Mini-course on Epistemic Game Theory
Lecture 2: Nash Equilibrium

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Singapore, September 2016

Introduction

- Nash equilibrium has dominated game theory for many years.
- But until the rise of Epistemic Game Theory it remained unclear what Nash equilibrium assumes about the reasoning of the players.
- In this lecture we will investigate Nash equilibrium from an epistemic point of view.
- We will see that Nash equilibrium requires more than just common belief in rationality.
- We show that Nash equilibrium can be epistemically characterized by

  common belief in rationality + simple belief hierarchy.

- However, the condition of a simple belief hierarchy is quite unnatural, and overly restrictive.

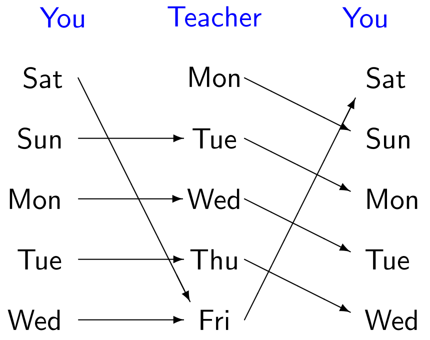
Example: Teaching a lesson

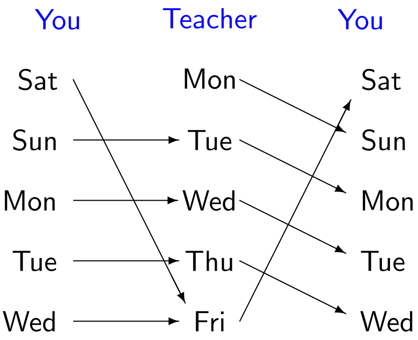
Story

- It is Friday, and your biology teacher tells you that he will give you a surprise exam next week.
- You must decide on what day you will start preparing for the exam.
- In order to pass the exam, you must study for at least two days.
- To write the perfect exam, you must study for at least six days. In that case, you will get a compliment by your father.
- Passing the exam increases your utility by 5.
- Failing the exam increases the teacher's utility by 5.
- Every day you study decreases your utility by 1, but increases the teacher's utility by 1.
- A compliment by your father increases your utility by 4.

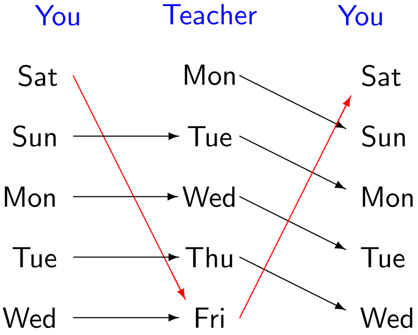
Teacher

You		Mon	Tue	Wed	Thu	Fri
	Sat	3, 2	2, 3	1, 4	0, 5	3, 6
	Sun	−1, 6	3, 2	2, 3	1, 4	0, 5
	Mon	0, 5	−1, 6	3, 2	2, 3	1, 4
	Tue	0, 5	0, 5	−1, 6	3, 2	2, 3
	Wed	0, 5	0, 5	0, 5	−1, 6	3, 2

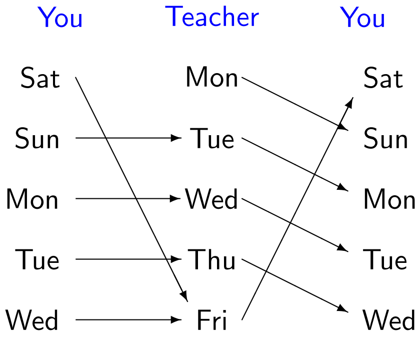




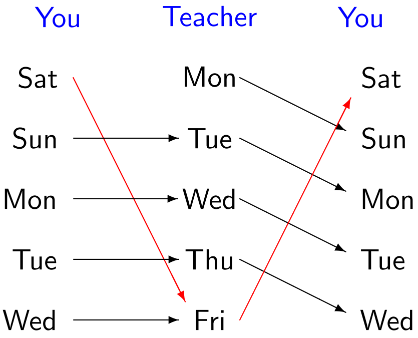
- Under common belief in rationality, you can rationally choose any day to start studying.
- Yet, some choices are supported by a simple belief hierarchy, whereas other choices are not.



