \{n, c\}. \end{aligned}$$

Round 4. The new sets of beliefs are

$$\begin{aligned}
B_1^4(v_1, r_1) &= (0.5) \cdot \Delta(C_2^3(v_2, r_2)) + (0.5) \cdot \Delta(C_2^3(v'_2, r'_2)) \\
&= (0.5) \cdot \Delta(\{n\}) + (0.5) \cdot \Delta(\{n, c\}) \\
&= \{\beta_1 \in \Delta(\{n, c\}) \mid \beta_1(c) \leq 0.5\}, \\
B_1^4(v'_1, r'_1) &= \Delta(C_2^3(v'_2, r'_2)) = \Delta(\{n, c\}), \\
B_2^4(v_2, r_2) &= \Delta(C_1^3(v_1, r_1)) = \Delta(f), \\
B_2^4(v'_2, r'_2) &= \Delta(C_1^3(v'_1, r'_1)) = \Delta(\{n, c\}).
\end{aligned}$$

Since no further choices can be eliminated from $C_1^3(v_1, r_1)$, $C_1^3(v'_1, r'_1)$, $C_2^3(v_2, r_2)$ and $C_2^3(v'_2, r'_2)$ we have that

$$\begin{aligned}
C_1^4(v_1, r_1) &= C_1^3(v_1, r_1) = \{f\}, & C_1^4(v'_1, r'_1) &= C_1^3(v'_1, r'_1) = \{n, c\}, \\
C_2^4(v_2, r_2) &= C_2^3(v_2, r_2) = \{n\}, & C_2^4(v'_2, r'_2) &= C_2^3(v'_2, r'_2) = \{n, c\},
\end{aligned}$$

and the procedure terminates.

We thus conclude that you can only rationally go to the *Faraway Beach* under common belief in rationality with the view v_1 and the belief hierarchy on views induced by r_1 .

Compare this to the case where we did not fix the belief hierarchy on views. As we saw in Section 4, you could rationally visit the *Faraway Beach* and the *Distant Beach* under common belief in rationality with the view v_1 if we allow for *any* belief hierarchy on views that is cognitively feasible for v_1 and expresses common belief in smaller views. Indeed, the epistemic model from Table 2 shows that under common belief in rationality with the view v_1 , you can rationally choose the *Distant Beach* if you hold the belief hierarchy induced by type t'_1 . In that belief hierarchy, you believe that Barbara has view v_2 , believe that Barbara believes that you have view v'_1 , believe that Barbara believes that you believe that Barbara has view v'_2 , believe that Barbara believes that you believe that Barbara believes that you have view v'_1 , and so on. Clearly, this belief hierarchy is different from the one induced by r_1 .

To conclude this section, we compare the case of fixed belief hierarchies on views to the case where these belief hierarchies are left free. Clearly, if for a given view v_i we look at each individual belief hierarchy on views h_i^{view} such that v_i together with h_i^{view} expresses common belief in smaller views, then this is the same as putting no restrictions on the belief hierarchy on views (except for common belief in smaller views). Consequently, if for every such belief hierarchy on views h_i^{view} we derive the choices that player i can rationally make under common belief in rationality with the view v_i and this particular belief hierarchy on views h_i^{view} (as defined in this section), then we should obtain exactly the choices that player i can rationally make under common belief in rationality with the view v_i (as defined in Section 4). We know by Theorem 5.2 that the choices that player i can rationally make under common belief in rationality with the

view v_i and the fixed belief hierarchy on views h_i^{view} are given by the iterated strict dominance procedure with fixed beliefs on views. On the other hand, Theorem 4.2 guarantees that the choices that player i can rationally make under common belief in rationality with the view v_i are given by the iterated strict dominance procedure for unawareness. Consequently, if for every possible belief hierarchy on views h_i^{view} , such that v_i together with h_i^{view} expresses common belief in smaller views, we run the iterated strict dominance procedure with fixed beliefs on views, and collect all the delivered choices for player i at view v_i , this will deliver exactly the same output as when we would run the iterated strict dominance procedure for unawareness (without fixed beliefs on views) and look at the delivered choices for player i at v_i .

6 Related Literature

Roughly speaking, the literature on unawareness can be divided into two categories. The first category explores the logical foundations of unawareness in a single agent and multi-agent setting, without an explicit reference to games, whereas the second category applies the logic of unawareness to games. For a survey of this literature we refer the reader to Schipper (2014),

An important question being addressed by the first category is how unawareness can be modeled in a meaningful way, both syntactically and semantically. See, for instance, Fagin and Halpern (1988), Dekel, Lipman and Rustichini (1998), Modica and Rustichini (1999), Halpern (2001), Heifetz, Meier and Schipper (2006, 2008, 2013a), Halpern and Rêgo (2008) and Li (2009).

A general conclusion in this literature is that in a multi-agent setting, every agent must be endowed with his own, *subjective* state space that only contains those objects he is aware of, and which therefore may be substantially smaller than the *full* state space. This principle is also reflected in our definition of a game with unawareness, and how we set up an epistemic model to encode belief hierarchies about choices and views.

To model a game with unawareness, we assume for every player a finite collection of possible views on the game. The implicit understanding is that a player with a certain view only has mental access to those choices that are part of his view, and to those views in the model that are smaller than his own. In other words, the subjective state space for a player with view v_i only contains the choices inside v_i , and the views for the opponents and himself that are contained in v_i .

Similarly, in the epistemic model we use to encode belief hierarchies on choices and views, the implicit understanding is that a player with view v_i only has mental access to choices in v_i , opponents' views that are contained in v_i , and types (hence, belief hierarchies) that only reason about views that are contained in v_i . The latter objects thus constitute the subjective state space for player i in the epistemic model if his view is v_i , and may thus be substantively smaller than the full epistemic model.

Papers in the second category deal specifically with static or dynamic games with unawareness, and can thus be seen as applications of the logic of unawareness. See, for instance, Feinberg

(2004, 2012), Čopič and Galeotti (2006), Rêgo and Halpern (2012), Heifetz, Meier and Schipper (2013b), Grant and Quiggin (2013), Halpern and Rêgo (2014), Meier and Schipper (2014) and Schipper (2017). Our paper clearly falls within this category as well.

As we already mentioned in Section 2, an important difference between our way of modelling games with unawareness and that of most other papers is that we do not exogenously specify a unique belief hierarchy on views for every player. In fact, of the abovementioned papers only Čopič and Galeotti (2006) and Meier and Schipper (2014) do not fix the belief hierarchies on views in their model. Moreover, we allow for *probabilistic* belief hierarchies on views, whereas most papers above – exceptions being Feinberg (2004), Rêgo and Halpern (2012), Halpern and Rêgo (2014), Heifetz, Meier and Schipper (2013a) and Meier and Schipper (2014) – restrict to deterministic belief hierarchies on views. We find such probabilistic beliefs on views important, as it allows for cases where a player is truly uncertain about the precise view held by an opponent.

