Incomplete Information and Equilibrium

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Abstract. In games with incomplete information Bayesian equilibrium constitutes the prevailing solution concept. We show that Bayesian equilibrium generalizes correlated equilibrium from complete to incomplete information. In particular, we provide an epistemic characterization of Bayesian equilibrium as well as of correlated equilibrium in terms of common belief in rationality and a common prior. Bayesian equilibrium is thus not the incomplete information counterpart of Nash equilibrium. To fill the resulting gap, we introduce the solution concept of generalized Nash equilibrium as the incomplete information analogue to Nash equilibrium, and show that it is more restrictive than Bayesian equilibrium. Besides, we propose a simplified tool to compute Bayesian equilibria.

Keywords: Bayesian equilibrium, common belief in rationality, common prior, correlated equilibrium, epistemic game theory, equilibrium, generalized Nash equilibrium, incomplete information, interactive epistemology, Nash equilibrium, simplified Bayesian equilibrium, solution concepts, static games.

1 Introduction

In game theory solution concepts are proposed for different classes of games to reduce the set of possible outcomes according to some decision-making criterion. The most basic class of games is static and exhibits complete information.

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A game is static if every player only chooses once in ignorance of the choices made by the opponents. According to the assumption of complete information, the payoffs of each player are transparent to his opponents. Indeed, the payoff structure of the game is supposed to be commonly known among the players.

For the class of static games with complete information, some of the most prevalent solution concepts are Nash's (1950) and (1951) equilibrium, Aumann's (1974) and (1987) correlated equilibrium, as well as Bernheim's (1984) and Pearce's (1984) rationalizability. Intuitively, Nash equilibrium requires the players' strategies to be mutually best responses. In terms of reasoning, the crucial ingredient is some correct-beliefs assumption, as the epistemic analysis of Nash equilibrium has shown (Aumann and Brandenburger, 1995; Polak, 1999; Perea, 2007; Barelli, 2009; Bach and Tsakas, 2014; Bonanno, 2017). Correlated equilibrium embeds a best response property in an information structure modelling strategic uncertainty. Aumann (1987) provides an epistemic foundation for correlated equilibrium in terms of universal rationality and a common prior. Also, Brandenburger and Dekel (1987) characterize a variant of correlated equilibrium without a common prior – a posteriori equilibrium – by common knowledge of rationality. Rationalizability iteratively deletes choices that fail to satisfy a best response requirement. From an epistemic perspective, rationalizability corresponds to common belief in rationality (Brandenburger and Dekel, 1987; Tan and Werlang, 1988). Note that for the general case of admitting correlated beliefs, rationalizability coincides with iterated strict dominance, which iteratively deletes choices that are strictly dominated. In terms of set inclusion the three solution concepts can be ordered: Nash equilibrium implies correlated equilibrium, which implies rationalizability. However, the converse does not hold.

In the more general class of games with incomplete information players face uncertainty about the opponents' utility functions. Harsanyi (1967-68) pioneered the analysis of this class of games and proposed the solution concept of Bayesian equilibrium. Intuitively, Bayesian equilibrium embeds a best-response property in a type structure that determines the belief hierarchies on the players' utility functions derived from a common prior. While Bayesian equilibrium constitutes the most prevalent solution concept for incomplete information games, more recently, new solution concepts have been developed based on the non-equilibrium complete information solution concepts of rationalizability due to Bernheim (1984) and Pearce (1984). Notably, weak and strong Δ-rationalizability have been introduced by Battigalli (2003) and further developed by Battigalli and Siniscalchi (2003) and (2007), Battigalli et al. (2011), Battigalli and Prestipino (2013), as well as Dekel and Siniscalchi (2015). Intuitively, Δ-rationalizability concepts iteratively delete strategy utility pairs by some best response requirement, and allow for exogenous restrictions on the first-order beliefs. Different incomplete information generalizations of rationalizability are Ely and Pęski (2006)'s interim rationalizability as well as Dekel et al. (2007)'s interim correlated rationalizability, respectively. The essential difference to weak and strong Δ-rationalizability lies in fixing the belief hierarchies on utilities. Besides, generalized iterated strict dominance by Bach and Perea (2016) iteratively reduces decision problems by some strict dominance requirement. In terms of reasoning, all of the incomplete information rationalizability and strict dominance solution concepts essentially correspond to common belief in rationality.

