Characterization of Consistent Assessments in Extensive Form Games*

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In this paper an algebraic characterization of consistent assessments in extensive form games (in the sense of Kreps and Wilson, 1982, *Econometrica* **50,** 863–894), is given. As a corollary, we show that consistency can be characterized by so-called "simple" sequences of assessments. The algebraic characterization is used to develop an algorithm which computes the set of consistent assessments. Moreover, the geometrical structure of the set of consistent assessments is described: It turns out to be the intersection of a finite product of simplices with a finite number of logarithmic cones. *Journal of Economic Literature* Classification Number: 210.
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1. INTRODUCTION

In an extensive form game, a combination of a behavior strategy profile and a belief system is called an assessment. Such an assessment is a sequential equilibrium (Kreps and Wilson, 1982) if it satisfies sequential rationality and consistency. A sequential equilibrium can be viewed as properly extending Selten's ideas of (subgame) perfectness (1965, 1975). The first condition, sequential rationality, is equivalent to a system of polynomial inequalities and is therefore straightforward to check. Checking consistency, however, is in general much harder because the definition requires sequences of assessments. A formal definition of consistent assessments is given in Section 2.

In Section 3 we present the central result of this paper (Theorem 3.1), which is a purely algebraic characterization of consistent assessments: It does not make any use of sequences and limits, but characterizes consistent assessments by two algebraic conditions. The first condition is a

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restriction pertaining to the supports of the strategies and beliefs. It turns out that checking this restriction is equivalent to solving a linear program, an insight which plays an important role in the algorithm indicated below. The second condition of the characterization implies that we can put mistake probabilities on the actions played with probability zero in such a way that the relative beliefs are equal to the relative realization probabilities of the corresponding nodes. The relationship between relative probabilities and consistency of assessments is also considered in Kohlberg and Reny (1991) and McLennan (1989a, 1989b). Before stating the proof of this theorem, we provide an example illustrating how this result can be used to check whether a given assessment is consistent.

This central result will be applied in several ways. First, as is shown at the end of Section 3, the proof of Theorem 3.1 can be used to prove that a consistent assessment can always be approximated by a sequence of completely mixed assessments of a very simple form, determined by just a few parameters.

Second, in Section 4, the characterization is used to develop an algorithm which computes the set of consistent assessments in a given extensive form game.

Finally, the characterization makes it possible to give a geometrical description of the set of consistent assessments in Section 5. As a byproduct of this description, it can be shown rather easily that the set of consistent assessments is semialgebraic, which means that it can be described by a finite number of polynomial inequalities. Since sequential rationality is equivalent to a finite system of polynomial inequalities, it follows directly that the set of sequential equilibria is semialgebraic, a result which has already been shown by Blume and Zame (1994).

Notation. $\mathbb{R}^* := \mathbb{R} \cup \{-\infty\}$. For every $x \in \mathbb{R}$ we define $x + (-\infty) := -\infty$. Moreover, $(-\infty) + (-\infty) := -\infty$ whereas $(-\infty) - (-\infty)$ is not defined.

For a finite set A, $\Delta(A)$ is the set of all probability distributions on A. For a matrix M we denote the transposed matrix by M^t.

2. MODEL AND DEFINITIONS

Notation in Extensive Form Games

In an extensive form game, the information sets are denoted by h, whereas H is the collection of all information sets. By X we denote the set of all nonterminal nodes. The set of terminal nodes is denoted by Z. At an information set h, $A(h)$ is the set of actions available at h. The sets $A(h)$ should be such that $A(h)$ and $A(h')$ are disjoint whenever h and h'

are different. We assume that the extensive form games considered have *perfect recall* (see Kuhn, 1953). A formal description of extensive form games can be found in Kreps and Wilson (1982).

Consistent Assessments

A *behavior strategy profile* (BSP) is a function σ which assigns to every information set h a probability distribution σ_h on $A(h)$. A *belief system* is a function β which assigns to every information set h a probability distribution β_h on the nodes in this information set. A combination (σ, β) of a BSP and a belief system is called an *assessment*.

An assessment (σ, β) is called *Bayesian consistent* if at every information set which is reached with positive probability the beliefs are derived according to Bayes' rule. So for every h with $\mathbb{P}_\sigma(h) > 0$ it must hold that

$$\beta_h(x) = \frac{\mathbb{P}_\sigma(x)}{\mathbb{P}_\sigma(h)}$$

for every $x \in h$. Here, $\mathbb{P}_\sigma(x)$ and $\mathbb{P}_\sigma(h)$ denote the probabilities that the node x and the information set h, respectively, are reached if σ is played.

An assessment (σ, β) is called *consistent* if there is a sequence $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ of completely mixed, Bayesian consistent assessments converging to (σ, β). Completely mixed means that every action is played with positive probability. Obviously, consistency implies Bayesian consistency.