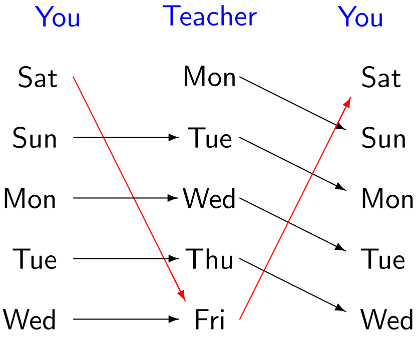
- Consider the belief hierarchy that supports your choices Saturday and Wednesday.
- This belief hierarchy is entirely generated by the belief σ_2 that the teacher puts the exam on Friday, and the belief σ_1 that you start studying on Saturday.
- We call such a belief hierarchy simple.
- In fact, $(\sigma_1, \sigma_2) =$ (Sat, Fri) is a Nash equilibrium.



- The belief hierarchies that support your choices Sunday, Monday and Tuesday are certainly not simple. Consider, for instance, the belief hierarchy that supports your choice Sunday. There,
- you believe that the teacher puts the exam on Tuesday,
- but you believe that the teacher believes that you believe that the teacher will put the exam on Wednesday.
- Hence, this belief hierarchy cannot be generated by a single belief σ_2 about the teacher's choice.



- One can show: Your choices Sunday, Monday and Tuesday cannot be supported by simple belief hierarchies that express common belief in rationality.
- Your choices Sunday, Monday and Tuesday cannot be optimal in any Nash equilibrium of the game.



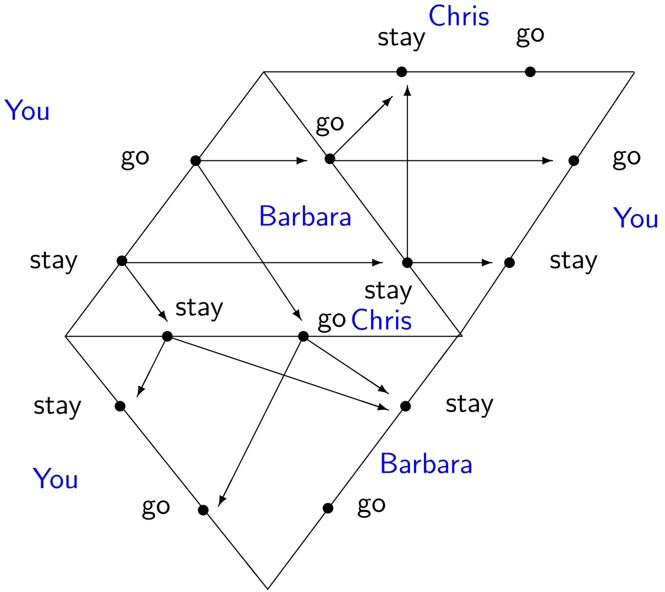
Summarizing

- Your choices Saturday and Wednesday are the only choices that are optimal for a simple belief hierarchy that expresses common belief in rationality.
- These are also the only choices that are optimal for you in any Nash equilibrium of the game.

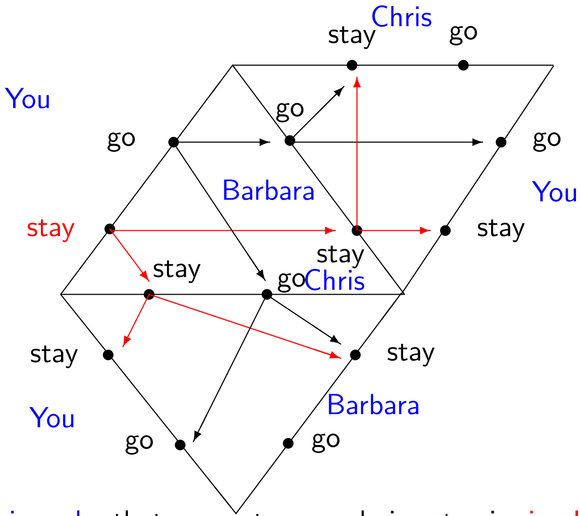
Example: Movie or party?

Story

- You have been invited to a party this evening, together with Barbara and Chris. But this evening, your favorite movie Once upon a time in America, starring Robert de Niro, will be on TV.
- Having a good time at the party gives you utility 3, watching the movie gives you utility 2, whereas having a bad time at the party gives you utility 0. Similarly for Barbara and Chris.
- You will only have a good time at the party if Barbara and Chris both join.
- Barbara and Chris had a fierce discussion yesterday. Barbara will only have a good time at the party if you join, but not Chris.
- Chris will only have a good time at the party if you join, but not Barbara.
- What should you do: Go to the party, or stay at home?