In terms of the approach adopted, this paper is one of the few to provide an *epistemic* analysis of the players' reasoning in games with unawareness, through the epistemic conditions of common belief in rationality. Another example is Guarino (2017, Chapter 3), who offers an epistemic characterization of *extensive-form rationalizability* (Pearce (1984), Battigalli (1997), Heifetz, Meier and Schipper (2013b)) for dynamic games with unawareness.

Like our paper, also Feinberg (2012) and Heifetz, Meier and Schipper (2013b) investigate the implications of common (strong) belief in rationality by studying the concepts of *rationalizability* and *extensive-form rationalizability*, respectively. One difference with our approach is that the latter papers do not investigate these concepts on an epistemic basis.

7 Concluding Remarks

The goal of this paper has been to investigate the reasoning of players in static games with unawareness through the basic concept of *common belief in rationality*. Our approach has been primarily *epistemic*, as we started by formulating the epistemic conditions that constitute common belief in rationality, and subsequently designed a recursive elimination procedure that characterizes exactly those choices that can rationally be made, for every possible view, under this epistemic concept. We did so for two scenarios: one in which we only restrict the possible views that may enter the players' belief hierarchies, and one in which we fix the players' belief hierarchies on views.

An interesting open question is how one can epistemically characterize various *equilibrium* concepts that have been proposed for games with unawareness, such as *action-awareness equilibrium* (Čopič and Galeotti (2006)), *extended Nash equilibrium* (Feinberg (2012)), *generalized Nash equilibrium* (Halpern and Rêgo (2014)), *generalized sequential equilibrium* (Rêgo and Halpern (2012)), *sequential equilibrium* (Grant and Quiggin (2013)), *equilibrium of Bayesian game with unawareness* (Meier and Schipper (2014)) and *self-confirming equilibrium* (Schipper (2017)).

Another problem that could be addressed in the future is how one could formulate the

backward induction concept of *common belief in future rationality* (Perea (2014)) for dynamic games with unawareness. Moreover, it could be explored how this concept would relate to the forward induction concept of *extensive-form rationalizability* as defined by Heifetz, Meier and Schipper (2013b) for dynamic games with unawareness. These, and other, open problems are left for future research.

8 Appendix

8.1 Proofs of Section 4

For the proofs of Section 4 we heavily rely on Lemma 3 in Pearce (1984). We will present this result below within the framework of decision problems, because we can then readily apply it for our specific purposes. Consider a decision problem (D_i, D_{-i}) , a choice $c_i \in D_i$ and a probabilistic belief $\beta_i \in \Delta(D_{-i})$ about the opponents' choice combinations. Choice c_i is said to be *optimal* for β_i within the decision problem (D_i, D_{-i}) if

$$\sum_{c_{-i} \in D_{-i}} \beta_i(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i} \in D_{-i}} \beta_i(c_{-i}) \cdot u_i(c'_i, c_{-i}) \text{ for all } c'_i \in D_i.$$

Lemma 3 in Pearce (1984) states that a choice is optimal for at least one belief, if and only if, the choice is not strictly dominated.

Lemma 8.1 (Pearce (1984)) *Consider a decision problem (D_i, D_{-i}) and an available choice $c_i \in D_i$. Then, c_i is optimal for some probabilistic belief within the decision problem (D_i, D_{-i}) , if and only if, c_i is not strictly dominated within the decision problem (D_i, D_{-i}) .*

As we will see, this result is the cornerstone to the proofs of Section 4.

Proof of Theorem 4.1. Note that in the iterated strict dominance procedure for unawareness, $C_i^{k+1}(v_i) \subseteq C_i^k(v_i)$ for every player i , every view $v_i \in V_i$ and every round $k \geq 0$. Since there are only finitely many choices and views in the game, the procedure must terminate after finitely many rounds. That is, there is some $K \geq 0$ such that $C_i^k(v_i) = C_i^K(v_i)$ and $C_{-i}^k(v_i) = C_{-i}^K(v_i)$ for every player i , view $v_i \in V_i$ and every $k \geq K$. As such, it is sufficient to show that $C_i^k(v_i)$ is always non-empty for every player i , every view $v_i \in V_i$ and every $k \geq 0$. We prove so by induction on k .

For $k = 0$ this is clear since $C_i^0(v_i) = C_i(v_i)$, which is non-empty.

Take now some $k \geq 1$ and assume that $C_j^{k-1}(v_j)$ is non-empty for every player j and every view $v_j \in V_j$. Consider some player i and some view $v_i \in V_i$. We show that $C_i^k(v_i)$ is non-empty.

For every opponent $j \neq i$, take some view $v_j \in V_j$ that is contained in v_i . Note that such view v_j exists by Definition 2.1. For every opponent $j \neq i$, take a choice $c_j \in C_j^{k-1}(v_j)$,

which is possible because $C_j^{k-1}(v_j)$ is non-empty by the induction assumption. Then, by construction, the choice combination $(c_j)_{j \neq i}$ is in $C_{-i}^k(v_i)$. Let the choice $c_i \in C_i(v_i)$ be optimal, among all choices in $C_i(v_i)$, for the belief β_i that assigns probability 1 to $(c_j)_{j \neq i}$. Hence, $\beta_i \in \Delta(C_{-i}^k(v_i))$. By Lemma 8.1 it then follows that c_i is not strictly dominated within the decision problem $(C_i(v_i), C_{-i}^k(v_i))$. In particular, c_i is not strictly dominated within the decision problem $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$, and hence $c_i \in C_i^k(v_i)$. We thus conclude that $C_i^k(v_i)$ is non-empty.

By induction, it follows that $C_i^k(v_i)$ is always non-empty for every player i , every view $v_i \in V_i$ and every round $k \geq 0$. As we have seen, this completes the proof. \blacksquare

Proof of Theorem 4.2. “Only if”: For every player i and every view $v_i \in V_i$, let $C_i^{cbr}(v_i)$ be the set of choices in $C_i(v_i)$ that player i can rationally make under common belief in rationality with the view v_i . We show, by induction on k , that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$ for every $k \geq 0$, every player i and every view $v_i \in V_i$.

For $k = 0$ this is obviously true since $C_i^0(v_i) = C_i(v_i)$.

