Generalized Nash Equilibrium without Common Belief in Rationality

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Abstract. This note considers generalized Nash equilibrium as an incomplete information analogue of Nash equilibrium and provides an epistemic characterization of it. It is shown that the epistemic conditions do not imply common belief in rationality. For the special case of complete information, an epistemic characterization of Nash equilibrium ensues as a corollary.

Keywords: common belief in rationality, epistemic characterization, epistemic game theory, generalized Nash equilibrium, incomplete information, interactive epistemology, Nash equilibrium, solution concepts, static games.

1 Introduction

In game theory Nash's (1950) and (1951) notion of equilibrium constitutes one of the most prevalent solution concepts for static games with complete information. In order to unveil the reasoning assumptions underlying Nash equilibrium, epistemic foundations have been provided for this solution concept by, for instance, Aumann and Brandenburger (1995), Perea (2007), Barelli (2009), Bach

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and Tsakas (2014), as well as Bonnano (2017). In each of these epistemic foundations some correct-beliefs assumption is needed to obtain Nash equilibrium. As correct-beliefs seems to be a rather demanding requirement, Nash equilibrium imposes strong conditions on the players' reasoning.

In static games with incomplete information, players face uncertainty about the opponents' utility functions. For this more general class of games the most widespread solution concept is Harsanyi's (1967-68) Bayesian equilibrium. In fact, as Bach and Perea (2017) show, Bayesian equilibrium does not generalize Nash equilibrium but correlated equilibrium to incomplete information. However, an incomplete information analogue to Nash equilibrium can be defined, by extending the mutual-optimality property to payoff uncertainty. Accordingly, a tuple consisting of beliefs about each player's choice and utility function is called a generalized Nash equilibrium, whenever each belief only assigns positive probability to choice utility function pairs such that the choice is optimal for the utility function and the product measure of the beliefs on the opponents' choices. Coinciding with the mutual-optimality property definition of Nash equilibrium in the case of complete information with mixed strategies interpreted as beliefs, the notion of generalized Nash equilibrium thus provides a direct generalization of Nash equilibrium to incomplete information.

As an illustration of the incomplete information solution concept of generalized Nash equilibrium, suppose a game between two players *Alice* and *Bob* who are both invited to a party. They need to – simultaneously and independently – choose the colour of their outfits black or pink, or alternatively, to stay at home. *Alice* prefers wearing the same colour as *Bob* to staying at home, but prefers staying at home to attending the party with a different colour than *Bob*. *Alice* is not sure about *Bob*'s preferences. She thinks that he either entertains the same preferences as her or that he prefers atttending the party with a different colour than her to staying at home, but prefers staying at home to attending the party with the same colour as her. The utility functions for *Alice* and *Bob* are provided in Figure 1, and an interactive representation of the game is given in Figure 2.

u_A	*black*	*pink*	*stay*
black	3	0	0
pink	0	3	0
stay	2	2	2

u_B	*black*	*pink*	*stay*
black	3	0	0
pink	0	3	0
stay	2	2	2

u'_B	*black*	*pink*	*stay*
black	0	3	0
pink	3	0	0
stay	2	2	2

Fig. 1. Utility functions of *Alice* and *Bob*.

Consider the two beliefs $(black, u_A)$ about *Alice*'s choice and utility function as well as $\frac{3}{4} \cdot (black, u_B) + \frac{1}{4} \cdot (pink, u'_B)\big)$ about *Bob*'s choice and utility function. Note that black is optimal for *Alice*'s utility function u_A, if she believes *Bob* to wear black with probability $\frac{3}{4}$ and pink with probability $\frac{1}{4}$. Also, black is optimal for *Bob*'s utility function u_B, if he believes *Alice* to wear black, and pink is optimal for *Bob*'s utility function u'_B, if he believes her to wear black.

		Bob		
		black	*pink*	*stay*
	black	3, 3	0, 0	0, 2
Alice	*pink*	0, 0	3, 3	0, 2
	stay	2, 0	2, 0	2, 2

		Bob		
		black	*pink*	*stay*
	black	3, 0	0, 3	0, 2
Alice	*pink*	0, 3	3, 0	0, 2
	stay	2, 0	2, 0	2, 2

Fig. 2. Interactive representation of the two-player game with incomplete information and utility functions as specified in Figure 1.

