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The weak sequential core for two-period economies

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Abstract We adapt the classical core concept to deal with situations involving time and uncertainty. We define the weak sequential core as the set of allocations that are stable against coalitional deviations ex ante, and moreover cannot be improved upon by any coalition after the resolution of uncertainty. We restrict ourselves to credible deviations, where a coalitional deviation cannot be counterblocked by some subcoalition. We study the relationship of the resulting core concept with other sequential core concepts, give sufficient conditions under which the weak sequential core is non-empty, but show that it is possible to give reasonable examples where it is empty.

Keywords Core · Assets

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1 Introduction

One of the central questions in the theory on resource allocation is which distributions are to be regarded as self-enforcing, or stable. There seems to exist a large consensus on a necessary condition for stable allocations, namely that no coalition should be able to improve upon it by gathering and redistributing its resources among the coalition members. The latter property is referred to as the *core condition*. While this property may have an unambiguous meaning in purely static contexts, it is not obvious, a-priori, how to formalize the principle in a framework, where there is initial uncertainty about some parameters of the economy, to be released at later stages, but where contracting at the initial state is incomplete. In fact, it is not as much the core property as the notion of feasibility that is ambiguous when renegotiation is allowed ex post. The reason is that, when a coalition considers to block a given allocation ex ante, it has to form expectations on feasible allocations at later stages.

It is thus necessary to extend the "classical" core concept to the framework mentioned above, requiring an allocation not only to be stable at the beginning, but in addition to remain stable against possible deviations after the resolution of uncertainty, or once the dynamic cooperation process is under way. Such sequential core concepts have been investigated by Gale (1978), Becker and Chakrabarti (1995), Repullo (1988), and Koutsougeras (1998) for specific economic environments, whereas Kranich et al. (2005) propose two core concepts for situations where agents face a finite sequence of transferable utility games: the *strong sequential core* and the *weak sequential core*.

Predtetchinski et al. (2002) apply an appropriately adapted version of the strong sequential core to two-stage economies in which trade in assets takes place at period zero and trade in commodities at period one. The payoff to each asset is uncertain at period zero, but is revealed before trade in commodities takes place. At period 0, only the contracts related to assets are binding. The idea in the strong sequential core concept is to select those state contingent allocations which are stable against coalitional deviations ex ante, and moreover cannot be improved upon by any coalition after the resolution of uncertainty. It may be the case, however, that the deviation by a coalition that promised improvement in ex ante terms, can in turn be improved upon by some sub-coalition once the true state is known. In this case, it could be argued that the coalitional deviation at hand is not "credible" since it may be counterblocked by some subcoalition. By imposing the weaker restriction that state contingent allocations should be robust solely against "credible" deviations, the concept of the *weak sequential core* is obtained, which is the subject of study in this paper.

The economic model under consideration resembles the set-up used in the literature on the core in economies with asymmetric information. For an excellent survey of that literature, see Forges et al. (2002). In both streams of the literature, one may consider an ex ante stage, before information is released, an interim stage, when individuals receive a private signal, and an ex post stage, when all private signals are available. In the special case considered in this paper, information is symmetric, and so the interim stage and the ex post stage coincide. The important distinction between this paper and the asymmetric information literature is that here only a (potentially) limited set of contracts, specified at the outset, is binding in the ex ante stage. In contrast, all incentive compatible contracts are binding in the asymmetric information literature. In our symmetric information set-up, this would correspond to the case where all possible state-contingent contracts are binding ex ante, and the weak sequential core would correspond to the classical core. When only a limited set of binding contracts is available ex ante, there is scope for renegotiation in the interim stage.

For static cooperative situations, it is known that the restriction to credible deviations by coalitions does not alter the definition of the core: an allocation can be blocked by some coalitional deviation if and only if it can be blocked by a credible coalitional deviation (see Ray 1989). Kranich et al. (2005) show, however, that this restriction becomes relevant if one turns to dynamic cooperative environments. Consequently, the weak sequential core for such economies is in general a strict superset of the strong sequential core.