Now, consider some $k \geq 1$ and assume that $C_i^{cbr}(v_i) \subseteq C_i^{k-1}(v_i)$ for every player i and every view $v_i \in V_i$. Consider some player i , some view v_i , and assume that $c_i \in C_i^{cbr}(v_i)$. By the induction assumption we know that $c_i \in C_i^{k-1}(v_i)$. As $c_i \in C_i^{cbr}(v_i)$, there is some epistemic model $M = (T_j, b_j)_{j \in I}$ and some type $t_i \in T_i$ such that (v_i, t_i) expresses common belief in smaller views, t_i expresses common belief in rationality, and c_i is optimal for (v_i, t_i) . Let $b_i^C(t_i)$ be the marginal of the belief $b_i(t_i)$ on C_{-i} . Then, in light of the above,

$$\sum_{c_{-i} \in C_{-i}(v_i)} b_i^C(t_i)(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c'_{-i} \in C_{-i}(v_i)} b_i^C(t_i)(c'_{-i}) \cdot u_i(c'_i, c_{-i}) \text{ for all } c'_i \in C_i(v_i). \quad (8.1)$$

Since (v_i, t_i) believes in smaller views, and t_i expresses common belief in rationality, we conclude that $b_i^C(t_i)((c_j)_{j \neq i}) > 0$ only if, for every $j \neq i$, choice c_j is in $C_j^{cbr}(v_j)$ for some view v_j that is contained in v_i . Since by the induction assumption we have that $C_j^{cbr}(v_j) \subseteq C_j^{k-1}(v_j)$, we conclude that $b_i^C(t_i)((c_j)_{j \neq i}) > 0$ only if, for every $j \neq i$, choice c_j is in $C_j^{k-1}(v_j)$ for some view v_j that is contained in v_i . Hence, by definition of the procedure, $b_i^C(t_i) \in \Delta(C_{-i}^k(v_i))$.

In view of (8.1) we thus conclude that $c_i \in C_i^{k-1}(v_i)$ is optimal for the belief $b_i^C(t_i) \in \Delta(C_{-i}^k(v_i))$ within the reduced decision problem $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$. By Lemma 8.1 it then follows that c_i is not strictly dominated for the reduced decision problem $(C_i^{k-1}(v_i), C_{-i}^k(v_i))$, and hence $c_i \in C_i^k(v_i)$, by definition of the procedure. As this holds for every $c_i \in C_i^{cbr}(v_i)$, we conclude that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$, which was to show. By induction on k we conclude that $C_i^{cbr}(v_i) \subseteq C_i^k(v_i)$ for every k , every player i and every view $v_i \in V_i$.

Now, consider a player i , a view $v_i \in V_i$, and a choice $c_i \in C_i(v_i)$ that can rationally be made under common belief in rationality with the view v_i . Then, $c_i \in C_i^{cbr}(v_i)$ and hence, by the analysis above, $c_i \in C_i^k(v_i)$ for every $k \geq 0$. Hence, (c_i, v_i) survives the procedure, which completes the proof of the “only if” direction.

“IF”: For every player i , and every view $v_i \in V_i$, let $C_i^\infty(v_i) := \cap_{k \geq 0} C_i^k(v_i)$ be the set of choices that survive the procedure for view v_i , and let $C_{-i}^\infty(v_i) := \cap_{k \geq 0} C_{-i}^k(v_i)$ be the set of opponents’ choice combinations that survive the procedure at v_i . By Theorem 4.1 we know that all these sets $C_i^\infty(v_i)$ and $C_{-i}^\infty(v_i)$ are non-empty. We show that every choice in $C_i^\infty(v_i)$ can rationally be made under common belief in rationality with the view v_i .

By construction, every choice $c_i \in C_i^\infty(v_i)$ is not strictly dominated within the decision problem $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$. Hence, by Lemma 8.1, there is for every choice $c_i \in C_i^\infty(v_i)$ some belief $\beta_i^{c_i, v_i} \in \Delta(C_{-i}^\infty(v_i))$ such that c_i is optimal for $\beta_i^{c_i, v_i}$ within the decision problem $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$. We will show that, in fact, c_i is optimal for $\beta_i^{c_i, v_i}$ within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$. Let $c_i^* \in C_i(v_i)$ be optimal for $\beta_i^{c_i, v_i}$ within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$. Then, by Lemma 8.1, c_i^* is not strictly dominated within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$, and hence c_i^* must be in $C_i^\infty(v_i)$. As c_i is optimal for $\beta_i^{c_i, v_i}$ within the decision problem $(C_i^\infty(v_i), C_{-i}^\infty(v_i))$, it follows that

$$\sum_{c_{-i} \in C_{-i}^\infty(v_i)} \beta_i^{c_i, v_i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i} \in C_{-i}^\infty(v_i)} \beta_i^{c_i, v_i}(c_{-i}) \cdot u_i(c_i^*, c_{-i}).$$

As c_i^* is optimal for $\beta_i^{c_i, v_i}$ within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$, it follows that c_i is optimal for $\beta_i^{c_i, v_i}$ within the decision problem $(C_i(v_i), C_{-i}^\infty(v_i))$ as well.

Moreover, since $\beta_i^{c_i, v_i} \in \Delta(C_{-i}^\infty(v_i))$ we know, by construction of the procedure, that $\beta_i^{c_i, v_i}$ only assigns positive probability to opponents’ choices c_j where $c_j \in C_j^\infty(v_j[c_i, v_i, c_j])$ for some view $v_j[c_i, v_i, c_j] \in V_j$ contained in v_i . On the basis of these beliefs $\beta_i^{c_i, v_i}$ and views $v_j[c_i, v_i, c_j]$ we now construct the following epistemic model $M = (T_i, b_i)_{i \in I}$. Let the set of types for every player i be given by

$$T_i := \{t_i^{c_i, v_i} \mid v_i \in V_i \text{ and } c_i \in C_i^\infty(v_i)\}.$$

Moreover, for every type $t_i^{c_i, v_i} \in T_i$, let the belief $b_i(t_i^{c_i, v_i})$ on $C_{-i} \times V_{-i} \times T_{-i}$ be given by

$$b_i(t_i^{c_i, v_i})((c_j, v_j, t_j)_{j \neq i}) := \begin{cases} \beta_i^{c_i, v_i}((c_j)_{j \neq i}), & \text{if } v_j = v_j[c_i, v_i, c_j] \text{ and } t_j = t_j^{c_j, v_j} \\ \text{for all } j \neq i, & \\ 0, & \text{otherwise.} \end{cases}$$

By construction, $t_i^{c_i, v_i}$ only assigns positive probability to combinations $(c_j, v_j, t_j^{c_j, v_j})$ for every opponent $j \neq i$, where $c_j \in C_j^\infty(v_j)$. Hence, in particular, $c_j \in C_j(v_j)$, which guarantees that this is a well-defined epistemic model according to Definition 3.1.