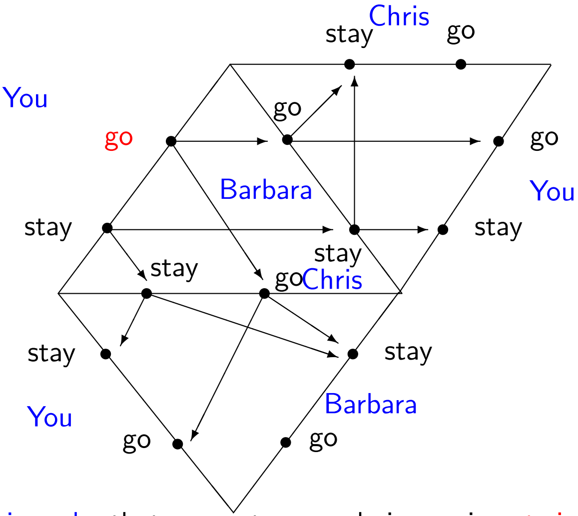
- Under common belief in rationality, you can go to the party or stay at home.



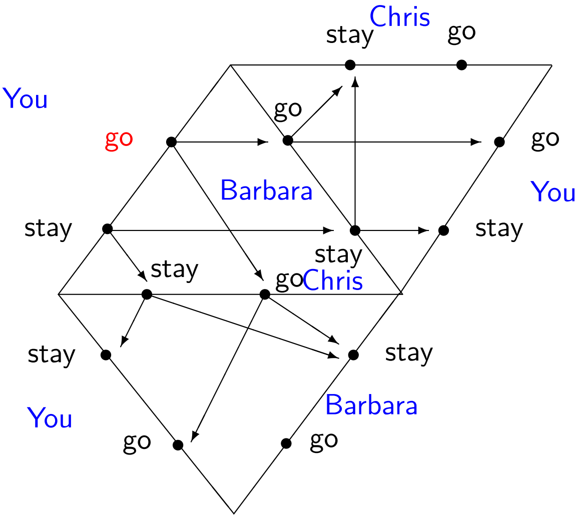
- The belief hierarchy that supports your choice stay is simple: It is completely generated by the beliefs

$$\sigma_1 = \text{ You stay, } \sigma_2 = \text{ Barbara stays, } \sigma_3 = \text{ Chris stays.}$$

- In fact, $(\sigma_1, \sigma_2, \sigma_3) = $ (stay, stay, stay) is a Nash equilibrium.



- The belief hierarchy that supports your choice go is not simple:
- You believe that Chris will go to the party.
- You believe that Barbara believes that Chris will stay at home.
- Hence, your belief hierarchy is not induced by a single belief σ_3 about Chris' choice.



- It can be shown: Your choice go cannot be supported by a simple belief hierarchy that expresses common belief in rationality.
- Your choice go is not optimal in any Nash equilibrium of the game.

- Show: Your choice go cannot be supported by a simple belief hierarchy that expresses common belief in rationality.
- Consider a simple belief hierarchy, generated by a combination of beliefs $(\sigma_1, \sigma_2, \sigma_3)$, that expresses common belief in rationality.
- We first show that $\sigma_1(go) = 0$.
- Assume that $\sigma_1(go) > 0$. Then, *go* must be optimal for you under the belief (σ_2, σ_3).
- For you, $u_1(go) = 3 \cdot \sigma_2(go) \cdot \sigma_3(go)$, whereas $u_1(stay) = 2$.
- Hence, $\sigma_2(go) \cdot \sigma_3(go) \geq 2/3$, which implies $\sigma_2(go) \geq 2/3$ and $\sigma_3(go) \geq 2/3$. This implies $\sigma_3(stay) \leq 1/3$.
- So, *go* must be optimal for Barbara under the belief (σ_1, σ_3).
- But for Barbara,

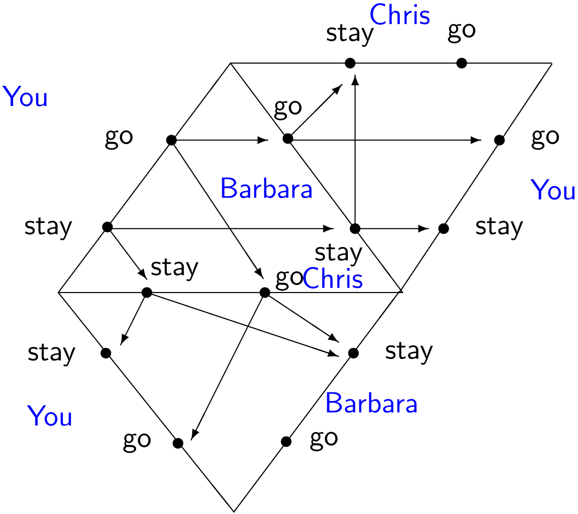
$$u_2(go) = 3 \cdot \sigma_1(go) \cdot \sigma_3(stay) \leq 1 < u_2(stay),$$

which means that *go* is not optimal for Barbara. Contradiction.