3. CHARACTERIZATION OF CONSISTENT ASSESSMENTS

Before formulating our main result, we need some further definitions. For an assessment (σ, β), $A^+(\sigma)$ denotes the set of actions played with positive probability and by $X^+(\beta)$ we mean the set of nodes with positive belief. The restriction of σ on the actions in $A^+(\sigma)$ is denoted by σ^+. By $A^0(\sigma)$ we denote the set of actions played with probability zero whereas $X^0(\beta)$ denotes the set of nodes with belief zero.

A *pseudo-BSP* is a system $\overline{\sigma} = (\overline{\sigma}_h)_{h \in H}$ of functions $\overline{\sigma}_h: A(h) \to [0, 1]$. In contrast with BSP's, the sum of the probabilities of the actions in $A(h)$ does not need to be equal to 1 in a pseudo-BSP. A pseudo-BSP $\overline{\sigma}$ is called completely mixed if every action is played with positive probability.

For a pseudo-BSP $\overline{\sigma}$, $\mathbb{P}_{\overline{\sigma}}(x)$ and $\mathbb{P}_{\overline{\sigma}}(h)$ are defined in a similar way as for a BSP.

The following theorem gives an algebraic characterization of consistent assessments. In the theorem, we denote by A_x the set of actions which occur on the path to the node x.

THEOREM 3.1. *Let Γ be an extensive form game and (σ, β) be an assessment. Then, (σ, β) is consistent if and only if:*

(1) *there are numbers $\{\varepsilon(a)\}_{a \in A^0(\sigma)} \in (0, 1)$ such that*

$$\prod_{a \in A^0(\sigma) \cap A_x} \varepsilon(a) = \prod_{a \in A^0(\sigma) \cap A_y} \varepsilon(a)$$

for all nodes $x, y \in X^+(\beta)$ in the same information set and

$$\prod_{a \in A^0(\sigma) \cap A_x} \varepsilon(a) < \prod_{a \in A^0(\sigma) \cap A_y} \varepsilon(a)$$

for all nodes x, y in the same information set with $x \in X^0(\beta)$ and $y \in X^+(\beta)$, and

(2) *σ^+ can be extended to a completely mixed pseudo-BSP $\bar{\sigma}$ such that*

$$\frac{\mathbb{P}_{\bar{\sigma}}(x)}{\mathbb{P}_{\bar{\sigma}}(y)} = \frac{\beta(x)}{\beta(y)}$$

for all nodes $x, y \in X^+(\beta)$ in the same information set.

Intuitively, condition (1) says that we can put mistake probabilities on the zero probability actions such that β places positive belief exactly on those nodes which are reached with maximum mistake probability. Therefore, this condition checks whether the combination $(A^+(\sigma), X^+(\beta))$ is possible in a consistent assessment. Condition (2) states that we can put mistake probabilities on the zero probability actions such that the relative probabilities of the nodes with positive belief are equal to the relative beliefs.

Condition (1) is somewhat related to the notion of a *b-labelling*, used by Kreps and Wilson (1982) in their Lemma A1. Furthermore, there is a connection between condition (2) and the mapping (μ^b, π^b) which can be found in Lemma A2 of the same paper.

Before proving this theorem, we give an example in order to illustrate the meaning of conditions (1) and (2) in the theorem. Moreover, this example shows how the characterization can be used to check whether a given assessment is consistent or not.

EXAMPLE 1. Consider an extensive form game with the extensive form structure shown in Fig. 1. This extensive form structure is also used by Kohlberg and Reny (1991) in their Fig. 5.

Consider an assessment (σ, β) with $\sigma(c) = \sigma(d) = 1$, $\beta(x_1) = \frac{1}{3}$, $\beta(y_1) = \frac{1}{5}$, and $\beta(z_1) = \frac{1}{2}$. We apply Theorem 3.1 in order to verify if (σ, β) is consistent.