The concept of the two-stage core, due to Koutsougeras (1998), is obtained under the assumption that if a deviation occurs before the true state is known, no exchange can take place within the deviating coalition once the state has been revealed. Members of the deviating coalition have completely pessimistic expectations concerning re-trading opportunities. In the weak sequential core, members of the deviating coalition will instead hold the more optimistic expectations that they are able to coordinate on a specific ex post core element. Under a number of weak conditions, the weak sequential core is a subset of the two-stage core.

In Predtetchinski et al. (2002) it is shown that the strong sequential core is empty for a large family of economies. In contrast, the weak sequential core will be shown to be non-empty in a number of important special cases. One instance is a finance economy, where the weak sequential core coincides with the two-stage core. Another is when a complete set of state-contingent contracts is available for trade. Then the weak sequential core, the strong sequential core, and the two-stage core coincide. Finally, we prove that the weak sequential core is non-empty for economies with two agents.

Surprisingly, we give an example with three agents and no assets where the weak sequential core is empty. While the strong sequential core has been shown to be weakly increasing in the number of assets, the comparative statics of the weak sequential core are more complicated: it may be non-empty in an economy with no assets, and become empty after introducing a single asset in the economy as we show by means of an example.

2 The model

We consider an economy with two periods of time referred to as $t = 0$ and $t = 1$, and a finite set $\mathcal{S} = \{1, \ldots, S\}$ of possible states in period $t = 1$. The probability $\rho_s > 0$ of realization of state $s \in \mathcal{S}$ is objectively known.

There is a finite set N of agents. Agents trade in J assets in period $t = 0$ and, conditional on the realization of the state, in L commodities in period $t = 1$. In state $s \in \mathcal{S}$, agent $i \in N$ has a consumption set $X^i_s \subset \mathbb{R}^L$. The ex-ante consumption set of agent i is given by $X^i = \times_{s \in \mathcal{S}} X^i_s$. Each agent i is further characterized by his vector $\omega^i_s \in X^i_s$ of endowments in state $s \in \mathcal{S}$ and his elementary utility function $u^i_s : X^i_s \to \mathbb{R}$. Agents are expected-utility maximizers, with $v^i : X^i \to \mathbb{R}$ the expected utility function defined by $v^i(x^i) = \sum_{s \in \mathcal{S}} \rho_s u^i_s(x^i_s)$.

We let A denote the $SL \times J$ matrix of asset payoffs. Generic entry A^j_{sl} of A specifies the quantity of commodity l paid by asset j in state $s \in \mathcal{S}$.

Thus, an economy is a tuple

$$E = \langle N, \mathcal{S}, J, L, A, (\rho_s, X^i_s, \omega^i_s, u^i_s)^{i \in N}_{s \in \mathcal{S}} \rangle.$$

Let E be the class of all economies. The institutional set-up is as follows.

1. In period $t = 0$, trade in assets takes place. Alternatively, one may think of these trades taking the form of *state-contingent contracts*. There is no endowment and therefore no consumption in period $t = 0$.
2. Nature randomly chooses the state. The execution of asset contracts takes place and results in an allocation x.
3. In period $t = 1$, trade in commodities takes place. Agents treat allocation x as their endowments. Trade in commodities results in a final allocation y.

Notice that the institutional setting is one of dynamic exchange without prices. Our analysis is therefore complementary to the extensive literature on constrained suboptimality of competitive equilibria when asset markets are incomplete (see for instance Geanakoplos and Polemarchakis 1986).

For $E \in \mathrm{E}$ and a coalition $M \subseteq N$ of agents, let $\mathrm{A}(M, E)$ be the set of *A-feasible* allocations for M, i.e. the allocations that can be achieved by trade in assets in period $t = 0$ within M. Thus,

$$\mathrm{A}(M, E) = \left\{ x \in \underset{i \in M}{\times} X^i \;\middle|\; \begin{array}{c} \sum_{i \in M} x^i = \sum_{i \in M} \omega^i, \\ x^i - \omega^i \in \langle A \rangle \text{ for all } i \in M \end{array} \right\},$$

where $\langle A \rangle$ denotes the linear space spanned by the columns of A. If there are no assets we define $\mathrm{A}(M, E)$ to be the one-point set $\{(\omega^i)_{i \in M}\}$.