The two beliefs $(black, u_A)$ and $\big(\frac{3}{4} \cdot (black, u_B) + \frac{1}{4} \cdot (pink, u'_B)\big)$ thus form a generalized Nash equilibrium.

This note provides an epistemic characterization of the incomplete information solution concept of generalized Nash equilibrium. Also, it is shown that the conditions actually do not imply common belief in rationality. Indeed, as in the complete information case of Nash equilibrium, the decisive property for players to reason in line with generalized Nash equilibrium is a correct-beliefs assumption. Besides, for complete information games an epistemic characterization of Nash equilibrium ensues as a corollary.

2 Generalized Nash Equilibrium

A game with incomplete information is modelled as a tuple $\Gamma = \big(I, (C_i)_{i \in I}, (U_i)_{i \in I}\big)$, where I is a finite set of players, C_i denotes player i's finite choice set, and the finite set U_i contains player i's utility functions, where a utility function $u_i : \times_{j \in I} C_j \to \mathbb{R}$ from U_i assigns a real number $u_i(c)$ to every choice combination $c \in \times_{j \in I} C_j$. Complete information obtains as a special case, if the set U_i is a singleton for every player $i \in I$.

Before the solution concept of generalized Nash equilibrium for games with incomplete information is defined, attention is restricted to complete information and the classical solution concept of Nash equilibrium is recalled. For a given game $\Gamma = \big(I, (C_i)_{i \in I}, (\{u_i\})_{i \in I}\big)$ with complete information, a tuple $(\sigma_i)_{i \in I} \in \times_{i \in I} \Delta(C_i)$ of probability measures constitutes a *Nash equilibrium*, whenever for all $i \in I$ and for all $c_i \in C_i$, if $\sigma_i(c_i) > 0$, then $\sum_{c_{-i} \in C_{-i}} \sigma_{-i}(c_{-i}) \cdot u_i(c_i, c_{-i})\big) \geq \sum_{c_{-i} \in C_{-i}} \sigma_{-i}(c_{-i}) \cdot u_i(c'_i, c_{-i})\big)$ for all $c'_i \in C_i$.[1] A direct generalization of Nash equilibrium to incomplete information obtains as follows.

Definition 1. *Let Γ be a game with incomplete information, and $(\beta_i)_{i \in I} \in \times_{i \in I}\big(\Delta(C_i \times U_i)\big)$ be a tuple of probability measures. The tuple $(\beta_i)_{i \in I}$ constitutes a* generalized Nash equilibrium, *whenever for all $i \in I$ and for all $(c_i, u_i) \in C_i \times U_i$, if $\beta_i(c_i, u_i) > 0$, then*

$$\sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c'_i, c_{-i})$$

[1] Given collection $\{X_i : i \in I\}$ of sets and probability measures $p_i \in \Delta(X_i)$ for all $i \in I$, the set X_{-i} refers to the product set $\times_{j \in I \setminus \{i\}} X_j$ and the probability measure p_{-i} refers to the product measure $\Pi_{j \in I \setminus \{i\}} p_j \in \Delta(X_{-i})$ on X_{-i}.

for all $c_i' \in C_i$.

Intuitively, the mutual-optimality property of the players' supports required by the complete information solution concept of Nash equilibrium is extended to the augmented uncertainty space of choices and utility functions. Note that in the specific case of complete information, i.e. $U_i = \{u_i\}$ for all $i \in I$, the notion of generalized Nash equilibrium reduces to Nash equilibrium. In other words, generalized Nash equilibrium imposes the analogous condition on the – due to payoff uncertainty extended – space $\times_{i \in I}\big(\Delta(C_i \times U_i)\big)$ that Nash equilibrium imposes on the space $\times_{i \in I}\Delta(C_i)$. Furthermore, Bach and Perea (2017) show that generalized Nash equilibrium is a refinement of Harsanyi's (1967-68) solution concept of Bayesian equilibrium. Note that for the game represented in Figure 2, the tuple $\big((black, u_A), \frac{3}{4} \cdot (black, u_B) + \frac{1}{4} \cdot (pink, u_B')\big)$ indeed constitutes a generalized Nash equilibrium.

In order to characterize decision-making in line with generalized Nash equilibrium, the notion of optimal choice in a generalized Nash equilibrium is defined next.