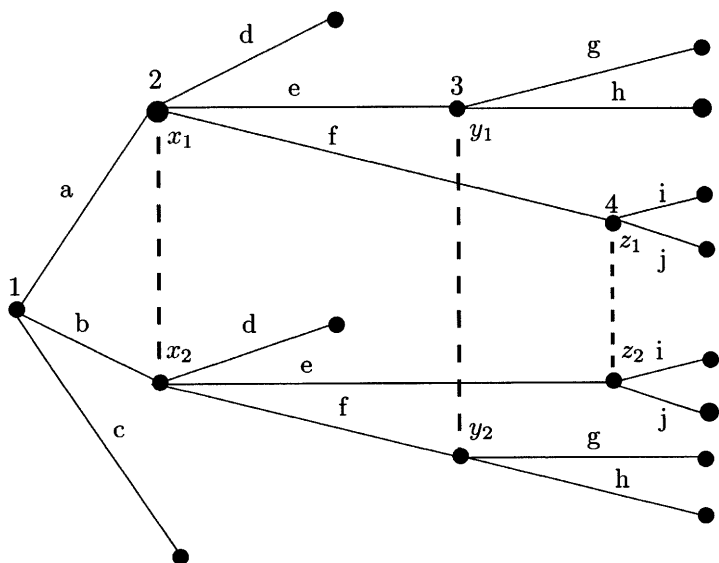FIGURE 1

We can extend σ^+ to a completely mixed pseudo-BSP $\bar{\sigma}$ by defining

$$\bar{\sigma}(a) = \tfrac{1}{3}, \ \bar{\sigma}(b) = \tfrac{2}{3}, \ \bar{\sigma}(e) = \tfrac{3}{10}, \text{ and } \bar{\sigma}(f) = \tfrac{6}{10}.$$

Obviously, it holds that

$$\frac{\mathbb{P}_{\bar{\sigma}}(x)}{\mathbb{P}_{\bar{\sigma}}(y)} = \frac{\beta(x)}{\beta(y)}$$

for all x, y lying in the same information set. Hence, condition (2) in Theorem 3.1 is satisfied.

Condition (1) in this theorem is also satisfied by choosing

$$\varepsilon(a) = \varepsilon(b) = \varepsilon(e) = \varepsilon(f) = \tfrac{1}{2}.$$

Therefore, we may conclude that (σ, β) is consistent.

Now, consider an assessment (σ, β) with $\sigma(c) = \sigma(d) = 1$, $\beta(x_1) = 0$, $\beta(y_1) = \tfrac{1}{3}$, and $\beta(z_1) = \tfrac{2}{3}$.

Assume that condition (1) in Theorem 3.1 would be satisfied for some $\varepsilon(a)$, $\varepsilon(b)$, $\varepsilon(c)$ and $\varepsilon(d)$. Since $\beta(x_1) = 0$ and $\beta(x_2) > 0$ it follows that $\varepsilon(a) < \varepsilon(b)$. Furthermore, it must hold that $\varepsilon(a) \cdot \varepsilon(e) = \varepsilon(b) \cdot \varepsilon(f)$ which implies that $\varepsilon(e) > \varepsilon(f)$. However, this would mean that $\varepsilon(a) \cdot \varepsilon(f) < \varepsilon(b) \cdot \varepsilon(e)$ which is a contradiction since $z_1, z_2 \in X^+(\beta)$. Hence, condition (1) can not be satisfied. By Theorem 3.1, (σ, β) is not a consistent assessment.

Proof of Theorem 3.1.   Let (σ, β) be an assessment. For convenience, we write A^+, A^0, X^+, and X^0 instead of $A^+(\sigma), A^0(\sigma), X^+(\beta)$, and $X^0(\beta)$. For every information set h, let $r(h)$ be the first node in h with positive belief. (We assume, for convenience, that the nodes in an information set are ordered.) For a node $x \in h$, let $r(x) := r(h)$. We construct the matrix M as follows. The rows of the matrix correspond with the nodes in X and the columns with the actions in A^0, so $M = (m_{x,a})_{x \in X, a \in A^0}$. The elements $m_{x,a}$ are given by

$$m_{x,a} := \begin{cases} 1 & \text{if } a \in A_x \text{ and } a \notin A_{r(x)} \\ -1 & \text{if } a \notin A_x \text{ and } a \in A_{r(x)} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we define the vector $s = (s_x)_{x \in X}$ by

$$s_x := \sum_{c \in C_x} \log \tau(c) + \sum_{a \in A^+ \cap A_x} \log \sigma(a) - \sum_{c \in C_{r(x)}} \log \tau(c)$$
$$- \sum_{a \in A^+ \cap A_{r(x)}} \log \sigma(a).$$

Here, C_x is the collection of chance moves on the path to x and $\tau(c)$ is the positive probability that the chance move c occurs.

Let the vector $b = (b_x)_{x \in X}$ be given by

$$b_x := \log \beta(x) - \log \beta(r(x)),$$

where $\log 0 := -\infty$. Note that b_x can be $-\infty$ since $\beta(x)$ can be 0.

From the definitions of M, b, and s, it can be shown that an assessment (σ, β) is consistent if and only if there is a sequence $(w^k)_{k \in \mathbb{N}}$ in $\mathbb{R}^{A^0}$ converging coordinatewise to $-\infty$ such that

$$b = s + \lim_{k \to \infty} M w^k. \tag{3.1}$$

Now, let M^+ and M^0 be the restrictions of M to the rows corresponding to nodes in X^+ and X^0 respectively and let s^+, b^+ be the restrictions of the vectors s, b to nodes in X^+. In the following lemma, we show that condition (3.1) is equivalent to two algebraic conditions.