For $E \in \mathrm{E}$, $M \subseteq N$, an *A-feasible* allocation $x \in \mathrm{A}(M, E)$, and a state $s \in \mathcal{S}$, we define an *ex post* sub-economy $r_s(x, M, E)$ as an economy following the realization of the state s and involving only the participants of coalition M. In this economy the set of agents is M, the number of commodities is L, the consumption set of agent $i \in M$ is X^i_s, the utility function is u^i_s, and the endowment is x^i_s. Formally,

$$r_s(x, M, E) = \langle M, L, (\rho_s, X^i_s, x^i_s, u^i_s)^{i \in M} \rangle.$$

The classical core $C(r_s(x, M, E))$ of $r_s(x, M, E)$ is the set of allocations $y_s \in \times_{i \in M} X^i_s$ such that

(1) $\sum_{i \in M} y^i_s = \sum_{i \in M} x^i_s$, and
(2) there exist no $Q \subseteq M$ and no $z_s \in \times_{i \in Q} X^i_s$ such that $\sum_{i \in Q} z^i_s = \sum_{i \in Q} x^i_s$ and $u^i_s(y^i_s) < u^i_s(z^i_s)$ for all $i \in Q$.

3 Three concepts of the sequential core

In this section we compare the concept of the weak sequential core to that of the strong sequential core and that of the two-stage core.

Definition 1 *An allocation* $\bar{y}$ *is an element of the weak sequential core of the economy* $E = (N, \mathcal{S}, J, L, A, \langle \rho_s, X^i_s, \omega^i_s, u^i_s \rangle^{i \in N}_{s \in \mathcal{S}})$, *denoted* $WSC(E)$, *if*

1. *there exists* $\bar{x} \in \mathrm{A}(N, E)$ *such that* $\bar{y}_s \in C(r_s(\bar{x}, N, E))$ *for all* $s \in \mathcal{S}$,
2. *there do not exist* $M \subseteq N$, $y \in \times_{i \in M} X^i$, *and* $x \in \mathrm{A}(M, E)$ *such that* $y_s \in C(r_s(x, M, E))$ *for all* $s \in \mathcal{S}$ *and* $v^i(y^i) > v^i(\bar{y}^i)$ *for all* $i \in M$.

Part 1 is the requirement that $\bar{y}$ be robust against potential deviations at $t = 1$, after the true state has been revealed. Part 2 is the requirement that $\bar{y}$ be robust against all *credible deviations* at $t = 0$, before the true state is known.

The three sequential core concepts – the weak sequential core, the strong sequential core, and the two-stage core – agree on the meaning of improvement by a coalition at $t = 1$. Since agents face no uncertainty and no prospects of future consumption at $t = 1$, the classical definition of an improving coalition seems to be appropriate, with the initial positions of coalitions being determined by the allocation $\bar{x}$. This results in the requirement that $\bar{y}_s \in C(r_s(\bar{x}, N, E))$ for all $s \in \mathcal{S}$.

The situation differs as we turn to $t = 0$. It is not obvious which deviations should be taken into account, and which not. Whence a discrepancy between the three sequential core concepts as to the definition of an improvement by a coalition at $t = 0$. The idea in the weak sequential core is to focus on credible deviations.

Suppose that a coalition M considers to deviate at $t = 0$ to allocation y, which is preferred, in expected terms, to allocation $\bar{y}$ by all of its members. In general, there may be counter-deviations from y by sub-coalitions of M at $t = 1$. Coalition M, however, might redistribute assets among its members at $t = 0$ in such a way that, given this redistribution, no deviations from y will be profitable at $t = 1$. If there does exist such a redistribution, then y can be regarded as a credible deviation. Otherwise it is not credible, for there may be counter-deviations from y at $t = 1$. The weak sequential core requires its elements to be robust only against credible deviations, rather than against all possible deviations in the classical sense.