Definition 2. *Let Γ be a game with incomplete information, $i \in I$ a player, and $u_i \in U_i$ some utility function of player i. A choice $c_i \in C_i$ of player i is* optimal for the utility function u_i in a generalized Nash equilibrium*, if there exists a generalized Nash equilibrium $(\beta_i)_{i \in I} \in \times_{i \in I}\big(\Delta(C_i \times U_i)\big)$ such that*

$$\sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c_i, c_{-i})\big) \geq \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c_i', c_{-i})$$

for all $c_i' \in C_i$.

3 Common Belief in Rationality

From the perspective of a single player there exist two basic sources of uncertainty with respect to Γ. A player faces strategic uncertainty, i.e. what choices his opponents make, as well as payoff uncertainty, i.e. what utility functions represent the opponents' preferences. The notion of an epistemic model provides the framework to formally describe the players' reasoning about these two sources of uncertainty.

Definition 3. *Let Γ be a game with incomplete information. An* epistemic model *of Γ is a tuple $\mathcal{M}^{\Gamma} = \big((T_i)_{i \in I}, (b_i)_{i \in I}\big)$, where for every player $i \in I$*

- *T_i is a finite set of types,*
- *$b_i : T_i \to \Delta(C_{-i} \times T_{-i} \times U_{-i})$ assigns to every type $t_i \in T_i$ a probability measure $b_i[t_i]$ on the set of opponents' choice type utility function combinations.*

Given a game and an epistemic model of it, belief hierarchies, marginal beliefs, as well as marginal belief hierarchies can be derived from every type. For instance, every type $t_i \in T_i$ induces a belief on the opponents' choice combinations by marginalizing the probability measure $b_i[t_i]$ on the space C_{-i}. For simplicity sake, no additional notation is introduced for marginal beliefs. It should always be clear from the context which belief $b_i[t_i]$ refers to.

Some further notions are now introduced. For that purpose consider a game Γ, an epistemic model $\mathcal{M}^\Gamma$ of it, and fix two players $i, j \in I$ such that $i \neq j$. A type $t_i \in T_i$ of i is said to *deem possible* some choice type utility function combination $(c_{-i}, t_{-i}, u_{-i}) \in C_{-i} \times T_{-i} \times U_{-i}$ of his opponents, if $b_i[t_i](c_{-i}, t_{-i}, u_{-i}) > 0$. Analogously, a type $t_i \in T_i$ deems possible some opponent j's type $t_j \in T_j$, if $b_i[t_i](t_j) > 0$. For each choice type utility function combination $(c_i, t_i, u_i) \in C_i \times T_i \times U_i$, the *expected utility* is given by

$$v_i(c_i, t_i, u_i) = \sum_{c_{-i} \in C_{-i}} \big(b_i[t_i](c_{-i}) \cdot u_i(c_i, c_{-i}) \big)$$

for every player $i \in I$.

Intuitively, an optimal choice yields at least as much payoff as all other choices, given what the player believes his opponents to select and given his utility function. Formally, optimality is a property of choices given a type utility function pair.

Definition 4. *Let Γ be a game with incomplete information, $\mathcal{M}^\Gamma$ an epistemic model of it, $i \in I$ some player, $u_i \in U_i$ some utility function of player i, and $t_i \in T_i$ some type of player i. A choice $c_i \in C_i$ is* optimal *for (t_i, u_i), if $v_i(c_i, t_i, u_i) \geq v_i(c'_i, t_i, u_i)$ for all $c'_i \in C_i$.*

A player believes in his opponents' rationality, if he only deems possible choice type utility function triples – for each of his opponents – such that the choice is optimal for the type utility function pair, respectively. Formally, a type $t_i \in T_i$ *believes in the opponents' rationality*, if t_i only deems possible choice type utility function combinations $(c_{-i}, t_{-i}, u_{-i}) \in C_{-i} \times T_{-i} \times U_{-i}$ such that c_j is optimal for (t_j, u_j) for every opponent $j \in I \setminus \{i\}$.

Iterating belief in rationality gives rise to the interactive reasoning concept of common belief in rationality.