Here, we give an epistemic characterization of Bayesian equilibrium in terms of common belief in rationality and a common prior. For the specific case of complete information, it is shown that the same conditions epistemically characterize correlated equilibrium. Thus, Bayesian equilibrium represents the incomplete information analogue to correlated equilibrium. In particular, Bayesian equilibrium does not generalize Nash equilibrium to incomplete information. In fact, Battigalli and Siniscalchi (2003) already point at this analogy between the solution concepts of Bayesian equilibrium and correlated equilibrium, which we prove formally from an epistemic perspective here. To fill the ensuing gap, we then introduce a new solution concept – generalized Nash equilibrium – as a direct generalization of Nash equilibrium from complete to incomplete information. Besides, an epistemic characterization is given of the former in terms of common belief in rationality and a simple belief hierarchy. Furthermore, the solution concepts of generalized iterated strict dominance, Bayesian equilibrium, and generalized Nash equilibrium can also be ordered in terms of set inclusion: generalized Nash equilibrium implies Bayesian equilibrium, which implies generalized iterated strict dominance. However, the converse does not hold. In addition, we introduce the notion of simplified Bayesian equilibrium, which we use to characterize the probability measures on choice utility combinations that arise from Bayesian equilibrium. Simplified Bayesian equilibrium also serves as a simple tool to identify Bayesian equilibria in specific games.

The table in Figure 1 classifies game-theoretic solution concepts in relation to their epistemic conditions and the underlying informational assumptions, where CBR denotes common belief in rationality and CPA denotes common prior assumption. The classification builds on contributions from the literature as well as on some of the results provided here. An enhanced understanding of the relationship between solution concepts thus emerges.

First of all, the epistemic conditions of common belief in rationality characterize rationalizability and – if correlated beliefs are admitted – iterated strict dominance (Brandenburger and Dekel, 1987; Tan and Werlang, 1988). In the more general case of incomplete information common belief in rationality epistemically characterizes the different Δ-rationalizability variants (Battigalli and Siniscalchi, 1999, 2002, and 2007; Battigalli et al., 2011; Battigalli and Prestipino, 2013) and generalized iterated strict dominance (Bach and Perea, 2016). Δ-rationalizability and generalized iterated strict dominance are thus the incomplete information analogues to the complete information solution concepts of rationalizability and iterated strict dominance.

Secondly, the epistemic conditions of common belief in rationality and a common prior characterize correlated equilibrium in the case of complete information (Theorem 1) and Bayesian equilibrium in the case of incomplete information (Theorem 2). It follows that Bayesian equilibrium constitutes the incomplete information analogue to correlated equilibrium (Corollary 1).

Epistemic Conditions	**Complete Information**	**Incomplete Information**
CBR	Rationalizability	Δ-Rationalizability
	&	&
	Iterated Strict Dominance	Generalized Iterated Strict Dominance
	$\not\Downarrow$ / $\Uparrow$	$\not\Downarrow$ / $\Uparrow$
CBR & CPA	Correlated Equilibrium	Bayesian Equilibrium
	$\not\Downarrow$ / $\Uparrow$	$\not\Downarrow$ / $\Uparrow$
CBR & Simple Belief Hierarchy	Nash Equilibrium	Generalized Nash Equilibrium

Fig. 1. Solution concepts for static games in relation to informational assumptions and epistemic conditions. The relationship between the concepts in terms of behavioural implication for complete and incomplete information, respectively, is also indicated.