Below we reproduce the definition of the strong sequential core from Predtetchinski et al. (2002).

Definition 2 *An allocation* $\bar{y}$ *is an element of the strong sequential core of the economy* $E = (N, \mathcal{S}, J, L, A, \langle \rho_s, X^i_s, \omega^i_s, u^i_s \rangle^{i \in N}_{s \in \mathcal{S}})$, *denoted* $SSC(E)$, *if*

1. *there exists* $\bar{x} \in \mathrm{A}(N, E)$ *such that* $\bar{y}_s \in C(r_s(\bar{x}, N, E))$ *for all* $s \in \mathcal{S}$,
2. *there do not exist* $M \subseteq N$ *and* $y \in \times_{i \in M} X^i$ *such that* $\sum_{i \in M} y^i = \sum_{i \in M} \omega^i$ *and* $v^i(y^i) > v^i(\bar{y}^i)$ *for all* $i \in M$.

The difference between Definitions 1 and 2 is that the latter does not require deviations of coalitions at $t = 0$ to be robust to possible counter-deviations by sub-coalitions at $t = 1$. Instead, the classical notion of improvement is adopted: a coalition improves upon $\bar{y}$ with y, if y is feasible and gives each member of the deviating coalition a higher expected utility than $\bar{y}$ does. Any element of the strong sequential core must therefore be stable against all the deviations in the classical sense, whether credible or not. Thus, the strong sequential core is a subset of the weak sequential core.

We proceed with the definition of the two-stage core, due to Koutsougeras (1998).

Definition 3 *An allocation $\bar{y}$ is an element of the two-stage core of the economy $E = (N, \mathcal{S}, J, L, A, \langle \rho_s, X_s^i, \omega_s^i, u_s^i \rangle_{s \in \mathcal{S}}^{i \in N})$, denoted $TSC(E)$, if*

1. *there exists $\bar{x} \in \mathrm{A}(N, E)$ such that $\bar{y}_s \in C(r_s(\bar{x}, N, E))$ for all $s \in \mathcal{S}$,*
2. *there do not exist $M \subseteq N$ and $x \in \mathrm{A}(M, E)$ such that $v^i(x^i) > v^i(\bar{y}^i)$ for all $i \in M$.*

Unlike Definitions 1 and 2, Definition 3 assumes that a coalition can only use A-feasible allocations to improve at $t = 0$: no exchange can take place within the deviating coalition at $t = 1$. The coalition that deviates at $t = 0$ is thus confined in its choice of consumption bundles to those that can be achieved by means of an exchange of assets alone.

The relationship between the three sequential core concepts is summarized in the following observation.

Observation 1 *Let $E \in \mathrm{E}$.*

1. *$SSC(E) \subseteq WSC(E)$ and $SSC(E) \subseteq TSC(E)$.*
2. *Suppose that the consumption set X_s^i is closed, convex and bounded from below, and the elementary utility function u_s^i is continuous and quasi-concave for all $i \in N$, $s \in \mathcal{S}$. Then $WSC(E) \subseteq TSC(E)$.*

In fact, the inclusion $\mathrm{WSC}(E) \subseteq \mathrm{TSC}(E)$ holds whenever $C(r_s(x, M, E)) \neq \emptyset$ for all $s \in \mathcal{S}$, $M \subseteq N$, and $x \in \mathrm{A}(M, E)$. This is guaranteed by the conditions of the second part of Observation 1. Otherwise, it may be more difficult to deviate according to the definition of the weak sequential core than according to the definition of the two-stage core. In this case the inclusion $\mathrm{WSC}(E) \subseteq \mathrm{TSC}(E)$ may not hold.