LEMMA 3.2.   *The assessment* (σ, β) *is consistent if and only if*

(1) *there is a vector* $w < 0$ *with* $M^+ w = 0$ *and* $M^0 w < 0$ *and*
(2) $b^+ \in s^+ + \text{Im}(M^+)$.

Proof. "$\Rightarrow$" Assume that (σ, β) is consistent. Then, we can find a sequence $(w^k)_{k \in \mathbb{N}}$ converging coordinatewise to $-\infty$ such that $b = s + \lim_{k \to \infty} Mw^k$. Let the vector v be given by $v := b - s$ and let v^+ be the restriction of v to the nodes in X^+. By construction, $v_x \in \mathbb{R}$ for all $x \in X^+$ and $v_x = -\infty$ for all $x \notin X^+$. Since $\lim_{k \to \infty} Mw^k = v$, it follows that the linear system of equations $Mx = v$ has an approximate solution, in the sense of Kohlberg and Reny (1991).[1] Now, we can use a remark in Kohlberg and Reny's paper, stating that if the system $Mx = v$ has an approximate solution, then the system restricted to the finite entries of v has a solution in the "normal" sense. In this case, this means that the system $M^+x = v^+$ has a solution, implying that $b^+ \in s^+ + \text{Im}(M^+)$.

Now, let $z \in \mathbb{R}^{A^0}$ with $M^+z = v^+$. Let $B := \{w \in \mathbb{R}^{A^0} \mid w \leq -1, \ M^0w \leq -1\}$ and $C := \{M^+w \mid w \in B\}$. (The inequality $w \leq -1$ should be read coordinatewise.) Obviously, B is a closed and convex set. Moreover, B is nonempty since $w^k \in B$ for large k. It follows that C is a nonempty closed convex set. Suppose that $0 \notin C$. Then, there exists a hyperplane which separates the sets C and $\{0\}$ strongly. In other words, we can find a nonzero vector p and a number $\alpha \in \mathbb{R}$ such that $p \cdot c > \alpha$ for all $c \in C$ and $p \cdot 0 < \alpha$. The last inequality implies that $\alpha > 0$. From the first inequality, it follows that $p \cdot M^+w > \alpha$ for every $w \in B$. Since $w^k - z \in B$ for large k, it follows that $\lim_{k \to \infty} p \cdot M^+(w^k - z) \geq \alpha$. We know that $\lim_{k \to \infty} M^+w^k = v^+ = M^+z$. Therefore $0 \geq \alpha$, which is a contradiction. So $0 \in C$, which implies that there is a $w \in \mathbb{R}^{A^0}$, $w \leq -1$, with $M^0w \leq -1$ and $M^+w = 0$.

"$\Leftarrow$" Let (1) and (2) in Lemma 3.2 be satisfied and let $z \in \mathbb{R}^{A^0}$ with $b^+ = s^+ + M^+z$. Define the sequence $(w^k)_{k \in \mathbb{N}}$ by $w^k := z + kw$. It is easy to check that $(w^k)_{k \in \mathbb{N}}$ converges coordinatewise to $-\infty$ and $b = s + \lim_{k \to \infty} Mw^k$, which implies that (σ, β) is consistent. ∎

Now, we are able to prove Theorem 3.1.

"$\Rightarrow$" Let (σ, β) be consistent. By (2) in Lemma 3.2 there is a $z \in \mathbb{R}^{A^0}$ such that $b^+ = s^+ + M^+z$. We define the pseudo-BSP $\bar{\sigma}$ by

$$\bar{\sigma}(a) := \begin{cases} \sigma(a) & \text{if } a \notin A^0 \\ \exp(z_a) & \text{if } a \in A^0. \end{cases}$$

Since

$$b_x = s_x + \sum_{a \in A^0} m_{x,a} z_a = s_s + \sum_{a \in A^0 \cap A_x} z_a - \sum_{a \in A^0 \cap A_{r(x)}} z_a$$

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[1]A linear system of equations $Mx = v$, where v can contain infinities, is said to have an approximate solution if there is a sequence $(x^k)_{k \in \mathbb{N}}$ satisfying $\lim_{k \to \infty} Mx^k = v$.

for every $x \in X^+$, we obtain by taking the exponential function on both sides and using the definitions of $b_x, v_x,$ and $m_{x,a}$ that

$$\frac{\beta(x)}{\beta(r(x))} = \frac{\prod\limits_{c \in C_x} \tau(c) \cdot \prod\limits_{a \in A^+ \cap A_x} \sigma(a)}{\prod\limits_{c \in C_{r(x)}} \tau(c) \cdot \prod\limits_{a \in A^+ \cap A_{r(x)}} \sigma(a)} \cdot \frac{\prod\limits_{a \in A^0 \cap A_x} \overline{\sigma}(a)}{\prod\limits_{a \in A^0 \cap A_{r(x)}} \overline{\sigma}(a)}$$

$$= \frac{\mathbb{P}_{\overline{\sigma}}(x)}{\mathbb{P}_{\overline{\sigma}}(r(x))}$$

for every $x \in X^+$ which implies condition (2) in Theorem 3.1.