- So we conclude that $\sigma_1(stay) = 1.$
- But then, for Barbara only stay can be optimal under the belief (σ_1, σ_3). Hence, $\sigma_2 = stay.$
- Similarly, for Chris only stay can be optimal under the belief (σ_1, σ_2). Consequently, $\sigma_3 = stay.$
- So, we must have that

$$\sigma_1 = stay, \ \sigma_2 = stay, \ \sigma_3 = stay.$$

- Under the belief (σ_2, σ_3), your only optimal choice is to stay at home.
- Hence, with a simple belief hierarchy that expresses common belief in rationality, your only optimal choice is to stay at home.



- Summarizing: Your choice stay is the only choice that is optimal for a simple belief hierarchy that expresses common belief in rationality.
- Your choice stay is the only choice that is optimal in a Nash equilibrium of the game.

Simple belief hierarchies

- A belief hierarchy is called simple if it is generated by a single combination of beliefs $\sigma_1, ..., \sigma_n$.

Definition (Belief hierarchy generated by $(\sigma_1, ..., \sigma_n)$)

For every player i, let σ_i be a probabilistic belief about i's choice.

The belief hierarchy for player i that is generated by $(\sigma_1, ..., \sigma_n)$ states that

(1) player i has belief σ_j about player j's choice,

(2) player i believes that player j has belief σ_k about player k's choice,

(3) player i believes that player j believes that player k has belief σ_l about player l's choice,

and so on.

Definition (Simple belief hierarchy)

Consider an epistemic model, and a type t_i within it.

Type t_i has a simple belief hierarchy, if its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.

- A player i with a simple belief hierarchy has the following properties:
- He believes that every opponent is correct about his belief hierarchy.
- He believes that every opponent j has the same belief about player k as he has.
- His belief about j's choice is stochastically independent from his belief about k's choice.

Nash equilibrium

- Nash (1950, 1951) phrased his equilibrium notion in terms of randomized choices (or, mixed strategies) $\sigma_1, ..., \sigma_n$, where $\sigma_i \in \Delta(C_i)$ for every player i.
- Following Aumann and Brandenburger (1995), we interpret $\sigma_1, ..., \sigma_n$ as beliefs.

Definition (Nash equilibrium)

A combination of beliefs $(\sigma_1, ..., \sigma_n)$, where $\sigma_i \in \Delta(C_i)$ for every player i, is a Nash equilibrium if for every player i, the belief σ_i only assigns positive probability to choices c_i that are optimal under the belief $\sigma_{-i} \in \Delta(C_{-i})$.

- Here, $\sigma_{-i} \in \Delta(C_{-i})$ is the probability distribution given by

$$\sigma_{-i}(c_{-i}) := \prod_{j \neq i} \sigma_j(c_j)$$

for every $c_{-i} = (c_j)_{j \neq i}$ in C_{-i}.

Theorem (Characterization of Nash equilibrium)

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

Then, type t_i expresses common belief in rationality, if and only if, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

- Proof. Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.
- Then, t_i's belief hierarchy can be generated within the following epistemic model $M = (T_j, b_j)_{j \in I}$:
- For every player j let $T_j := \{t_j\}$, and

$$b_j(t_j)(c_{-j}, t_{-j}) := \prod_{k \neq j} \sigma_k(c_k) \text{ for every } c_{-j} = (c_k)_{k \neq j} \text{ in } C_{-j}.$$

- Suppose first that t_i expresses common belief in rationality.
- We show that $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

Theorem (Characterization of Nash equilibrium)

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

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- Proof. For every player j let $T_j := \{t_j\}$, and

$$b_j(t_j)(c_{-j}, t_{-j}) := \prod_{k \neq j} \sigma_k(c_k) \text{ for every } c_{-j} = (c_k)_{k \neq j} \text{ in } C_{-j}.$$

- Take some opponent $j \neq i$, and some c_j with $\sigma_j(c_j) > 0$. Then, t_i assigns positive probability to (c_j, t_j).
- As t_i believes in j's rationality, c_j must be optimal for t_j. Hence, c_j is optimal for σ_{-j}.