Definition 5. *Let Γ be a game with incomplete information, $\mathcal{M}^\Gamma$ an epistemic model of it, and $i \in I$ some player.*

- *A type $t_i \in T_i$ expresses* 1-fold belief in rationality*, if t_i believes in the opponents' rationality.*
- *A type $t_i \in T_i$ expresses* k-fold belief in rationality *for some $k > 1$, if t_i only deems possible types $t_j \in T_j$ for all $j \in I \setminus \{i\}$ such that t_j expresses $k-1$-fold belief in rationality.*
- *A type $t_i \in T_i$ expresses* common belief in rationality*, if t_i expresses k-fold belief in rationality for all $k \geq 1$.*

A player satisfying common belief in rationality entertains a belief hierarchy in which the rationality of all players is not questioned at any level. Observe that if an epistemic model for every player only contains types that believe in the opponents' rationality, then every type also expresses common belief in rationality. This fact is useful when constructing epistemic models with types expressing common belief in rationality.

4 Epistemic Characterization of Generalized Nash Equilibrium

Before the incomplete information solution concept of generalized Nash equilibrium can be characterized epistemically, some further epistemic notions need to be invoked. For this purpose, consider a game with incomplete information Γ, some epistemic model $\mathcal{M}^\Gamma$ of it, and fix some player $i \in I$.

A type $t_i \in T_i$ of player i is said to have *projective beliefs*, if for every opponent $j \in I \setminus \{i\}$ it is the case that $b_i[t_i](t_j) > 0$ implies that $b_i[t_i](c_k, u_k) = b_j[t_j](c_k, u_k)$ for all $c_k \in C_k \times U_k$ and for all $k \in I \setminus \{i, j\}$. Intuitively, a player with projective beliefs thinks that every opponent shares his belief on every other player's choice utility function combination.

Moreover, a type $t_i \in T_i$ of player i is said to have *independent beliefs*, if $b_i[t_i](c_{-i}, u_{-i}, t_{-i}) = \Pi_{j \in I \setminus \{i\}} b_i[t_i](c_j, u_j, t_j)$ for all $(c_{-i}, t_{-i}, u_{-i}) \in C_{-i} \times T_{-i} \times U_{-i}$. Intuitively, a player with independent beliefs excludes the possibility that his opponents' choice utility function pairs could be correlated.

In addition, for every opponent $j \in I \setminus \{i\}$, a type $t_i \in T_i$ believes that j is *correct* about i's belief about the opponents' choice utility function combinations, if $b_i[t'_i](c_{-i}, u_{-i}) = b_i[t_i](c_{-i}, u_{-i})$ for all $t'_i \in \text{supp}(b_j[t_j])$, for all $t_j \in \text{supp}(b_i[t_i])$, and for all $(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}$.

Furthermore, a type $t_i \in T_i$ of player i is said to have *connected beliefs*, if for two opponents $j, k \in I \setminus \{i\}$ such that $j \neq k$, it is the case that $t_k \in \text{supp}(b_j[t_j])$ or $t_j \in \text{supp}(b_k[t_k])$ for all $t_j, t_k \in \text{supp}(b_i[t_i])$

Besides, for every opponent $j \in I \setminus \{i\}$, a type $t_i \in T_i$ of player i is said to believe that j expresses a certain property, if t_i only deems possible types $t_j \in T_j$ of player j that express the property.

Using these epistemic notions, the following epistemic characterization of generalized Nash equilibrium ensues.

Theorem 1. *Let Γ be a game with incomplete information, $i \in I$ some player, and $u_i^* \in U$ some utility function of player i. A choice $c_i^* \in C_i$ is optimal for u_i^* in a generalized Nash equilibrium, if and only if, there exists an epistemic model $\mathcal{M}^\Gamma$ of Γ with a type $t_i \in T_i$ of player i such that c_i^* is optimal for (t_i, u_i^*) and t_i satisfies the following conditions:*

- *(i) t_i has projective beliefs,*
- *(ii) t_i believes that every opponent $j \in I \setminus \{i\}$ has projective beliefs,*
- *(iii) t_i has independent beliefs,*
- *(iv) t_i believes that every opponent $j \in I \setminus \{i\}$ has independent beliefs,*
- *(v) t_i believes in the opponents' rationality,*

(*vi*) t_i *believes that every opponent* $j \in I \setminus \{i\}$ *believes in the opponents' rationality,*
(*vii*) t_i *believes that every opponent* $j \in I \setminus \{i\}$ *deems possible* t_i*,*
(*viii*) t_i *believes that every opponent* $j \in I \setminus \{i\}$ *is correct about* i*'s belief about the opponents' choice utility function combinations,*
(*ix*) t_i *believes that every opponent* $j \in I \setminus \{i\}$ *believes that* i *is correct about* j*'s belief about the opponents' choice utility function combinations.*
(*x*) t_i *has connected beliefs.*