Since $M^+ w = 0$ it follows that

$$\sum_{a \in A^0 \cap A_x} w_a - \sum_{a \in A^0 \cap A_{r(x)}} w_a = 0$$

for every $x \in X^+$. Taking the exponential function on both sides leads to the equation

$$\frac{\prod\limits_{a \in A^0 \cap A_x} \exp w_a}{\prod\limits_{a \in A^0 \cap A_{r(x)}} \exp w_a} = 1$$

for every $x \in X^+$. Since $M^0 w < 0$, we can show in a similar way that

$$\frac{\prod\limits_{a \in A^0 \cap A_x} \exp w_a}{\prod\limits_{a \in A^0 \cap A_{r(x)}} \exp w_a} < 1$$

for every $x \in X^0$. Finally, we define the constants $\varepsilon(a)$ by $\varepsilon(a) := \exp(w_a)$ for every $a \in A^0$.

Since the proof in the other direction is similar, the proof of Theorem 3.1 is complete.   ∎

Consistency and Simple Sequences of Assessments

As a corollary of Theorem 3.1, we show that we can restrict ourselves to a very special class of sequences of completely mixed assessments if we want to check whether a given assessment is consistent or not. These sequences, which we call simple, have the property that they are completely determined by assigning two parameters to every action. As a consequence, the infinitely dimensional problem of checking consistency is reduced to a finitely dimensional problem.

A sequence $(\sigma^k, \beta^k)_{k \in \mathbb{N}}$ of assessments is called *simple* if for every action a there are numbers $\bar{\sigma}(a) > 0$ and $\varepsilon(a) \in (0, 1]$ such that

$$\sigma^k(a) = R^k(h) \cdot \bar{\sigma}(a) \cdot (\varepsilon(a))^k$$

for every $k \in \mathbb{N}$. Here, h is the information set with $a \in A(h)$ and $R^k(h)$ is the normalizing constant given by $R^k(h) = [\sum_{a' \in A(h)} \bar{\sigma}(a') \cdot (\varepsilon(a'))^k]^{-1}$.

COROLLARY 3.3.   *An assessment (σ, β) is consistent if and only if there is a simple sequence (σ^k, β^k) of completely mixed, Bayesian consistent assessments converging to (σ, β).*

Proof.   We only have to prove the "only if" part, since the "if" part is true by definition. Let (σ, β) be a consistent assessment. By Theorem 3.1 we can find numbers $\varepsilon(a)$ and $\bar{\sigma}(a)$ for every $a \in A^0(\sigma)$ such that the conditions (1) and (2) in this theorem are satisfied. For every $a \in A^+(\sigma)$ we define $\bar{\sigma}(a) := \sigma(a)$ and $\varepsilon(a) := 1$. Since the simple sequence (σ^k, β^k) of completely mixed, Bayesian consistent assessments induced by these numbers converges to (σ, β) the proof is complete.   ∎

The notion of simple sequences of assessments is somewhat related to the sequences used by Kreps and Wilson (1982) in the proof of Lemma A2. However, the sequences in Kreps and Wilson are constructed in a different way.

4. AN ALGORITHM

In this section we provide an algorithm to compute the set of consistent assessments in an extensive form game. First, we introduce some further notation and discuss several lemmas which play an essential role in the development of the algorithm.

In an extensive form game Γ, we denote the set of consistent assessments by $\mathcal{A}^c$. For given sets A^+ and X^+, $\mathcal{A}^c(A^+, X^+)$ denotes the set of consistent assessments (σ, β) with $A^+(\sigma) = A^+$ and $X^+(\beta) = X^+$. Obviously,

$$\mathcal{A}^c = \bigcup_{A^+, X^+} \mathcal{A}^c(A^+, X^+).$$

In the proof of Theorem 3.1 we constructed for a given assessment (σ, β) a matrix M. However, this matrix depends on the sets $A^+(\sigma)$ and $X^+(\beta)$ only. This means that we can construct such a matrix M for every possible combination A^+, X^+. Furthermore, we introduced for a given assessment (σ, β) the vectors b and s, where b depends on β and s depends on σ.

Therefore, we denote these vectors by $b(\beta)$ and $s(\sigma)$ respectively. By $b^+(\beta)$ and $s^+(\sigma)$ we denote restrictions of $b(\beta)$ and $s(\sigma)$ to X^+.

From Lemma 3.2 it follows that an assessment (σ, β) can only be consistent if there is a vector $w < 0$ with $M^+w = 0$ and $M^0w < 0$. Obviously, this problem is equivalent to the problem "Is there a vector $w \leq -1$ with $M^+w = 0$ and $M^0w \leq -1$?".[2] The latter problem is an LP problem and can therefore be solved efficiently (by using the simplex method, for example). Combining this insight with Lemma 3.2 leads to the following lemma, which turns out to be the keystone for our algorithm.