Moreover, $t_i^{c_i, v_i}$ only assigns positive probability to opponents’ views $v_j[c_i, v_i, c_j]$ which are contained in v_i , which implies that $(v_i, t_i^{c_i, v_i})$ believes in smaller views. As every type in the epistemic model only assigns positive probability to view-type pairs of the sort $(v_j, t_j^{c_j, v_j})$, we conclude that $(v_i, t_i^{c_i, v_i})$ expresses common belief in smaller views for every player i and every type $t_i^{c_i, v_i} \in T_i$.

Note that every type $t_i^{c_i, v_i}$ has the belief $\beta_i^{c_i, v_i}$ about the opponents’ choices. Since we have seen above that c_i is optimal for $\beta_i^{c_i, v_i}$ among all choices in $C_i(v_i)$, it follows that c_i is optimal

for $(v_i, t_i^{c_i, v_i})$ as well. By construction, every type $t_i^{c_i, v_i}$ only assigns positive probability to combinations $(c_j, v_j, t_j^{c_j, v_j})$ for every opponent $j \neq i$, where $c_j \in C_j^\infty(v_j)$. Since we have seen that c_j is optimal for $(v_j, t_j^{c_j, v_j})$, it follows that every type $t_i^{c_i, v_i}$ in the epistemic model believes in the opponents' rationality. As a consequence, every type in the epistemic model expresses *common* belief in rationality.

Take now some player i , and some choice-view pair (c_i, v_i) that survives the procedure. Then, $c_i \in C_i^\infty(v_i)$. Consider the type $t_i^{c_i, v_i} \in T_i$ in the epistemic model constructed above. We have seen above that c_i is optimal for $(v_i, t_i^{c_i, v_i})$, that $(v_i, t_i^{c_i, v_i})$ expresses common belief in smaller views, and that type $t_i^{c_i, v_i}$ expresses common belief in rationality. It thus follows that c_i can rationally be chosen under common belief in rationality with the view v_i . This completes the proof. \blacksquare

8.2 Belief Hierarchies on Views Induced by Types

8.2.1 Epistemic Models for Views

Consider an epistemic model for views $M^{view} = (R_i, p_i)_{i \in I}$. We show how, for every player i and every view-type $r_i \in R_i$, we can derive the induced belief hierarchy $h_i^{view}(r_i)$ on views. Formally, this belief hierarchy can be written as an infinite sequence of beliefs $h_i^{view}(r_i) = (h_i^1(r_i), h_i^2(r_i), \dots)$, where $h_i^1(r_i)$ is the induced first-order belief, $h_i^2(r_i)$ is the induced second-order belief, and so on.

We will inductively define, for every n , the n -th order beliefs induced by types r_i in M^{view} , building upon the $(n-1)$ -th order beliefs that have been defined in the preceding step. We start by defining the first-order beliefs.

For every player i , and every type $r_i \in R_i$, define the first-order belief $h_i^1(r_i) \in \Delta(V_{-i})$ by

$$h_i^1(r_i)(v_{-i}) := p_i(r_i)(\{v_{-i}\} \times R_{-i}) \text{ for all } v_{-i} \in V_{-i}.$$

Now, suppose that $n \geq 2$, and assume that the $(n-1)$ -th order beliefs $h_i^{n-1}(r_i)$ have been defined for all players i , and every type $r_i \in R_i$. Let

$$h_i^{n-1}(R_i) := \{h_i^{n-1}(r_i) \mid r_i \in R_i\}$$

be the finite set of $(n-1)$ -th order beliefs for player i induced by types in R_i . For every $h_i^{n-1} \in h_i^{n-1}(R_i)$, let

$$R_i[h_i^{n-1}] := \{r_i \in R_i \mid h_i^{n-1}(r_i) = h_i^{n-1}\}$$

be the set of types in R_i that have the $(n-1)$ -th order belief h_i^{n-1} .

Let $h_{-i}^{n-1}(R_{-i}) := \times_{j \neq i} h_j^{n-1}(R_j)$, and for a given $h_{-i}^{n-1} = (h_j^{n-1})_{j \neq i}$ in $h_{-i}^{n-1}(R_{-i})$ let $R_{-i}[h_{-i}^{n-1}] := \times_{j \neq i} R_j[h_j^{n-1}]$.

For every type $r_i \in R_i$, let the n -th order belief $h_i^n(r_i) \in \Delta(V_{-i} \times h_{-i}^{n-1}(R_{-i}))$ be given by

$$h_i^n(r_i)(v_{-i}, h_{-i}^{n-1}) := p_i(r_i)(\{v_{-i}\} \times R_{-i}[h_{-i}^{n-1}])$$

for every $v_{-i} \in V_{-i}$ and every $h_{-i}^{n-1} \in h_{-i}^{n-1}(R_{-i})$.

Finally, for every type $r_i \in R_i$, we denote by

$$h_i^{view}(r_i) := (h_i^n(r_i))_{n \in \mathbb{N}}$$

the *belief hierarchy on views* induced by r_i .

8.2.2 Epistemic Models for Choices and Views

Consider an epistemic model for choices and views $M = (T_i, b_i)_{i \in I}$. We show how, for every player i and every type $t_i \in T_i$, we can derive the induced belief hierarchy $h_i^{view}(t_i)$ on views. Formally, this belief hierarchy can be written as an infinite sequence of beliefs $h_i^{view}(t_i) = (h_i^1(t_i), h_i^2(t_i), \dots)$, where $h_i^1(t_i)$ is the induced first-order belief on views, $h_i^2(t_i)$ is the induced second-order belief on views, and so on.

We will inductively define, for every n , the n -th order beliefs on views induced by types t_i in M , building upon the $(n-1)$ -th order beliefs on views that have been defined in the preceding step. We start by defining the first-order beliefs.

For every player i , and every type $t_i \in T_i$, define the first-order belief on views $h_i^1(t_i) \in \Delta(V_{-i})$ by

$$h_i^1(t_i)(v_{-i}) := b_i(t_i)(C_{-i} \times \{v_{-i}\} \times T_{-i}) \text{ for all } v_{-i} \in V_{-i}.$$

Now, suppose that $n \geq 2$, and assume that the $(n-1)$ -th order beliefs on views $h_i^{n-1}(t_i)$ have been defined for all players i , and every type $t_i \in T_i$. Let

$$h_i^{n-1}(T_i) := \{h_i^{n-1}(t_i) \mid t_i \in T_i\}$$

be the finite set of $(n-1)$ -th order beliefs for player i induced by types in T_i . For every $h_i^{n-1} \in h_i^{n-1}(T_i)$, let

$$T_i[h_i^{n-1}] := \{t_i \in T_i \mid h_i^{n-1}(t_i) = h_i^{n-1}\}$$

be the set of types in T_i that have the $(n-1)$ -th order belief h_i^{n-1} .