Proof. For the *only if* direction of the theorem, let c_i^* be optimal for u_i^* in a generalized Nash equilibrium $(\beta_j)_{j \in I}$. Construct an epistemic model $\mathcal{M}^{\Gamma} = \big((T_j)_{j \in I}, (b_j)_{j \in I}\big)$ of Γ, where $T_j := \{t_j\}$ and $b_j[t_j](c_{-j}, t_{-j}, u_{-j}) := \beta_{-j}(c_{-j}, u_{-j})$ for all $(c_{-j}, u_{-j}) \in C_{-j} \times U_{-j}$ and for all $j \in I$.

As

$$v_i(c_i^*, t_i, u_i^*) = \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i^*(c_i^*, c_{-i})$$

$$\geq \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i^*(c_i, c_{-i}) = v_i(c_i, t_i, u_i^*)$$

for all $c_i \in C_i$, it is the case that c_i^* is optimal for (t_i, u_i^*).

Observe that by definition of the marginal beliefs of $b_k[t_k]$ about the opponents' choice type utility function combinations to be the product measure $\Pi_{l \in I \setminus k} \beta_l$ for all $k \in I$, it directly holds that every type has projective and independent beliefs. It thus also directly follows that every type believes every opponent to have projective and independent beliefs.

Consider some opponent $j \in I \setminus \{i\}$ of player i and a choice type utility function tuple $(c_j, t_j, u_j) \in C_j \times \{t_j\} \times U_j$ of player j such that $b_i[t_i](c_j, t_j, u_j) > 0$. Then, $\beta_j(c_j, u_j) > 0$ and

$$v_j(c_j, t_j, u_j) = \sum_{(c_{-j}, u_{-j}) \in C_{-j} \times U_{-j}} \beta_{-j}(c_{-j}, u_{-j}) \cdot u_j(c_j, c_{-j})$$

$$\geq \sum_{(c_{-j}, u_{-j}) \in C_{-j} \times U_{-j}} \beta_{-j}(c_{-j}, u_{-j}) \cdot u_j(c_j', c_{-j}) = v_j(c_j', t_j, u_j)$$

for all $c_j' \in C_j$, by construction of $b_i[t_i]$ and by virtue of $(\beta_j)_{j \in I}$ being a generalized Nash equilibrium. Thus, c_j is optimal for (t_j, u_j). Therefore, t_i believes in the opponents' rationality. Analogeously, it can be shown that every type t_j of every player $j \in I \setminus \{i\}$ also believes in the opponents' rationality. As $b_i[t_i](t_j) = 1$ for all $j \in I \setminus \{i\}$, it follows that t_i believes his opponents to believe in the opponents' rationality.

Note that it directly holds that t_i believes every opponent $j \in I \setminus \{i\}$ to deem possible his true type t_i, as there exists only this single type of i in the epistemic model $\mathcal{M}^{\Gamma}$.

Moreover, t_i's marginal belief on $C_{-i} \times U_{-i}$ coincides with $\Pi_{j \in I \setminus \{i\}} \beta_j$. Since $b_i[t_i](t_j) = 1$ and $b_j[t_j](t_i) = 1$ holds for every opponent $j \in I \setminus \{i\}$ of player

i, type t_i believes that every opponent j believes that i's marginal belief on $C_{-i} \times U_{-i}$ is indeed given by $\Pi_{j \in I \setminus \{i\}} \beta_j$. Analogeously, it can be shown that the single type $t_j \in T_j$ for every player $j \in I \setminus \{i\}$ believes that every respective opponent $k \in I \setminus \{j\}$ is correct about j's marginal belief on $C_{-j} \times U_{-j}$. As for all $j \in I \setminus \{i\}$ it is the case that $b_i[t_i](t_j) = 1$ and t_j believes that i is correct about j's marginal beliefs on $C_{-j} \times U_{-j}$, it follows that t_i believes every opponent j to believe that i is correct about j's marginal belief on $C_{-j} \times U_{-j}$.

Finally, as there exists only one type for each player, every type must have connected beliefs.

For the *if* direction of the theorem, consider an epistemic model $\mathcal{M}^\Gamma$ of Γ with a type $t_i \in T_i$ of player i that satisfies conditions $(i) - (x)$ and such that c_i^* is optimal for (t_i, u_i^*).