LEMMA 4.1.   *Let A^+, X^+ be given and M be the corresponding matrix.*
If there is a vector $w \leq -1$ with $M^+w = 0$ and $M^0w \leq -1$ then $\mathcal{A}^c(A^+, X^+)$ is equal to the set of assessments (σ, β) with $A^+(\sigma) = A^+$, $X^+(\beta) = X^+$, and $b^+(\beta) \in s^+(\sigma) + \mathrm{Im}(M^+)$.
Otherwise, $\mathcal{A}^c(A^+, X^+)$ is empty.

The proof follows directly from Lemma 3.2.

LEMMA 4.2.   *There is a basis for $\mathrm{Ker}((M^+)^t)$ which consists of integer vectors.*

Proof.   Since $(M^+)^t$ is an integer matrix, we can transform $(M^+)^t$ with the Gauss-elimination method into a rational upper-triangular matrix. Obviously, the kernel of this triangular matrix (and, hence, the kernel of $(M^+)^t$) has a basis consisting of rational vectors. By multiplying these vectors with an appropriate integer, we obtain a basis consisting of integer vectors.   ∎

LEMMA 4.3.   *Let $n^1, \ldots, n^r$ be a basis for $\mathrm{Ker}((M^+)^t)$ consisting of integer vectors. Then,*

$$\mathrm{Im}(M^+) = \{z \mid n^i \cdot z = 0 \ for \ i = 1, \ldots, r\}.$$

The proof of this lemma is straightforward.

From Lemma 4.1 and Lemma 4.3 it follows that $b^+(\beta) - s^+(\sigma) \in \mathrm{Im}(M^+)$ if and only if $n^i \cdot b^+(\beta) = n^i \cdot s^+(\sigma)$ for all i. This result leads to the following corollary.

COROLLARY 4.4.   *Let $n^1, \ldots, n^r$ be a basis for $\mathrm{Ker}((M^+)^t)$. If there is a vector $w \leq -1$ with $M^+w = 0$ and $M^0w \leq -1$ then $\mathcal{A}^c(A^+, X^+)$ is equal to the set of assessments (σ, β) with $A^+(\sigma) = A^+$, $X^+(\beta) = X^+$, and*

$$n^i \cdot b^+(\beta) = n^i \cdot s^+(\sigma) \qquad for \ all \ i.$$

Otherwise, $\mathcal{A}^c(A^+, X^+)$ is empty.

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[2] The inequalities $w \leq -1$ and $M^0w \leq -1$ should be read coordinatewise.

Now, we are able to construct an algorithm which generates the set of consistent assessments. The algorithm is based on the following steps.

Step 1. Choose A^+, X^+ and compute the corresponding matrix M.

Step 2. Solve the LP-problem "Is there a vector $w \leq -1$ with $M^+ w = 0$ and $M^0 w \leq -1$?" with the simplex method.

If the answer is "yes," then go to step 3.

If the answer is "no," then $\mathscr{A}^c(A^+, X^+)$ is empty and go to Step 1 (until every combination A^+, X^+ has been chosen).

Step 3. Compute a basis $n^1, \ldots, n^r$ for $\mathrm{Ker}((M^+)^t)$ consisting of integer vectors with the Gauss-elimination method. Then $\mathscr{A}^c(A^+, X^+)$ is equal to the set of assessments (σ, β) with $A^+(\sigma) = A^+$, $X^+(\beta) = X^+$ and $n^i \cdot b^+(\beta) = n^i \cdot s^+(\sigma)$ for all i.

Go to Step 1 (until every combination A^+, X^+ has been chosen).

If we translate the linear equations in Step 3 into the original strategies σ and the original beliefs β by taking the exponential function on both sides, we obtain a system of polynomial equations in σ and β. A different algorithm to compute such polynomial equations can be found in Kohlberg and Reny (1991).

In the following example, we apply the algorithm in order to compute one particular set $\mathscr{A}^c(A^+, X^+)$ for the extensive form game of Example 1.

EXAMPLE 2. Let Γ be an extensive form game with the extensive form structure of Fig. 1.

Step 1. Choose $A^+ = \{c, d, g, h, i, j\}$ and $X^+ = \{x_1, x_2, y_1, y_2, z_1, z_2\}$. Hence, $A^0 = \{a, b, e, f\}$ and X^0 is empty. The nodes $r(x)$ are given by

$$r(x_1) = r(x_2) = x_1, \qquad r(y_1) = r(y_2) = y_1, \qquad r(z_1) = r(z_2) = z_1.$$

The corresponding matrix M is given by

$$M = \begin{bmatrix} & a & b & e & f & \\ 0 & 0 & 0 & 0 & x_1 \\ -1 & 1 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & y_1 \\ -1 & 1 & -1 & 1 & y_2 \\ 0 & 0 & 0 & 0 & z_1 \\ -1 & 1 & 1 & -1 & z_2 \end{bmatrix}.$$

Step 2. Since $X^+ = X$, we have that $M^+ = M$ and $M^0 = \varnothing$. There is a vector $w \leq -1$ with $M^+ w = 0$. (Take, for example, $w = [-1, -1, -1, -1]$.)