Theorem (Characterization of Nash equilibrium)

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- Proof. For every player j let $T_j := \{t_j\}$, and

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- Next, take some c_i with $\sigma_i(c_i) > 0$. Then, t_j assigns positive probability to (c_i, t_i).
- As t_i believes that j believes in i's rationality, c_i must be optimal for t_i. Hence, c_i is optimal for σ_{-i}.
- Hence, $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

Theorem (Characterization of Nash equilibrium)

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- Suppose next that $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.
- We show that t_i expresses common belief in rationality.
- It is sufficient to show that t_j believes in the opponents' rationality for every player j.

Theorem (Characterization of Nash equilibrium)

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

Then, type t_i expresses common belief in rationality, if and only if, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

- Proof. For every player j let $T_j := \{t_j\}$, and

$$b_j(t_j)(c_{-j}, t_{-j}) := \prod_{k \neq j} \sigma_k(c_k) \text{ for every } c_{-j} = (c_k)_{k \neq j} \text{ in } C_{-j}.$$

- Consider some type t_j, and suppose that t_j assigns positive probability to (c_k, t_k).

- Then, $\sigma_k(c_k) > 0$. Since $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium, c_k is optimal for the belief σ_{-k}.

- Hence, c_k is optimal for t_k. Therefore, t_j believes in k's rationality.

Theorem (Characterization of Nash equilibrium)

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

Then, type t_i expresses common belief in rationality, if and only if, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

- Proof. For every player j let $T_j := \{t_j\}$, and

$$b_j(t_j)(c_{-j}, t_{-j}) := \prod_{k \neq j} \sigma_k(c_k) \text{ for every } c_{-j} = (c_k)_{k \neq j} \text{ in } C_{-j}.$$

- We have shown that all types in the epistemic model believe in the opponents' rationality.
- Hence, type t_i expresses common belief in rationality. ■

Behavioral characterization of Nash equilibrium

- We have seen that a Nash equilibrium corresponds to the beliefs that generate a simple belief hierarchy expressing common belief in rationality.
- We now wish to characterize the choices that are optimal in Nash equilibrium.

Definition (Choices optimal in a Nash equilibrium)

A choice c_i is a optimal in a Nash equilibrium if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

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A choice c_i is a optimal in a Nash equilibrium if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

- Observation 1: If there is a Nash equilibrium $(\sigma_1, ..., \sigma_n)$ with $\sigma_i(c_i) > 0$, then c_i is optimal in a Nash equilibrium.
- Proof: Take a Nash equilibrium $(\sigma_1, ..., \sigma_n)$ with $\sigma_i(c_i) > 0$. Since $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium, c_i is optimal under the belief σ_{-i}.
- Hence, c_i is optimal in the Nash equilibrium $(\sigma_1, ..., \sigma_n)$. ■

Definition (Choices optimal in a Nash equilibrium)

A choice c_i is a optimal in a Nash equilibrium if there is some Nash equilibrium $(\sigma_1, ..., \sigma_n)$ where c_i is optimal for player i under the belief σ_{-i}.

- Observation 2: A choice c_i that is optimal in a Nash equilibrium need not always receive positive probability in a Nash equilibrium.
- Proof: Consider the game

	c	d
a	$2, 0$	$0, 1$
b	$1, 0$	$1, 0$

.

- Then, $(b, \frac{1}{2}c + \frac{1}{2}d)$ is a Nash equilibrium.
- Since a is optimal under the belief $\frac{1}{2}c + \frac{1}{2}d$, choice a is optimal in the Nash equilibrium $(b, \frac{1}{2}c + \frac{1}{2}d)$.
- However, there is no Nash equilibrium (σ_1, σ_2) with $\sigma_1(a) > 0$.
- Indeed, if $\sigma_1(a) > 0$, then only d is optimal for player 2, and hence $\sigma_2 = d$.
- But then, only b can be optimal for player 1, hence $\sigma_1 = b$. This is a contradiction. ■

Theorem (Behavioral characterization of Nash equilibrium)

A choice c_i is optimal in a Nash equilibrium, if and only if, c_i is optimal for a simple belief hierarchy that expresses common belief in rationality.

- Proof:
- Let c_i be optimal in a Nash equilibrium $(\sigma_1, ..., \sigma_n)$. Let t_i be a type whose simple belief hierarchy is generated by $(\sigma_1, ..., \sigma_n)$.
- Then, we know from the previous theorem that t_i expresses common belief in rationality.
- As c_i is optimal for σ_{-i}, it follows that c_i is optimal for t_i.
- Hence, c_i is optimal for a simple belief hierarchy that expresses common belief in rationality.