Let $h_{-i}^{n-1}(T_{-i}) := \times_{j \neq i} h_j^{n-1}(T_j)$, and for a given $h_{-i}^{n-1} = (h_j^{n-1})_{j \neq i}$ in $h_{-i}^{n-1}(T_{-i})$ let $T_{-i}[h_{-i}^{n-1}] := \times_{j \neq i} T_j[h_j^{n-1}]$.

For every type $t_i \in T_i$, let the n -th order belief on views $h_i^n(t_i) \in \Delta(V_{-i} \times h_{-i}^{n-1}(T_{-i}))$ be given by

$$h_i^n(t_i)(v_{-i}, h_{-i}^{n-1}) := b_i(t_i)(C_{-i} \times \{v_{-i}\} \times T_{-i}[h_{-i}^{n-1}])$$

for every $v_{-i} \in V_{-i}$ and every $h_{-i}^{n-1} \in h_{-i}^{n-1}(T_{-i})$.

Finally, for every type $t_i \in T_i$, we denote by

$$h_i^{view}(t_i) := (h_i^n(t_i))_{n \in \mathbb{N}}$$

the *belief hierarchy on views* induced by t_i .

8.3 Proofs of Section 5

Proof of Theorem 5.1. Note that $C_i^{k+1}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$ for every player i , every view $v_i \in V_i$, every view-type $r_i \in R_i$ such that (v_i, r_i) expresses common belief in smaller views, and every round $k \geq 0$. Since there are only finitely many choices, views and view-types in the game, the procedure must terminate after finitely many rounds. That is, there is some $K \geq 0$ such that for every $k \geq K$, $C_i^k(v_i, r_i) = C_i^K(v_i, r_i)$ for every player i , every view v_i , and every view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views. As such, it is sufficient to show that $C_i^k(v_i, r_i)$ is always non-empty for every player i , every view $v_i \in V_i$, every view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views, and every $k \geq 0$. We prove so by induction on k .

For $k = 0$ this is clear since $C_i^0(v_i, r_i) = C_i(v_i)$, which is non-empty.

Take now some $k \geq 1$ and assume that $C_j^{k-1}(v_j, r_j)$ is non-empty for every player j , every view $v_j \in V_j$ and every view-type $r_j \in R_j$ for which (v_j, r_j) expresses common belief in smaller views. Consider some player i , some view $v_i \in V_i$ and some view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views. Then, $B_i^k(v_i, r_i)$ is non-empty, since the choice sets $C_j^{k-1}(v_j, r_j)$ are non-empty for every $j \neq i$, every $v_j \in V_j$ and every $r_j \in R_j$ for which (v_j, r_j) expresses common belief in smaller views.

Now, take some $b_i \in B_i^k(v_i, r_i)$ and some choice $c_i \in C_i(v_i)$ that is optimal for b_i among choices in $C_i(v_i)$. Then, c_i will also be optimal for b_i among choices in $C_i^{k-1}(v_i, r_i)$, and hence $c_i \in C_i^k(v_i, r_i)$. We thus conclude that $C_i^k(v_i, r_i)$ is non-empty.

By induction, it follows that $C_i^k(v_i, r_i)$ is always non-empty for every player i , every view $v_i \in V_i$, every view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views, and every round $k \geq 0$. As we have seen, this completes the proof. \blacksquare

Proof of Theorem 5.2. “Only if”: Assume, without loss of generality, that different types in M^{view} induce different belief hierarchies on views. For every player i , every view $v_i \in V_i$, and every view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views, let $C_i^{cbr}(v_i, r_i)$ be the set of choices in $C_i(v_i)$ that player i can rationally make under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i . We show, by induction on k , that $C_i^{cbr}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$ for every $k \geq 0$, every player i , every view $v_i \in V_i$, and every view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views.

For $k = 0$ this is obviously true since $C_i^0(v_i, r_i) = C_i(v_i)$.

Now, consider some $k \geq 1$ and assume that $C_i^{cbr}(v_i, r_i) \subseteq C_i^{k-1}(v_i, r_i)$ for every player i , every view $v_i \in V_i$ and every view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views. Consider some player i , some view v_i , some view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views, and assume that $c_i \in C_i^{cbr}(v_i, r_i)$. Then, there is some epistemic model $M = (T_j, b_j)_{j \in I}$ and some type $t_i \in T_i$ such that $h_i^{view}(t_i) = h_i^{view}(r_i)$, type t_i expresses common belief in rationality, and such that c_i is optimal for (v_i, t_i) .

Let $b_i^C(t_i)$ be the marginal of the belief $b_i(t_i)$ on C_{-i} . Later, we will show that $b_i^C(t_i) \in$

$B_i^k(v_i, r_i)$. In order to do so, we need two preliminary observations.

First, since $h_i^{view}(t_i) = h_i^{view}(r_i)$, there is for every opponent j , and every view-type r_j that receives positive probability under $p_i(r_i)$, some set of types $T_j(r_j)$ such that

$$h_j^{view}(t_j) = h_j^{view}(r_j) \text{ for all } t_j \in T_j(r_j), \quad (8.2)$$

and

$$b_i(t_i)(\times_{j \neq i}(C_j \times \{v_j\} \times T_j(r_j))) = p_i(r_i)((v_j, r_j)_{j \neq i}) \quad (8.3)$$

for all $(v_j, r_j)_{j \neq i}$ in $V_{-i} \times R_{-i}$ with $p_i(r_i)((v_j, r_j)_{j \neq i}) > 0$. Here, we use the assumption above that different types in M^{view} induce different belief hierarchies on views.

Second, since t_i expresses common belief in rationality, we have that $b_i(t_i)((c_j, v_j, t_j)_{j \neq i}) > 0$ only if for every opponent $j \neq i$, type t_j expresses common belief in rationality, and c_j is optimal for (v_j, t_j) . Note that in this case, there must be some $r_j \in R_j$ with $t_j \in T_j(r_j)$, in view of (8.3). Hence, by (8.2), we know that $h_j^{view}(t_j) = h_j^{view}(r_j)$. Together with the facts that c_j is optimal for (v_j, t_j) , and t_j expresses common belief in rationality, it follows that $c_j \in C_j^{cbr}(v_j, r_j)$ in this case. By the induction assumption, $C_j^{cbr}(v_j, r_j) \subseteq C_j^{k-1}(v_j, r_j)$. We thus conclude that

$$b_i(t_i)((c_j, v_j, t_j)_{j \neq i}) > 0 \text{ only if } t_j \in T_j(r_j) \text{ and } c_j \in C_j^{k-1}(v_j, r_j) \quad (8.4)$$

for all opponents $j \neq i$.