Step 3. The transposed matrix $((M^+)^t)$ is given by

$$(M^+)^t = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

By Gauss-elimination, this matrix can be transformed to the upper-triangular matrix

$$\begin{bmatrix} 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the integer vectors $n^1, n^2,$ and n^3 are given by

$$n^1 := [1, -2, 0, 1, 0, 1], n^2 := [0, -2, 1, 1, 0, 1], n^3 := [0, -2, 0, 1, 1, 1]$$

form a basis for $\text{Ker}((M^+)^t)$.

For every assessment (σ, β) we have by definition

$$b_x^+(\beta) = \log \beta(x) - \log \beta(r(x))$$

and

$$s_x^+(\sigma) = \sum_{a \in A^+ \cap A_x} \log \sigma(a) - \sum_{a \in A^+ \cap A_{r(x)}} \log \sigma(a)$$

for every $x \in X$. Hence, $b_x^+(\beta) = s_x^+(\sigma) = 0$ if $x \in \{x_1, y_1, z_1\}$. Therefore, the equations $n^i \cdot b^+(\beta) = n^i \cdot s^+(\sigma)$ are all equivalent to the equation $n^1 \cdot b^+(\beta) = n^1 \cdot s^+(\sigma)$. This equation is given by

$$-2(\log \beta(x_2) - \log \beta(x_1)) + 1(\log \beta(y_2) - \log \beta(y_1))$$
$$+ 1(\log \beta(z_2) - \log \beta(z_1)) = 0$$

since $s_x^+(\sigma) = 0$ for all $x \in X$. If we take the exponential function on both sides, we obtain the equation

$$\frac{\beta(x_2)^{-2}}{\beta(x_1)^{-2}} \frac{\beta(y_2)}{\beta(y_1)} \frac{\beta(z_2)}{\beta(z_1)} = 1$$

which is equivalent to

$$\beta(x_1)^2 \beta(y_2) \beta(z_2) = \beta(x_2)^2 \beta(y_1) \beta(z_1).$$

Finally, we may conclude that $\mathscr{A}^c(A^+, X^+)$ is the set of assessments (σ, β) with $A^+(\sigma) = A^+$, $X^+(\beta) = X^+$, and

$$\beta(x_1)^2 \beta(y_2) \beta(z_2) = \beta(x_2)^2 \beta(y_1) \beta(z_1).$$

5. STRUCTURE OF THE SET OF CONSISTENT ASSESSMENTS

In this section, we give a geometrical description of the set of consistent assessments. To this purpose, we use some results derived in the previous section.

Let Γ be an extensive form game and the sets A^+, X^+ be fixed. By Corollary 4.4 we know that either $\mathscr{A}^c(A^+, X^+)$ is empty or $\mathscr{A}^c(A^+, X^+)$ is the set of assessments (σ, β) with $A^+(\sigma) = A^+$, $X^+(\beta) = X^+$, and

$$n^i \cdot b^+(\beta) = n^i \cdot s^+(\sigma)$$

for all i. If we take the exponential function on both sides and use the definitions of $b^+(\beta)$ and $s^+(\sigma)$, the equation $n^i \cdot b^+(\beta) = n^i \cdot s^+(\sigma)$ is equivalent to the equation

$$\frac{\prod_{x \in X^+} \beta(x)^{n_x^i}}{\prod_{x \in X^+} \beta(r(x))^{n_x^i}} = \frac{\prod_{x \in X^+} \left[\prod_{c \in C_x} \tau(c) \cdot \prod_{a \in A^+ \cap A_x} \sigma(a) \right]^{n_x^i}}{\prod_{x \in X^+} \left[\prod_{c \in C_{r(x)}} \tau(c) \cdot \prod_{a \in A^+ \cap A_{r(x)}} \sigma(a) \right]^{n_x^i}}.$$

This equation can be written in the form

$$\prod_{x \in X} \beta(x)^{m_x^i} \cdot \prod_{a \in A} \sigma(a)^{m_a^i} = c^i \cdot \prod_{x \in X} \beta(x)^{l_x^i} \cdot \prod_{a \in A} \sigma(a)^{l_a^i} \quad (5.1)$$

where c^i is a constant and $m_x^i, m_a^i, l_x^i, l_a^i$ are nonnegative integers.