Proof Part 1 is obvious. To prove Part 2, let $\bar{y} \in WSC(E)$. Suppose that $\bar{y} \notin TSC(E)$. Then there are $M \subseteq N$ and $x \in \mathrm{A}(M, E)$ such that $v^i(x^i) > v^i(\bar{y}^i)$ for all $i \in M$. The assumptions of Observation 1 guarantee that $C(r_s(x, M, E)) \neq \emptyset$ for all $s \in \mathcal{S}$. For $s \in \mathcal{S}$, let $y_s \in C(r_s(x, M, E))$. Now, $v^i(y^i) \geq v^i(x^i) > v^i(\bar{y}^i)$ for all $i \in M$. This implies, however, that $\bar{y} \notin WSC(E)$, a contradiction. We conclude that $\bar{y} \in TSC(E)$. □

4 Some properties of the weak sequential core

In this section we consider three special cases: the case where a complete set of state-contingent contracts is available, the case of a finance economy, and the case of a two-agent economy.

We start with the case where for each commodity l and each state s there is a contract specifying the delivery of commodity l contingent on the occurrence of state s. The total number of state-contingent contracts is therefore SL. When all of these state-contingent contracts are available, A is the identity matrix. More generally, we can require the span of A to be the whole space $\mathbb{R}^{SL}$. If this is indeed the case, then any coalitional deviation in the classical sense and, in particular, any credible coalitional deviation, can be implemented directly by an appropriate trade in assets. Upon the realization of the state, asset contracts are executed, and no retrading of commodities is needed. It follows that the weak sequential core and the strong sequential core coincide with the two-stage core.

Observation 2 *Let $E \in \mathrm{E}$. Suppose that X^i_s is closed and bounded from below, and u^i_s is continuous for all $i \in N$, $s \in \mathcal{S}$. If $\langle A \rangle = \mathbb{R}^{SL}$, then $SSC(E) = WSC(E) = TSC(E)$.*

Proof If $\langle A \rangle = \mathbb{R}^{SL}$, then $SSC(E) = TSC(E)$, as is obvious from Definitions 2 and 3. By Observation 1, $SSC(E) \subseteq WSC(E)$. We now prove the inclusion $SSC(E) \supseteq WSC(E)$.

Let $\bar{y} \in \mathrm{WSC}(E)$ and suppose $\bar{y} \notin \mathrm{SSC}(E)$. Then there exists a coalitional deviation from $\bar{y}$ at $t = 0$ in the classical sense (see Definition 2). Let M be a deviating coalition and y be a profitable deviation. We show that the coalition M has a credible deviation from $\bar{y}$, thus obtaining a contradiction.

Consider the following optimization problem:

$$\begin{aligned}
&\text{maximize } \textstyle\sum_{i \in M} v^i(z^i) \\
&\text{subject to } z^i \in X^i \text{ and } v^i(z^i) \geq v^i(y^i) \quad \text{for all } i \in M, \\
&\qquad\qquad \textstyle\sum_{i \in M} z^i = \sum_{i \in M} \omega^i.
\end{aligned}$$

Under the conditions of Observation 2, the admissible set of this problem is compact and the objective function is continuous. Therefore, the problem has a solution, say $\bar{z} \in \times_{i \in M} X^i$. We prove that $\bar{z}_s \in C(r_s(\bar{z}, M, E))$ for all $s \in \mathcal{S}$.

Suppose not. Then there exist $\sigma \in \mathcal{S}$, $Q \subseteq M$, and $\dot{z}_\sigma \in \times_{i \in Q} X^i_\sigma$ such that $\sum_{i \in Q} \dot{z}^i_\sigma = \sum_{i \in Q} \bar{z}^i_\sigma$ and $u^i_\sigma(\dot{z}^i_\sigma) > u^i_\sigma(\bar{z}^i_\sigma)$ for all $i \in Q$. Define $z \in \times_{i \in M} X^i$ by the equation

$$z^i_s = \begin{cases} \dot{z}^i_\sigma & \text{if } s = \sigma \text{ and } i \in Q, \\ \bar{z}^i_s & \text{otherwise.} \end{cases}$$