Thirdly, common belief in rationality and a simple belief hierarchy characterize Nash equilibrium in the case of complete information (Perea, 2012) and generalized Nash equilibrium in the case of incomplete information (Theorem 5). Consequently, generalized Nash equilibrium represents the incomplete information counterpart to Nash equilibrium.

For the three different epistemic conditions, the behavioural relationship between the solution concepts is also indicated. Indeed, Theorem 6 and Theorem 4 establish that optimal choice in a generalized Nash equilibrium implies optimal choice in a Bayesian equilibrium, which in turn implies optimal choice in Δ-rationalizability and generalized iterated strict dominance, respectively. However, the converse does not hold (Remark 1 and Remark 3). In the specific case of complete information these relationships apply between the analogous solution concepts of Nash equilibrium, correlated equilibrium, and rationalizability as well as iterated strict dominance. Note that the relationships between the solution concepts provided here are stated behaviourally. With a beliefs interpretation of solution concepts in the sense that mixed choices are viewed as beliefs, a behavioural comparison of solution concepts seems natural.

We proceed as follows. In Section 2, the epistemic framework for games with incomplete information is set out. Section 3 restricts attention to complete information and provides an epistemic characterization of correlated equilibrium in terms of common belief in rationality and a common prior. In Section 4, Harsanyi's seminal solution concept of Bayesian equilibrium is considered and also epistemically characterized by common belief in rationality and a common prior. A corollary establishing the equivalence of Bayesian equilibrium and correlated equilibrium directly follows. Besides, the notion of simplified Bayesian equilibrium is introduced and used to characterize the probability measures on choice utility combinations that arise from Bayesian equilibria. This tool simplifies the computation of Bayesian equilibria in specific games. Furthermore, it is shown that Bayesian equilibrium is more restrictive than generalized iterated strict dominance. In Section 5, the solution concept of generalized Nash equilibrium is introduced as the incomplete information analogue to Nash equilibrium, and an epistemic characterization in terms of common belief in rationality and a simple belief hierarchy is provided. It is also shown that generalized Nash equilibrium is behaviourally more restrictive than Bayesian equilibrium. Finally, Section 6 offers some concluding remarks.

2 Preliminaries

A game with incomplete information is modelled as a tuple $\Gamma = \big(I, (C_i)_{i\in I}, (U_i)_{i\in I}\big)$, where I is a finite set of players, C_i denotes player i's finite choice set, and the finite set U_i contains player i's utility functions. A utility function $u_i : \times_{j\in I} C_j \rightarrow \mathbb{R}$ from U_i assigns a real number $u_i(c)$ to every choice combination $c \in \times_{j\in I} C_j$. From the perspective of a single player there exist two basic sources of uncertainty with respect to Γ. A player faces strategic uncertainty, i.e. what choices his opponents make, as well as payoff uncertainty, i.e. what utility functions represent the opponents' preferences.

Reasoning in games is usually modelled by belief hierarchies about the underlying space of uncertainty. Due to Harsanyi (1967-68) types can be used as implicit representations of belief hierarchies. The notion of an epistemic model provides the framework to formally describe reasoning in games.

Definition 1. *Let Γ be a game with incomplete information. An* epistemic model *of Γ is a tuple $\mathcal{M}^{\Gamma} = \big((T_i)_{i\in I}, (b_i)_{i\in I}\big)$, where for every player $i \in I$*

- *T_i is a finite set of types,*
- *$b_i : T_i \rightarrow \Delta(C_{-i} \times T_{-i} \times U_{-i})$ assigns to every type $t_i \in T_i$ a probability measure $b_i[t_i]$ on the set of opponents' choice type utility function combinations.*

Given a game and an epistemic model of it, belief hierarchies, marginal beliefs, as well as marginal belief hierarchies can be derived from every type. For instance, every type $t_i \in T_i$ induces a belief on the opponents' choice combinations by marginalizing the probability measure $b_i[t_i]$ on the space C_{-i}. Note that no additional notation is introduced for marginal beliefs, in order to keep notation as sparse as possible. It should always be clear from the context which belief $b_i[t_i]$ refers to.