We will now use (8.3) and (8.4) to prove that $b_i^C(t_i) \in B_i^k(v_i, r_i)$. That is, we must show that

$$b_i^C(t_i) = \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \beta_i^{(v_j, r_j)_{j \neq i}}, \quad (8.5)$$

where $\beta_i^{(v_j, r_j)_{j \neq i}} \in \Delta(\times_{j \neq i} C_j^{k-1}(v_j, r_j))$ for all $(v_j, r_j)_{j \neq i}$ with $p_i(r_i)((v_j, r_j)_{j \neq i}) > 0$.

Let

$$(V_{-i} \times R_{-i})^* := \{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i} \mid p_i(r_i)((v_j, r_j)_{j \neq i}) > 0\}.$$

For every $(v_j, r_j)_{j \neq i} \in (V_{-i} \times R_{-i})^*$, define $\beta_i^{(v_j, r_j)_{j \neq i}}$ by

$$\beta_i^{(v_j, r_j)_{j \neq i}}((c_j)_{j \neq i}) := \frac{b_i(t_i)(\times_{j \neq i} \{c_j\} \times \{v_j\} \times T_j(r_j))}{p_i(r_i)((v_j, r_j)_{j \neq i})}. \quad (8.6)$$

Then, it may be verified that $\beta_i^{(v_j, r_j)_{j \neq i}}$ is a probability distribution on C_{-i} , since

$$\beta_i^{(v_j, r_j)_{j \neq i}}(C_{-i}) = \frac{b_i(t_i)(\times_{j \neq i}(C_j \times \{v_j\} \times T_j(r_j)))}{p_i(r_i)((v_j, r_j)_{j \neq i})} = 1$$

because of (8.3).

We next show that $\beta_i^{(v_j, r_j)_{j \neq i}}$ only assigns positive probability to $(c_j)_{j \neq i} \in \times_{j \neq i} C_j^{k-1}(v_j, r_j)$. Indeed, suppose that $\beta_i^{(v_j, r_j)_{j \neq i}}((c_j)_{j \neq i}) > 0$. Then, by (8.6), $b_i(t_i)(\times_{j \neq i} \{c_j\} \times \{v_j\} \times T_j(r_j)) > 0$, and hence we conclude by (8.4) that $c_j \in C_j^{k-1}(v_j, r_j)$ for every $j \neq i$. Hence, $(c_j)_{j \neq i} \in \times_{j \neq i} C_j^{k-1}(v_j, r_j)$. We may thus conclude that

$$\beta_i^{(v_j, r_j)_{j \neq i}} \in \Delta(\times_{j \neq i} C_j^{k-1}(v_j, r_j)) \text{ for every } (v_j, r_j)_{j \neq i} \in (V_{-i} \times R_{-i})^*. \quad (8.7)$$

We finally show (8.5). By definition, for every $(c_j)_{j \neq i}$ in C_{-i} , we have that

$$\begin{aligned} b_i^C(t_i)((c_j)_{j \neq i}) &= \sum_{(v_j, t_j)_{j \neq i} \in V_{-i} \times T_{-i}} b_i(t_i)((c_j, v_j, t_j)_{j \neq i}) \\ &= \sum_{(v_j, r_j)_{j \neq i} \in (V_{-i} \times R_{-i})^*} b_i(t_i)(\times_{j \neq i} \{c_j\} \times \{v_j\} \times T_j(r_j)) \\ &= \sum_{(v_j, r_j)_{j \neq i} \in (V_{-i} \times R_{-i})^*} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \frac{b_i(t_i)(\times_{j \neq i} \{c_j\} \times \{v_j\} \times T_j(r_j))}{p_i(r_i)((v_j, r_j)_{j \neq i})} \\ &= \sum_{(v_j, r_j)_{j \neq i} \in (V_{-i} \times R_{-i})^*} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \beta_i^{(v_j, r_j)_{j \neq i}}((c_j)_{j \neq i}), \end{aligned}$$

which implies (8.5). Here, the second equality follows from (8.3), whereas the fourth equality follows from (8.6). But then, we conclude from (8.5) and (8.7) that $b_i^C(t_i) \in B_i^k(v_i, r_i)$.

Remember from above that c_i is optimal for (v_i, t_i) . Hence, c_i is optimal for the marginal belief $b_i^C(t_i) \in B_i^k(v_i, r_i)$ among choices in $C_i(v_i)$, which implies that $c_i \in C_i^k(v_i, r_i)$. As this holds for every $c_i \in C_i^{cbr}(v_i, r_i)$, we conclude that $C_i^{cbr}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$. By induction on k , we may then conclude that $C_i^{cbr}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$ for every k .

Now, take some choice c_i that can rationally be made under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i . Then, by definition, $c_i \in C_i^{cbr}(v_i, r_i)$. By the conclusion above that $C_i^{cbr}(v_i, r_i) \subseteq C_i^k(v_i, r_i)$ for every k , it follows that $c_i \in C_i^k(v_i, r_i)$ for every k . Hence, (c_i, v_i, r_i) survives the procedure. This completes the ‘‘only if’’ direction.

‘‘IF’’: For every player i , every view $v_i \in V_i$, and every view-type $r_i \in R_i$ for which (v_i, r_i) expresses common belief in smaller views, let $C_i^\infty(v_i, r_i) := \cap_{k \geq 0} C_i^k(v_i, r_i)$ be the set of choices that survive the procedure for view v_i and view-type r_i , and let $B_i^\infty(v_i, r_i) := \cap_{k \geq 0} B_i^k(v_i, r_i)$ be the set of beliefs that survive the procedure at v_i and r_i . By Theorem 5.1 we know that all these sets $C_i^\infty(v_i, r_i)$ and $B_i^\infty(v_i, r_i)$ are non-empty. We show that every choice in $C_i^\infty(v_i, r_i)$ can rationally be made under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i .

By construction, every choice $c_i \in C_i^\infty(v_i, r_i)$ is optimal for some belief $\beta_i^{c_i, v_i, r_i} \in B_i^\infty(v_i, r_i)$ among choices in $C_i^\infty(v_i, r_i)$. We will show that, in fact, c_i is optimal for $\beta_i^{c_i, v_i, r_i}$ among choices in

$C_i(v_i)$. Let $c_i^* \in C_i(v_i)$ be optimal for $\beta_i^{c_i, v_i, r_i}$ among choices in $C_i(v_i)$. Then, c_i^* is in $C_i^\infty(v_i, r_i)$. As c_i is optimal for $\beta_i^{c_i, v_i, r_i}$ among choices in $C_i^\infty(v_i, r_i)$, it follows that

$$\sum_{c_{-i} \in C_{-i}(v_i)} \beta_i^{c_i, v_i, r_i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i} \in C_{-i}(v_i)} \beta_i^{c_i, v_i, r_i}(c_{-i}) \cdot u_i(c_i^*, c_{-i}).$$

As c_i^* is optimal for $\beta_i^{c_i, v_i, r_i}$ among choices in $C_i(v_i)$, it follows that c_i is optimal for $\beta_i^{c_i, v_i, r_i}$ among choices in $C_i(v_i)$ as well.