Then z is feasible for coalition M and $v^i(z^i) \geq v^i(\bar{z}^i)$ for all $i \in M$, with strict inequality for all $i \in Q$. This contradicts, however, the definition of $\bar{z}$ as a solution to the optimization problem above. We see that $\bar{z}_s \in C(r_s(\bar{z}, M, E))$ for all $s \in \mathcal{S}$. Furthermore, $v^i(\bar{z}^i) \geq v^i(y^i) > v^i(\bar{y}^i)$ for all $i \in M$. Thus $\bar{z}$ is a credible deviation from $\bar{y}$ by M. □

Next we examine the case of a finance economy. A finance economy is a special case of a two-period economy, where a unique commodity is available in each $s \in \mathcal{S}$, i.e. $L = 1$. If $X^i_s = \mathbb{R}_+$ and the utility functions u^i_s are all increasing, then for all $s \in \mathcal{S}$, all $M \subseteq N$, and all $x \in \mathrm{A}(M, E)$ the set $C(r_s(x, M, E))$ consists of the point x_s alone. This leads to the following observation.

Observation 3 *Let $E \in \mathrm{E}$. If $L = 1$, $X^i_s = \mathbb{R}_+$, and u^i_s is increasing for all $i \in N$ and all $s \in \mathcal{S}$, then $WSC(E) = TSC(E)$.*

In a finance economy with no assets, the initial allocation need not be robust against ex ante coalitional deviations in the classical sense, in which case the strong sequential core is empty. This shows that the weak sequential core may be a strict superset of the strong sequential core.

We summarize Observations 1–3 in Table 1. Each column of this table gives sufficient conditions for an inclusion specified in the top row to hold. A star means "no assumptions".

Table 1 The summary of results (Observations 1–3)

	SSC ⊆ WSC SSC ⊆ TSC	TSC ⊆ SSC	WSC ⊆ SSC	WSC ⊆ TSC	WSC = TSC
$\langle A \rangle$	$*$	$\mathbb{R}^{SL}$	$\mathbb{R}^{SL}$	$*$	$*$
X_s^i	$*$	$*$	Closed bounded below	Closed bounded below convex	$L = 1,\ X_s^i = \mathbb{R}_+$
u_s^i	$*$	$*$	Continuous	Continuous quasi-concave	Increasing

To conclude the present section we formulate an existence result for the weak sequential core in case of an economy with two agents. This result is in sharp contrast with the findings on the strong sequential core. As is demonstrated in Predtetchinski et al. (2002), the strong sequential core may be empty in an economy satisfying all assumptions of Observation 4.

Observation 4 *Let $E \in \mathrm{E}$. Suppose that the economy E satisfies the following assumptions: the set of agents is $N = \{a, b\}$, the consumption sets X_s^i are closed and bounded from below, and the elementary utility functions u_s^i are continuous. Then $WSC(E) \neq \emptyset$.*

Proof The following notation will be useful for the proof:

$$X_s(N) = \left\{ y_s \in \times_{i \in N} X_s^i \,\middle|\, \sum_{i \in N} y_s^i = \sum_{i \in N} \omega_s^i \right\}$$

$$X(N) = \times_{s \in \mathcal{S}} X_s(N)$$

$$Z(N) = \left\{ (x, y) \in \mathrm{A}(N, E) \times X(N) \,\middle|\, \begin{array}{c} v^i(\omega^i) \leq v^i(y^i), \quad i \in N, \\ u_s^i(x_s^i) \leq u_s^i(y_s^i), \quad i \in N,\ s \in \mathcal{S} \end{array} \right\}.$$

Observe that $(\omega, \omega) \in Z(N)$. Moreover, under the assumptions of Observation 4, both $X(N)$ and $\mathrm{A}(N, E)$ are compact sets. It follows that $Z(N)$ is compact as well.

Let α^a and α^b be non–negative real numbers at least one of which is positive. Let $(\bar{x}, \bar{y}) \in Z(N)$ be a maximizer of the function $\alpha^a v^a(y^a) + \alpha^b v^b(y^b)$ over $Z(N)$. In the remainder of the proof we show that $\bar{y} \in WSC(E)$.