Here, payoff uncertainty is treated symmetrically to strategic uncertainty. As the latter only concerns the respective *opponents'* choices, the former is also defined with regard to the respective *opponents'* utility functions only. Our approach thus also follows Harsanyi's (1967-68) model, in which a player's payoff uncertainty only concerns his opponents. However, the case of players being uncertain about their *own* payoffs could be accommodated in Definition 1 by extending the space of uncertainty for every player $i \in I$ from $C_{-i} \times T_{-i} \times U_{-i}$ to $C_{-i} \times T_{-i} \times (\times_{j\in I} U_j)$. Alternatively, a reasoner's actual utility function could be defined as the expectation over the set U_i. This modelling choice does not affect any of the subsequent results.

Since the epistemic model according to Definition 1 treats the sources of uncertainty – choices and utilities – symmetrically, our approach differs from Ely and Pęski (2006) as well as Dekel et al. (2007). Indeed, the latter models formalize incomplete information by fixing the belief hierarchies on the utilities before reasoning about choice is considered. Besides, we follow a one-player perspective approach, which considers game theory as an interactive extension of decision theory. Accordingly, all epistemic concepts – including iterated ones – are defined as mental states inside the mind of a single person. A one-player approach seems natural in the sense that reasoning is formally represented by epistemic concepts and any reasoning process prior to choice does indeed take place entirely *within* the reasoner's mind. Formally, this approach is parsimonious in the sense that states, describing the beliefs of all players, do not have to be introduced.

Some further notions and notation are now introduced. For that purpose consider a game Γ, an epistemic model $\mathcal{M}^{\Gamma}$ of it, and fix two players $i, j \in I$ such that $i \neq j$.

A type $t_i \in T_i$ of i is said to *deem possible* some choice type utility function combination (c_{-i}, t_{-i}, u_{-i}) of his opponents, if $b_i[t_i]$ assigns positive probability to (c_{-i}, t_{-i}, u_{-i}). Analogously, a type $t_i \in T_i$ deems possible some opponent type $t_j \in T_j$, if $b_i[t_i]$ assigns positive probability to t_j.

For each choice type utility function combination (c_i, t_i, u_i), the *expected utility* is given by

$$v_i(c_i, t_i, u_i) = \sum_{c_{-i} \in C_{-i}} \big(b_i[t_i](c_{-i}) \cdot u_i(c_i, c_{-i})\big).$$

Intuitively, the common prior assumption in economics states that every belief in models with multiple agents is derived from a single probability distribution, the so-called common prior. In our framework all beliefs are furnished by epistemic models. The common prior assumption thus imposes a condition on epistemic models, which requires all beliefs to be derived from a single probability distribution on the basic space of uncertainty and the players' types.

Definition 2. *Let Γ be a game with incomplete information, and $\mathcal{M}^{\Gamma}$ an epistemic model of it. The epistemic model $\mathcal{M}^{\Gamma}$ satisfies the* common prior assumption*, if there exists a probability measure $\varphi \in \Delta\big(\times_{j \in I} (C_j \times T_j \times U_j)\big)$ such that for every player $i \in I$, and for every type $t_i \in T_i$ it is the case that $\varphi(t_i) > 0$ and*

$$b_i[t_i](c_{-i}, t_{-i}, u_{-i}) = \frac{\varphi(c_i, c_{-i}, t_i, t_{-i}, u_i, u_{-i})}{\varphi(c_i, t_i, u_i)}$$

for all $(c_i, u_i) \in C_i \times U_i$ with $\varphi(c_i, t_i, u_i) > 0$, and for all $(c_{-i}, t_{-i}, u_{-i}) \in C_{-i} \times T_{-i} \times U_{-i}$.

Accordingly, every type's induced belief function obtains from a single probability measure – the common prior – via Bayesian updating. Note that the common prior is defined on the full space of uncertainty, i.e. on the set of all the players' choice type utility function combinations, while belief functions are defined on the space of respective opponents' choice type utility function combinations. The common prior assumption could be interpreted by means of an interim stage set-up, in which every player $i \in I$ observes the triple (c_i, t_i, u_i) on which he then conditionalizes.