Construct a tuple $(\beta_j)_{j \in I} \in \Delta\big(\times_{j \in I} (C_j \times U_j)\big)$ of probability measures such that $\beta_j(c_j, u_j) := b_i[t_i](c_j, u_j)$ for all $(c_j, u_j) \in C_j \times U_j$ and for all $j \in I \setminus \{i\}$, and $\beta_i(c_i, u_i) := b_m[\hat{t}_m](c_i, u_i)$ for all $(c_i, u_i) \in C_i \times U_i$ and for some $m \in I \setminus \{i\}$ and for some $\hat{t}_m \in T_m$ with $b_i[t_i](\hat{t}_m) > 0$.

We first show that for all players $j, k \in I \setminus \{i\}$, for every type $t_j \in T_j$ such that $b_i[t_i](t_j) > 0$ and for every type $t_k \in T_k$ such that $b_i[t_i](t_k) > 0$, it is the case that $b_j[t_j](c_i, u_i) = b_k[t_k](c_i, u_i)$ for all $(c_i, u_i) \in C_i \times U_i$. Fix some $(c_i, u_i) \in C_i \times U_i$. Suppose that $j = k$ and consider $t_j, t'_j \in T_j$ with $b_i[t_i](t_j) > 0$ and $b_i[t_i](t'_j) > 0$. Towards a contradiction assume that $b_j[t_j](c_i, u_i) \neq b_j[t'_j](c_i, u_i)$. By condition (vii), it is the case that $b_j[t_j](t_i) > 0$. Hence, t_j deems it possible that i is not correct about j's belief about i's choice utility function combination, a contradiction with condition (ix). Now, suppose that $j \neq k$ and consider $t_j \in T_j$ as well as $t_k \in T_k$ with $b_i[t_i](t_j) > 0$ and $b_i[t_i](t_k) > 0$. By condition (x) and without loss of generality, it is the case that $b_j[t_j](t_k) > 0$. By condition (ii), it follows that $b_j[t_j](c_i, u_i) = b_k[t_k](c_i, u_i)$.

Next, we show that $(\beta_j)_{j \in I}$ constitutes a generalized Nash equilibrium. Consider player i and suppose that $\beta_i(c_i, u_i) > 0$. Then, $b_m[\hat{t}_m](c_i, u_i) > 0$, and there thus exists a type $t'_i \in T_i$ of player i such that $b_m[\hat{t}_m](c_i, t'_i, u_i) > 0$. By conditions $(viii)$ and (iii), it follows that $b_i[t'_i](c_{-i}, u_{-i}) = b_i[t_i](c_{-i}, u_{-i}) = \beta_{-i}(c_{-i}, u_{-i})$. By condition (vi), c_i is optimal for (t'_i, u_i), and hence c_i is optimal for (t_i, u_i). Therefore,

$$\sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c_i, c_{-i}) = v_i(c_i, t_i, u_i)$$

$$\geq v_i(c'_i, t_i, u_i) = \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i(c'_i, c_{-i})$$

for all $c'_i \in C_i$.

Now, consider some player $j \in I \setminus \{i\}$ and suppose that $\beta_j(c_j, u_j) > 0$ for some $(c_j, u_j) \in C_j \times U_j$. Then, $b_i[t_i](c_j, u_j) > 0$, and consequently $b_i[t_i](c_j, t_j, u_j) > 0$ for some type $t_j \in T_j$ of player j with $b_i[t_i](t_j) > 0$. By condition (i), it holds that $b_j[t_j](c_k, u_k) = b_i[t_i](c_k, u_k) = \beta_k(c_k, u_k)$ for all $(c_k, u_k) \in C_k \times U_k$ and for all $k \in I \setminus \{i, j\}$. Since $\beta_i(c_i, u_i) = b_m[\hat{t}_m](c_i, u_i)$ for all $(c_i, u_i) \in C_i \times U_i$, and as