First we show that $\bar{y}_s \in C(r_s(\bar{x}, N, E))$ for all $s \in \mathcal{S}$. Since $u_s^i(\bar{x}_s^i) \leq u_s^i(\bar{y}_s^i)$, neither player a nor player b is able to improve upon $\bar{y}$ at $t = 1$. Now suppose that the grand coalition is able to improve upon $\bar{y}$ at $t = 1$ by deviating in some state $\sigma \in \mathcal{S}$. Then there is $z_\sigma \in X_\sigma(N)$ such that $u_\sigma^i(\bar{y}_\sigma^i) < u_\sigma^i(z_\sigma^i)$ for $i = a, b$. Define $\bar{z} \in X(N)$ as follows:

$$\bar{z}_s^i = \begin{cases} z_\sigma^i, & \text{if } s = \sigma, \\ \bar{y}_s^i, & \text{otherwise.} \end{cases}$$

Then $u_s^i(\bar{y}_s^i) \leq u_s^i(\bar{z}_s^i)$ for $i = a, b$ with strict inequality in state σ. It follows that $v^i(\bar{y}^i) < v^i(\bar{z}^i)$ for $i = a, b$ and that $(\bar{x}, \bar{z}) \in Z(N)$. This contradicts the definition of $(\bar{x}, \bar{y})$.

Now we show that the pair of allocations $(\bar{x}, \bar{y})$ satisfies the second part of Definition 1. Since $v^i(\omega^i) \leq v^i(\bar{y}^i)$, neither player can improve upon $\bar{y}$ by deviating to his endowment at $t = 0$. Suppose that the grand coalition is able to improve upon $\bar{y}$ by deviating at $t = 0$. Let (x, y) denote an improving deviation, i.e. a pair of allocations such that $x \in \mathrm{A}(N, E)$, $y_s \in C(r_s(x, N, E))$ for all $s \in \mathcal{S}$, and $v^i(\bar{y}^i) < v^i(y^i)$ for $i = a, b$. Then y is ex ante individually rational. Since y_s is individually rational in the ex post subeconomies, $u_s^i(x_s^i) \leq u_s^i(y_s^i)$. We see that $(x, y) \in Z(N)$. This contradicts the definition of $(\bar{x}, \bar{y})$. □

5 Examples

The purpose of the examples reported below is to demonstrate some limitations of the concept of the weak sequential core. In Examples 1 and 2, two economies with three agents and no assets are shown to have an empty weak sequential core. In contrast, the economy in Example 3, also with three agents and no assets, has a non-empty weak sequential core. It is shown, however, that the weak sequential core becomes empty, when an appropriately specified asset is introduced into the economy.

Example 1 Let E be an economy with two equally probable states ($\mathcal{S} = \{1, 2\}$, $\rho_1 = \rho_2 = 0.5$), two goods in each state ($L = 2$), a set $N = \{a, b, c\}$ of agents, and no assets ($J = 0$). The consumption set X_s^i of each agent in every state is the non-negative orthant of the space $\mathbb{R}^2$. The elementary utility functions and the endowments in state $s = 1$ are given by

$$\omega_1 = (\omega_1^a, \omega_1^b, \omega_1^c) = \begin{pmatrix} 100 & 0 & 0 \\ 0 & 100 & 100 \end{pmatrix},$$

$$u_1^a(z^a) = z_2^a, \; u_1^b(z^b) = z_1^b + 2z_2^b - 200, \; \text{and } u_1^c(z^c) = z_1^c + 2z_2^c - 200.$$

In state $s = 2$, the utility functions and the endowments are

$$\omega_2 = (\omega_2^a, \omega_2^b, \omega_2^c) = \begin{pmatrix} 0 & 100 & 0 \\ 100 & 0 & 100 \end{pmatrix},$$

$$u_2^a(z^a) = z_1^a + 2z_2^a - 200, \; u_2^b(z^b) = z_2^b, \; \text{and } u_2^c(z^c) = z_1^c + 2z_2^c - 200.$$

We show $WSC(E) = \emptyset$. The set $C(r_1(\omega, N, E))$ consists, in utility space, of the unique vector $(50, 0, 0)$. This utility vector is supported by any allocation of commodities where the entire endowment of good 1 is allocated among agents b and c, with one unit of good 1 exchanged for half a unit of good 2, so that agent a receives 50 units of good 2. Similarly, $C(r_2(\omega, N, E))$ consists of the utility vector $(0, 50, 0)$ alone. Thus, the only ex ante utility vector that can be achieved by the grand coalition is $(25, 25, 0)$.