Intuitively, an optimal choice yields at least as much payoff as all other options, given what the player believes his opponents to choose as well as given his utility function. Formally, optimality is a property of choices given a type utility function pair.

Definition 3. *Let Γ be a game with incomplete information, $\mathcal{M}^{\Gamma}$ an epistemic model of it, $i \in I$ some player, $u_i \in U_i$ some utility function for player i, and $t_i \in T_i$ some type of player i. A choice $c_i \in C_i$ is* optimal *for the type utility function pair (t_i, u_i), if*

$$v_i(c_i, t_i, u_i) \geq v_i(c_i', t_i, u_i)$$

for all $c_i' \in C_i$.

A player believes in rationality, if he only deems possible choice type utility function triples – for each of his opponents – such that the choice is optimal for the type utility function pair, respectively.

Definition 4. *Let Γ be a game with incomplete information, $\mathcal{M}^{\Gamma}$ an epistemic model of it, and $i \in I$ some player. A type $t_i \in T_i$* believes in rationality, *if t_i only deems possible choice type utility function combinations $(c_{-i}, t_{-i}, u_{-i}) \in C_{-i} \times T_{-i} \times U_{-i}$ such that c_j is optimal for (t_j, u_j) for every opponent $j \in I \setminus \{i\}$.*

Note that essentially belief in rationality imposes restrictions on the first two layers of a player's belief hierarchy, since the player's belief about his opponents' choices and utility functions as well as the player's belief about his opponents' beliefs about their respective opponents' choices are affected.

The conditions on interactive reasoning can be taken to further layers in belief hierarchies.

Definition 5. *Let Γ be a game with incomplete information, $\mathcal{M}^{\Gamma}$ an epistemic model of it, and $i \in I$ some player.*

- *A type $t_i \in T_i$ expresses* 1-fold belief in rationality, *if t_i believes in rationality.*
- *A type $t_i \in T_i$ expresses* k-fold belief in rationality *for some $k > 1$, if t_i only deems possible types $t_j \in T_j$ for all $j \in I \setminus \{i\}$ such that t_j expresses $k-1$-fold belief in rationality.*
- *A type $t_i \in T_i$ expresses* common belief in rationality, *if t_i expresses k-fold belief in rationality for all $k \geq 1$.*

A player satisfying common belief in rationality entertains a belief hierarchy in which the rationality of all players is not questioned at any level. Observe that if an epistemic model for every player only contains types that believe in rationality, then every type also expresses common belief in rationality. This fact is useful when constructing epistemic models with types expressing common belief in rationality.

A type t_j is called *belief reachable* from a type t_i, if there exists a finite sequence $(t^1, \ldots, t^N)$ of types with $N \in \mathbb{N}$, where $t^{n+1} \in \text{supp}(b_k[t^n])$ with $k \in I$ such that $t^n \in T_k$, as well as $t^1 = t_i$ and $t^N = t_j$ for some players $i, j \in I$. Intuitively, if a type t_j is belief reachable from a type t_i, the former is not excluded in the interactive reasoning by the latter.

The following lemma ensures that belief reachability preserves common belief in rationality.

Lemma 1. *Let Γ be a game with incomplete information, $\mathcal{M}^\Gamma$ an epistemic model of it, $i, j \in I$ some players, $t_i \in T_i$ a type of player i, and $t_j \in T_j$ a type of player j. If t_i expresses common belief in rationality and t_j is belief reachable from t_i, then t_j expresses common belief in rationality.*

Proof. Assume that t_j is belief reachable from t_i in $N > 1$ steps, i.e. there exists a finite sequence $(t^1, \ldots, t^N)$ of types with $t^{n+1} \in \text{supp}(b_k[t^n])$ as well as $t^1 = t_i$ and $t^N = t_j$. Towards a contradiction suppose that t_j does not express common belief in rationality. Then, there exists $k > 0$ such that t_j does not express k-fold belief in rationality. However, as t_i deems possible t_j at the N-level of its induced belief hierarchy, t_i thus violates $(N + k)$-fold belief in rationality and a fortiori common belief in rationality, a contradiction. ■

The choice rule of rationality and the reasoning concept of common belief in rationality together with a utility function define rational choice under common belief in rationality.