$b_i[t_i](t_j) > 0$, it follows from above that $b_j[t_j](c_i, u_i) = b_m[\hat{t}_m](c_i, u_i) = \beta_i(c_i, u_i)$ for all $(c_i, u_i) \in C_i \times U_i$. By condition (iv), it thus holds that $b_j[t_j](c_{-j}, u_{-j}) = \beta_{-j}(c_{-j}, u_{-j})$. Moreover, by condition (v), the choice c_j is optimal for (t_j, u_j), and thus

$$\sum_{(c_{-j},u_{-j})\in C_{-j}\times U_{-j}} \beta_{-j}(c_{-j}, u_{-j}) \cdot u_j(c_j, c_{-j}) = v_j(c_j, t_j, u_j)$$

$$\geq v_j(c'_j, t_j, u_j) = \sum_{(c_{-j},u_{-j})\in C_{-j}\times U_{-j}} \beta_{-j}(c_{-j}, u_{-j}) \cdot u_j(c'_j, c_{-j})$$

holds for all $c'_j \in C_j$. Consequently, $(\beta_j)_{j\in I}$ constitutes a generalized Nash equilibrium.

Since $b_i[t_i](c_{-i}) = \beta_{-i}(c_{-i})$ and c_i^* is optimal for (t_i, u_i^*), it is the case that

$$\sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i^*(c_i^*, c_{-i}) = v_i(c_i^*, t_i, u_i^*)$$

$$\geq v_i(c_i, t_i, u_i^*) = \sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \beta_{-i}(c_{-i}, u_{-i}) \cdot u_i^*(c_i, c_{-i})$$

for all $c_i \in C_i$. As $(\beta_j)_{j\in I}$ constitutes a generalized Nash equilibrium, c_i^* is optimal for u_i^* in a generalized Nash equilibrium. ■

The preceeding theorem shows that correct-beliefs conditions are inherently linked to the incomplete information solution concept of generalized Nash equilibrium. In fact, conditions $(vii) - (ix)$ together form the correct-beliefs assumption that is needed. Intuivitely, with the presence of incomplete information the correct-beliefs assumption naturally does not only apply to strategic but also to payoff uncertainty.

However, only two layers of common belief in rationality are needed for the epistemic characterization of generalized Nash equilibrium. In fact, the epistemic conditions of Theorem 1 do not even imply common belief in rationality.

Remark 1. There exists a game Γ with incomplete information, an epistemic model $\mathcal{M}^\Gamma$ of Γ, $i \in I$ some player, and some type $t_i \in T_i$ of player i such that t_i satisfies conditions $(i) - (x)$ of Theorem 1, but t_i does not express common belief in rationality.

As complete information is a special case of incomplete information, the following example of a two person complete information game establishes Remark 1.

Example 1. Consider the two player game between Alice in Bob represented in Figure 3.

Construct an epistemic model $\mathcal{M}^\Gamma$ of Γ given by $T_{Alice} = \{t_A, t'_A, t''_A\}$ and $T_{Bob} = \{t_B, t'_B\}$ with $b_{Alice}[t_A] = (c, t_B)$, $b_{Alice}[t'_A] = (c, t'_B)$, and $b_{Alice}[t''_A] = (d, t_B)$, as well as $b_{Bob}[t_B] = 0.5 \cdot (a, t_A) + 0.5 \cdot (a, t'_A)$, and $b_{Bob}[t'_B] = (a, t''_A)$. Observe that t_A satisfies conditions $(i) - (x)$ of Theorem 1. However, t_A does

		Bob	
		c	d
Alice	a	0, 0	0, 0
	b	0, 0	1, 0

Fig. 3. A two player game between Alice and Bob.

not express common belief in rationality, as t_A believes that t_B deems possible that Alice is of type t'_A, which believes that Bob is of type t'_B, which in turn believes Alice to be of type t''_A and to choose a, i.e. which believes Alice to choose irrationally. ♣

Restricting attention to the specific class of complete information games, the epistemic characterization of generalized Nash equilibrium provides an epistemic characterization of its complete information analogue, i.e. of Nash equilibrium. The result is a direct consequence of Theorem 1, if payoff uncertainty is eliminated.

Corollary 1. *Let Γ be a game with complete information, and $i \in I$ some player. A choice $c_i \in C_i$ is optimal in a Nash equilibrium, if and only if, there exists an epistemic model $\mathcal{M}^{\Gamma}$ of Γ with a type $t_i \in T_i$ of player i such that c_i is optimal for t_i and t_i satisfies the conditions $(i) - (x)$ of Theorem 1.*

With Corollary 1 a new characterization is added to the epistemic analysis of Nash equilibrium. In particular, its conditions do not imply common belief in rationality.

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