This vector is blocked by the coalition $\{a, b\}$. Indeed, in state $s = 1$, the vector $(0, 100)$ belongs to $C(r_1(\omega, \{a, b\}, E))$. It is supported by the allocation of commodities where the entire endowment of good 1 is allocated to agent b, while a gets nothing. In state $s = 2$ the vector $(100, 0)$ of ex post utilities belongs to $C(r_2(\omega, \{a, b\}, E))$. Since the corresponding ex ante utilities are $(50, 50)$, coalition $\{a, b\}$ has a credible deviation. Thus the weak sequential core is empty. □

It is even possible to construct an economy with empty weak sequential core where the elementary utility functions are state-independent, strongly monotone, and strictly concave. Below we report one example of such an economy.

Example 2 Consider an economy E with two states occurring with positive probability ($\mathcal{S} = \{1, 2\}$), three goods in each state ($L = 3$), the set $N = \{a, b, c\}$ of agents and no assets ($J = 0$). The consumption set X^i_s of each agent in every state is given by the strictly positive orthant of $\mathbb{R}^3$, and the state–independent elementary utility functions are given by

$$u^i_s(z^i) = \ln(z^i_1) + \ln(z^i_2) + \ln(z^i_3), \quad i \in N, \ s \in \mathcal{S}.$$

The endowments are

$$\omega_1 = (\omega^a_1,\ \omega^b_1,\ \omega^c_1) = \begin{pmatrix} 1 & \epsilon & 1 \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}$$

$$\omega_2 = (\omega^a_2,\ \omega^b_2,\ \omega^c_2) = \begin{pmatrix} \epsilon & 1 & 1 \\ 1 & \epsilon & 1 \\ 1 & \epsilon & 1 \end{pmatrix}.$$

It turns out that the weak sequential core is empty whenever $0 < \epsilon \leq 0.021$. The proof can be found in the discussion paper by Predtetchinski et al. (2002). □

Finally, we report an example of an economy where the weak sequential core is non-empty if no assets are present, and becomes empty when an appropriately specified asset is introduced.

Example 3 Consider an economy E with two equally probable states ($\mathcal{S} = \{1, 2\}$, $\rho_1 = \rho_2 = 0.5$), two goods in each state ($L = 2$), and the set $N = \{a, b, c\}$ of agents. The consumption set X^i_s of each agent in every state is given by the strictly positive orthant of $\mathbb{R}^2$, and the state-independent elementary utility functions are given by

$$u^i_s(z^i) = \ln(z^i_1) + \ln(z^i_2), \quad i \in N, \ s \in \mathcal{S}.$$

The endowments are

$$\omega_1 = (\omega^a_1,\ \omega^b_1,\ \omega^c_1) = \begin{pmatrix} 1 & \epsilon & 1 \\ 1 & \epsilon & 1 \end{pmatrix}$$

$$\omega_2 = (\omega^a_2,\ \omega^b_2,\ \omega^c_2) = \begin{pmatrix} 1 & 1 & \epsilon \\ 1 & \epsilon & 1 \end{pmatrix}.$$

If there are no assets, then the weak sequential core is non-empty and is given by the set of allocations of the form (ω_1, y_2), where $y_2 \in C(r_2(\omega, N, E))$. Suppose now that an asset is introduced, with payoffs

$$A_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. the asset is a contract for the delivery of commodity 1 in state $s = 1$. Then $WSC(E) = \emptyset$ whenever $0 < \epsilon \leq 0.038$. The proof can be found in the discussion paper by Predtetchinski et al. (2002). □

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