Definition 6. *Let Γ be a game with incomplete information, $i \in I$ some player, and $u_i \in U_i$ some utility function of player i. A choice $c_i \in C_i$ of player i is* rational for utility function u_i under common belief in rationality*, if there exists an epistemic model $\mathcal{M}^\Gamma$ of Γ with a type $t_i \in T_i$ of player i such that c_i is optimal for (t_i, u_i) and t_i expresses common belief in rationality.*

Note that rational choice under common belief in rationality imposes conditions on choice utility function pairs. Besides, a choice c_i is called *rational for utility function u_i under common belief in rationality with a common prior*, if there exists an epistemic model $\mathcal{M}^\Gamma$ of Γ satisfying the common prior assumption and containing a type $t_i \in T_i$ of player i such that c_i is optimal for (t_i, u_i) and t_i expresses common belief in rationality.

The formal framework for complete information games obtains as a special case. Since payoff uncertainty vanishes, the sets U_i become singletons for every player $i \in I$. Consequently, a game $\Gamma = \big(I, (C_i)_{i \in I}, (U_i)_{i \in I}\big)$ is with complete information, whenever $\mid U_i \mid = 1$ for all $i \in I$. Note that beliefs about payoffs then become redundant and can be removed from the formal framework. Consequently, the standard complete information case ensues with a game being represented by the tuple $\Gamma = \big(I, (C_i)_{i \in I}, (u_i)_{i \in I}\big)$, where $u_i : \times_{j \in I} C_j \to \mathbb{R}$ is player i's utility function for every player $i \in I$, and an epistemic model of it by the tuple $\mathcal{M}^\Gamma = \big((T_i)_{i \in I}, (b_i)_{i \in I}\big)$, where for every player $i \in I$ the belief function $b_i : T_i \to \Delta(C_{-i} \times T_{-i})$ assigns to every type $t_i \in T_i$ a probability measure $b_i[t_i]$ on the opponents' choice type combinations. Accordingly, the formal definition of the concepts – common prior assumption, optimality, (common) belief in rationality, rational choice for a given utility function under common belief in rationality – can readily be applied to complete information.

3 Epistemic Characterization of Correlated Equilibrium

For static games with complete information the solution concept of correlated equilibrium has been introduced by Aumann (1974) and given an epistemic foundation in terms of universal rationality and a common prior by Aumann (1987). Loosely speaking, in a correlated equilibrium each player's choice is required to satisfy a best response property given a probability measure on his opponents' choice combinations derived from a common prior via Bayesian updating.

The notion of correlated equilibrium is embedded in the epistemic framework of Aumann models, which describe the players' knowledge and beliefs in terms of information partitions.

Definition 7. *Let Γ be a game with complete information. An* Aumann model *of Γ is a tuple $\mathcal{A}^\Gamma = \big(\Omega, \pi, (\mathcal{I}_i)_{i \in I}, (\hat{\sigma}_i)_{i \in I}\big)$, where*

- *Ω is a finite set of all possible worlds,*
- *$\pi \in \Delta(\Omega)$ is a common prior probability measure on the set of all possible worlds,*
- *$\mathcal{I}_i$ is an information partition on Ω for every player $i \in I$ such that $\pi\big(\mathcal{I}_i(\omega)\big) > 0$ for all $\omega \in \Omega$, where $\mathcal{I}_i(\omega)$ denotes the cell of $\mathcal{I}_i$ containing ω,*
- *$\hat{\sigma}_i : \Omega \to C_i$ is an $\mathcal{I}_i$-measurable choice function for every player $i \in I$.*