Moreover, since

$$\beta_i^{c_i, v_i, r_i} \in B_i^\infty(v_i, r_i) = \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \Delta(\times_{j \neq i} C_j^\infty(v_j, r_j)),$$

there is for every $(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}$ that receives positive probability under $p_i(r_i)$, some belief $\gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}] \in \Delta(\times_{j \neq i} C_j^\infty(v_j, r_j))$ such that

$$\beta_i^{c_i, v_i, r_i} = \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]. \quad (8.8)$$

On the basis of these beliefs $\beta_i^{c_i, v_i, r_i}$ we now construct the following epistemic model $M = (T_i, b_i)_{i \in I}$. Let the set of types for every player i be given by

$$T_i = \{t_i^{c_i, v_i, r_i} \mid v_i \in V_i, r_i \in R_i \text{ such that } (v_i, r_i) \text{ expresses common belief in smaller views, and } c_i \in C_i^\infty(v_i, r_i)\}.$$

Moreover, for every type $t_i^{c_i, v_i, r_i} \in T_i$, let the belief $b_i(t_i^{c_i, v_i, r_i})$ on $C_{-i} \times V_{-i} \times T_{-i}$ be given by

$$b_i(t_i^{c_i, v_i, r_i})(c_j, v_j, t_j)_{j \neq i} := \begin{cases} \frac{p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]}{\sum_{(c_j, v_j, t_j)_{j \neq i} \in C_{-i} \times V_{-i} \times T_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]}, & \text{if } t_j = t_j^{c_j, v_j, r_j} \text{ for all } j \neq i, \\ 0, & \text{otherwise} \end{cases} \quad (8.9)$$

Suppose that type $t_i^{c_i, v_i, r_i}$ assigns positive probability to some combination $(c_j, v_j, t_j^{c_j, v_j, r_j})_{j \neq i}$. Then, by (8.9), the belief $\gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]$ assigns positive probability to the choice-combination $(c_j)_{j \neq i}$. Since $\gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}] \in \Delta(\times_{j \neq i} C_j^\infty(v_j, r_j))$, it follows that $c_j \in C_j^\infty(v_j, r_j)$, and hence $c_j \in C_j(v_j)$, for every $j \neq i$. Therefore, this is a well-defined epistemic model according to Definition 3.1.

Moreover, by (8.9) it must be the case that $p_i(r_i)$ assigns positive probability to the combination $(v_j, r_j)_{j \neq i}$. As (v_i, r_i) believes in smaller views, it follows that v_j is contained in v_i for every opponent $j \neq i$. Hence, we conclude that $(v_i, t_i^{c_i, v_i, r_i})$ believes in smaller views too. Since all types in the epistemic model only assign positive probability to view-type pairs of the sort

$(v_j, t_j^{c_j, v_j, r_j})$, we conclude that $(v_i, t_i^{c_i, v_i, r_i})$ expresses common belief in smaller views for all types $t_i^{c_i, v_i, r_i}$ in the model.

We next show that every type $t_i^{c_i, v_i, r_i}$ holds the belief $\beta_i^{c_i, v_i, r_i}$ about the opponents' choices. Let $b_i^C(t_i^{c_i, v_i, r_i})$ be the marginal belief of type $t_i^{c_i, v_i, r_i}$ on C_{-i} . Then, for every $(c_j)_{j \neq i} \in C_{-i}$ we have that

$$\begin{aligned}
b_i^C(t_i^{c_i, v_i, r_i})((c_j)_{j \neq i}) &= \sum_{(v_j, t_j)_{j \neq i} \in V_{-i} \times T_{-i}} b_i(t_i^{c_i, v_i, r_i})((c_j, v_j, t_j)_{j \neq i}) \\
&= \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} b_i(t_i^{c_i, v_i, r_i})((c_j, v_j, t_j^{c_j, v_j, r_j})_{j \neq i}) \\
&= \sum_{(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]((c_j)_{j \neq i}) \\
&= \beta_i^{c_i, v_i, r_i}((c_j)_{j \neq i}),
\end{aligned}$$

where the second and third equality follow from (8.9), and the last equality follows from (8.8). Hence, we conclude that $t_i^{c_i, v_i, r_i}$ holds the belief $\beta_i^{c_i, v_i, r_i}$ about the opponents' choices.

Note that, by construction, $c_i \in C_i^\infty(v_i, r_i)$ for every type $t_i^{c_i, v_i, r_i} \in T_i$. Since we have seen above that c_i is optimal for $\beta_i^{c_i, v_i, r_i}$ among choices in $C_i(v_i)$, and that $t_i^{c_i, v_i, r_i}$ holds the belief $\beta_i^{c_i, v_i, r_i}$ about the opponents' choices, it follows that c_i is optimal for $(v_i, t_i^{c_i, v_i, r_i})$. We use this to show that every type $t_i^{c_i, v_i, r_i}$ believes in the opponents' rationality. Suppose that $b_i(t_i^{c_i, v_i, r_i})((c_j, v_j, t_j^{c_j, v_j, r_j})_{j \neq i}) > 0$. Then, as we have just seen, c_j is optimal for $(v_j, t_j^{c_j, v_j, r_j})$, and hence $t_i^{c_i, v_i, r_i}$ indeed believes in the opponents' rationality. As this holds for all types in the epistemic model M , we conclude that all types in M express *common* belief in rationality.

We finally show that every type $t_i^{c_i, v_i, r_i}$ has the belief hierarchy on views induced by the view-type r_i . For every $(v_j, r_j)_{j \neq i} \in V_{-i} \times R_{-i}$ we have that

$$\begin{aligned}
\sum_{(c_j)_{j \neq i} \in C_{-i}} b_i(t_i^{c_i, v_i, r_i})((c_j, v_j, t_j^{c_j, v_j, r_j})_{j \neq i}) &= \sum_{(c_j)_{j \neq i} \in C_{-i}} p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \\
&\quad \cdot \gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]((c_j)_{j \neq i}) \\
&= p_i(r_i)((v_j, r_j)_{j \neq i}) \cdot \\
&\quad \cdot \sum_{(c_j)_{j \neq i} \in C_{-i}} \gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]((c_j)_{j \neq i}) \\
&= p_i(r_i)((v_j, r_j)_{j \neq i}), \tag{8.10}
\end{aligned}$$

where the first equality follows from (8.9), and the last equality follows from the fact that $\gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]$ is a probability distribution on C_{-i} , and hence

$$\sum_{(c_j)_{j \neq i} \in C_{-i}} \gamma_i^{c_i, v_i, r_i}[(v_j, r_j)_{j \neq i}]((c_j)_{j \neq i}) = 1.$$

Equation (8.10) thus states that the probability that type $t_i^{c_i, v_i, r_i}$ assigns to the set of tuples $\{(v_j, t_j^{c_j, v_j, r_j})_{j \neq i} \mid (c_j)_{j \neq i} \in C_{-i}\}$ is the same as the probability that view-type r_i assigns to the tuple $(v_j, r_j)_{j \neq i}$. Since this holds for every type $t_i^{c_i, v_i, r_i}$ in the epistemic model M , we conclude that every type $t_i^{c_i, v_i, r_i}$ in M has the belief hierarchy on views induced by r_i . That is, $h_i^{view}(t_i^{c_i, v_i, r_i}) = h_i^{view}(r_i)$ for every type $t_i^{c_i, v_i, r_i}$ in M .