Theorem (Behavioral characterization of Nash equilibrium)

A choice c_i is optimal in a Nash equilibrium, if and only if, c_i is optimal for a simple belief hierarchy that expresses common belief in rationality.

- Proof:
- Let c_i be optimal for a type t_i that has a simple belief hierarchy generated by $(\sigma_1, ..., \sigma_n)$, and that expresses common belief in rationality.
- Then, we know from the previous theorem that $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.
- Since c_i is optimal for t_i, the choice c_i is optimal for σ_{-i}.
- Hence, c_i is optimal in the Nash equilibrium $(\sigma_1, ..., \sigma_n)$. ■

Characterization of simple belief hierarchies

- We have seen that Nash equilibrium can be characterized by common belief in rationality with a simple belief hierarchy.
- Which epistemic conditions characterize a simple belief hierarchy?
- We focus on the case of two players.

Characterization of simple belief hierarchies

- If a type t_i has a simple belief hierarchy induced by (σ_1, σ_2), then t_i believes that
- opponent j is correct about his belief hierarchy,
- opponent j believes that i is correct about j's belief hierarchy.
- Following Perea (2007), we show that these two conditions characterize simple belief hierarchies for the case of two players.

Definition (Correct beliefs)

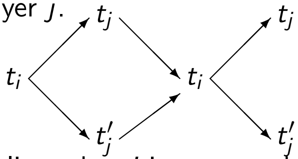
Type t_i believes that j is correct about his beliefs if t_i only assigns positive probability to types t_j that assign probability 1 to his actual type t_i.

Theorem (Characterization of types with a simple belief hierarchy in two-player games)

Consider a game with two players.

A type t_i for player i has a simple belief hierarchy, if and only if, t_i believes that j is correct about his beliefs, and believes that j believes that i is correct about j's beliefs.

- Proof. Suppose that type t_i believes that j is correct about his beliefs, and believes that j believes that i is correct about j's beliefs.
- Show: Type t_i assigns probability 1 to a single type t_j for player j.
- Suppose that t_i would assign positive probability to two different types t_j and t_j' for player j.



- Then, t_j would not believe that i is correct about j's beliefs. Contradiction.

Theorem (Characterization of types with a simple belief hierarchy in two-player games)

Consider a game with two players.

A type t_i for player i has a simple belief hierarchy, if and only if, t_i believes that j is correct about his beliefs, and believes that j believes that i is correct about j's beliefs.

- So, we know that t_i assigns probability 1 to some type t_j for player j, and t_j assigns probability 1 to t_i.
- Let σ_j be the belief that t_i has about j's choice, and let σ_i be the belief that t_j has about i's choice.

$$t_i \xrightarrow[\sigma_j]{} t_j \xrightarrow[\sigma_i]{} t_i$$

- But then, t_i's belief hierarchy is generated by (σ_i, σ_j). So, t_i has a simple belief hierarchy. ■

- Be careful: If we have more than two players, then these conditions are no longer enough to induce simple belief hierarchies.
- In a game with more than two players, we need to impose the following extra conditions:
- type t_i believes that player j has the same belief about player k as t_i has;
- type t_i's belief about player j's choice must be stochastically independent from his belief about player k's choice.

Theorem (Behavioral characterization of Nash equilibrium for two players)

Consider a game with two players.

Then, a choice c_i is optimal in a Nash equilibrium, if and only if, it is optimal for a type t_i that

(a) expresses common belief in rationality,
(b) believes that j is correct about his beliefs, and
(c) believes that j believes that i is correct about j's beliefs.

- Based on Perea (2007).
- Condition (a) can be weakened to:

  (a1) type t_i believes in j's rationality,
  (a2) type t_i believes that j believes in i's rationality.

- Similar results can be found in Tan and Werlang (1988), Brandenburger and Dekel (1987 / 1989), Aumann and Brandenburger (1995), Polak (1999) and Asheim (2006).

How reasonable is Nash equilibrium?

- We have seen that a Nash equilibrium makes the following assumptions:
- you believe that your opponents are correct about the beliefs that you hold;
- you believe that player j holds the same belief about player k as you do;
- your belief about player j's choice is stochastically independent from your belief about player k's choice.
- Each of these conditions is actually very questionable.
- Therefore, Nash equilibrium is not such a natural concept after all.

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