Let m^i be the vector $((m_a^i)_{a \in A}, (m_x^i)_{x \in X})$ and l^i be the vector $((l_a^i)_{a \in A}, (l_x^i)_{x \in X})$. By $(\sigma, \beta)^{m^i}$ we denote the expression

$$\prod_{a \in A} \sigma(a)^{m_a^i} \cdot \sum_{x \in X} \beta(x)^{m_x^i}.$$

We treat similarly $(\sigma, \beta)^{l^i}$. Using this notation, Eq. (5.1) can be written in the form

$$(\sigma, \beta)^{m^i} = c^i \cdot (\sigma, \beta)^{l^i}$$

and we obtain the following theorem.

THEOREM 5.1. *For every pair A^+, X^+ either $A^c(A^+, X^+)$ is empty or $A^c(A^+, X^+)$ is determined by finitely many equations of the form*

$$(\sigma, \beta)^{m^i} = c^i \cdot (\sigma, \beta)^{l^i},$$

where m^i and l^i are nonnegative integer vectors.

For a nonnegative vector $v \in \mathbb{R}^n$, we denote by $\log v$ the vector in $(\mathbb{R}^*)^n$ obtained by taking the coordinatewise logarithm in v. (Note that $z_i = -\infty$ if $v_i = 0$.) For a set $S \subset \mathbb{R}^n$ of nonnegative vectors the set $\log S \subset (\mathbb{R}^*)^n$ is defined in the obvious way.

A set $C \subset (\mathbb{R}^*)^n$ is called a *cone with vertex* if for every $c^1, c^2 \in C$ and every $a, b > 0$ we have that $ac^1 + bc^2 \in C$. We call a set $C \subset (\mathbb{R}^*)^n$ a *cone* if there is a vector $v \in \mathbb{R}^n$ and a cone C' with vertex 0 such that $C = v + C'$.

A set $L \subset \mathbb{R}^n$ is said to be a *logarithmic cone* if $\log L$ is a cone. Hence, a logarithmic cone can be transformed into a cone by taking the coordinatewise logarithm.

THEOREM 5.2. *The set of consistent assessments is the intersection of a finite product of simplices with a finite number of logarithmic cones.*

Proof. For a given pair A^+, X^+ we denote by $A(A^+, X^+)$ the set of assessments (σ, β) with $A^+(\sigma) = A^+$ and $X^+(\beta) = X^+$. By Theorem 5.1 we know that $A^c(A^+, X^+)$ is either empty or is equal to the set of assessments in $A(A^+, X^+)$ which satisfy finitely many equations of the form

$$(\sigma, \beta)^{m^i} = c^i \cdot (\sigma, \beta)^{l^i}. \tag{5.2}$$

Assume that $A^c(A^+, X^+)$ is not empty. For a nonnegative vector (σ, β) of the same size as an assessment we define $A^+(\sigma)$ and $X^+(\beta)$ in the obvious way. By $L(A^+, X^+)$ we denote the set of nonnegative vectors (σ, β) with $A^+(\sigma) = A^+$, $X^+(\beta) = X^+$, and which satisfies the equations (5.2).

Since the variables $\sigma(a), \beta(x)$ with $a \notin A^+$, $x \notin X^+$ do not appear in these equations, the set $L(A^+, X^+)$ can be written as

$$L(A^+, X^+) = \{y > 0 \mid B(\log y) = d\} \times \{\mathbf{0}\}$$

for some appropriate matrix B and vector d. Here, $\mathbf{0}$ denotes the vector of zeroes corresponding to the restriction of (σ, β) on $A^0(\sigma) \times X^0(\beta)$. It can be seen easily that $L(A^+, X^+)$ is a logarithmic cone.

If we denote the set of all assessments by $\mathcal{A}$, we obtain

$$\mathcal{A}^c = \mathcal{A} \cap \left[\bigcup_{A^+, X^+} L(A^+, X^+) \right].$$

Since $\mathcal{A}$ is a finite product of simplices, it follows that $\mathcal{A}^c$ is the intersection of a finite product of simplices with a finite number of logarithmic cones.[3] ∎

In Blume and Zame (1994) it has been shown that the set of sequential equilibria is a semialgebraic set. (A set is called semialgebraic if it is the finite union of sets determined by a finite number of polynomial inequalities. These inequalities may be strict or non-strict.) Using our insights about consistent assessments, this result can be shown within a few lines.

Obviously, the sets $L(A^+, X^+)$ in the proof of Theorem 5.2 are semialgebraic sets. Since $\mathcal{A}$ is also semialgebraic, it follows that the set of consistent assessments is semialgebraic. We already know that the set of sequentially rational assessments is semialgebraic since sequential rationality is equivalent to a finite number of polynomial inequalities. Hence, the set of sequential equilibria is the intersection of two semialgebraic sets and is therefore semialgebraic itself.

COROLLARY 5.3. *The set of sequential equilibria is a semialgebraic set.*

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