Take now some player i , and some triple (c_i, v_i, r_i) that survives the procedure. Then, $c_i \in C_i^\infty(v_i, r_i)$. Consider the type $t_i^{c_i, v_i, r_i} \in T_i$ in the epistemic model constructed above. We have seen above that c_i is optimal $(v_i, t_i^{c_i, v_i, r_i})$, that $(v_i, t_i^{c_i, v_i, r_i})$ expresses common belief in smaller views, that type $t_i^{c_i, v_i, r_i}$ expresses common belief in rationality, and that $h_i^{view}(t_i^{c_i, v_i, r_i}) = h_i^{view}(r_i)$. It thus follows that c_i can rationally be chosen under common belief in rationality with the view v_i and the belief hierarchy on views induced by r_i . This completes the proof. ■

References

- [1] Bach, C.W. and A. Perea (2017), Incomplete information and generalized iterated strict dominance, Epicenter Working Paper No. 7.
- [2] Battigalli, P. (1997), On rationalizability in extensive games, *Journal of Economic Theory* **74**, 40–61.
- [3] Bernheim, B.D. (1984), Rationalizable strategic behavior, *Econometrica* **52**, 1007–1028.
- [4] Brandenburger, A. and E. Dekel (1987), Rationalizability and correlated equilibria, *Econometrica* **55**, 1391–1402.
- [5] Čopič, J. and A. Galeotti (2006), Awareness as an equilibrium notion: Normal-form games, Working paper.
- [6] Dekel, E., Fudenberg, D. and S. Morris (2007), Interim correlated rationalizability, *Theoretical Economics* **1**, 15–40.
- [7] Dekel, E., Lipman, B.L. and A. Rustichini (1998), Standard state-space models preclude unawareness, *Econometrica* **66**, 159–173.
- [8] Ely, J.C. and M. Peşki (2006), Hierarchies of belief and interim rationalizability, *Theoretical Economics* **1**, 19–65.
- [9] Fagin, R. and J.Y. Halpern (1988), Belief, awareness and limited reasoning, *Artificial Intelligence* **34**, 39–72.
- [10] Feinberg, Y. (2004), Subjective reasoning – games with unawareness, Stanford Graduate School of Business, Research Paper No. 1875.

- [11] Feinberg, Y. (2012), Games with unawareness, Stanford Graduate School of Business, Research Paper No. 2122.
- [12] Grant, S. and J. Quiggin (2013), Inductive reasoning about unawareness, *Economic Theory* **54**, 717–755.
- [13] Guarino, P. (2017), Essays on epistemic game theory: Large conditional type structures and unawareness, Maastricht University, PhD Dissertation.
- [14] Halpern, J.Y. (2001), Alternative semantics for unawareness, *Games and Economic Behavior* **37**, 321–339.
- [15] Halpern, J.Y. and L.C. Rêgo (2008), Interactive unawareness revisited, *Games and Economic Behavior* **62**, 232–262.
- [16] Halpern, J.Y. and L.C. Rêgo (2014), Extensive games with possibly unaware players, *Mathematical Social Sciences* **70**, 42–58.
- [17] Harsanyi, J.C. (1967–1968), Games with incomplete information played by “bayesian” players, I–III, *Management Science* **14**, 159–182, 320–334, 486–502.
- [18] Heifetz, A., Meier, M. and B.C. Schipper (2006), Interactive unawareness, *Journal of Economic Theory* **130**, 78–94.
- [19] Heifetz, A., Meier, M. and B.C. Schipper (2008), A canonical model for interactive unawareness, *Games and Economic Behavior* **62**, 304–324.
- [20] Heifetz, A., Meier, M. and B.C. Schipper (2013a), Unawareness, beliefs, and speculative trade, *Games and Economic Behavior* **77**, 100–121.
- [21] Heifetz, A., Meier, M. and B.C. Schipper (2013b), Dynamic unawareness and rationalizable behavior, *Games and Economic Behavior* **81**, 50–68.
- [22] Heinsalu, S. (2014), Universal type structures with unawareness, *Games and Economic Behavior* **83**, 255–266.
- [23] Hu, H. and H.W. Stuart, Jr. (2001) An epistemic analysis of the Harsanyi transformation, *International Journal of Game Theory* **30**, 517–525.
- [24] Li, J. (2009), Information structures with unawareness, *Journal of Economic Theory* **144**, 977–993.
- [25] Meier, M. and B.C. Schipper (2014), Bayesian games with unawareness and unawareness perfection, *Economic Theory* **56**, 219–249.

- [26] Modica, S. and A. Rustichini (1999), Unawareness and partitional information structures, *Games and Economic Behavior* **27**, 265–298.
- [27] Pearce, D.G. (1984), Rationalizable strategic behavior and the problem of perfection, *Econometrica* **52**, 1029–1050.
- [28] Perea, A. (2012), *Epistemic Game Theory: Reasoning and Choice*, Cambridge University Press.
- [29] Perea, A. (2014), Belief in the opponents' future rationality, *Games and Economic Behavior* **83**, 231–254.
- [30] Rêgo, L.C. and J.Y. Halpern (2012), Generalized solution concepts in games with possibly unaware players, *International Journal of Game Theory* **41**, 131–155.
- [31] Schipper, B.C. (2014), Unawareness – a gentle introduction to both the literature and the special issue, *Mathematical Social Sciences* **70**, 1–9.
- [32] Schipper, B.C. (2017), Self-confirming games: unawareness, discovery, and equilibrium, Working paper.
- [33] Spohn, W. (1982), How to make sense of game theory, in W. Stegmüller, W. Balzer and W. Spohn (eds.), *Philosophy of Economics*, Springer Verlag, pp. 239–270.
- [34] Tan, T. and S.R.C. Werlang (1988), The Bayesian foundations of solution concepts of games, *Journal of Economic Theory* **45**, 370–391.