Note that the players' beliefs in Aumann models are obtained via Bayesian conditionalization on the common prior given a player's information. More precisely, an event $E \subseteq \Omega$ consists of possible worlds, and player i's belief in E at a world ω is defined as $b_i(E, \omega) := \pi\big(E \mid \mathcal{I}_i(\omega)\big) = \frac{\pi\left(E \cap \mathcal{I}_i(\omega)\right)}{\pi\left(\mathcal{I}_i(\omega)\right)}$. Besides, the $\mathcal{I}_i$-measurability of $\hat{\sigma}_i$, i.e. $\hat{\sigma}_i(\omega) = \hat{\sigma}_i(\omega')$ for all $\omega' \in \mathcal{I}_i(\omega)$ and for all $\omega \in \Omega$, ensures that i knows his own choice. In Aumann models the expected utility of a choice c_i of player i at a world ω is obtained as $\hat{v}_i(c_i, \hat{\sigma}_{-i}, \omega) := \sum_{\omega' \in \mathcal{I}_i(\omega)} \frac{\pi(\omega')}{\pi\left(\mathcal{I}_i(\omega)\right)} \cdot u_i\big(c_i, \hat{\sigma}_{-i}(\omega')\big)$.

Within the framework of Aumann models the notion of correlated equilibrium can now be formally defined as follows.

Definition 8. *Let Γ be a game with complete information, and $\mathcal{A}^\Gamma$ an Aumann model of it with choice functions $\hat{\sigma}_i : \Omega \to C_i$ for every player $i \in I$. The tuple*

$(\hat{\sigma}_i)_{i \in I}$ *of choice functions constitutes a* correlated equilibrium*, if for every world* $\omega \in \Omega$*, and for every player* $i \in I$*, it is the case that*

$$\hat{v}_i\big(\hat{\sigma}_i(\omega), \hat{\sigma}_{-i}, \omega\big) \geq \hat{v}_i(c_i, \hat{\sigma}_{-i}, \omega)$$

for every choice $c_i \in C_i$*.*

Intuitively, a choice function tuple constitutes a correlated equilibrium, if at every world that receives positive probability by the common prior, for every player the choice function specifies a best response given the common prior conditionalized on the player's information and given the opponents' choice functions.

From a behavioural viewpoint it is ultimately of interest what choices a player can make given a particular line of reasoning and decision-making fixed by specific epistemic assumptions or by a specific solution concept. The notion of optimal choice in a correlated equilibrium is thus defined next.

Definition 9. *Let* Γ *be a game with complete information, and* $i \in I$ *some player. A choice* $c_i \in C_i$ *of player* i *is* optimal in a correlated equilibrium*, if there exists an Aumann model* $\mathcal{A}^{\Gamma}$ *of* Γ *such that the tuple* $(\hat{\sigma}_i)_{i \in I}$ *constitutes a correlated equilibrium and with some world* $\omega \in \Omega$ *such that*

$$\hat{v}_i\big(c_i, \hat{\sigma}_{-i}, \omega\big) \geq \hat{v}_i(c_i', \hat{\sigma}_{-i}, \omega)$$

for all $c_i' \in C_i$*.*

An epistemic characterization of correlated equilibrium in terms of common belief in rationality and a common prior is now given.

Theorem 1. *Let* Γ *be a game with complete information,* $i \in I$ *some player, and* $c_i^* \in C_i$ *some choice of player* i*. The choice* c_i^* *is optimal in a correlated equilibrium, if and only if, the choice* c_i^* *is rational under common belief in rationality with a common prior.*

Proof. For the *only if* direction of the theorem, suppose that c_i^* is optimal in a correlated equilibrium. Let $\mathcal{A}^{\Gamma}$ be an Aumann model and $(\hat{\sigma}_j)_{j \in I}$ a correlated equilibrium, in which c_i^* is optimal. For every player $j \in I$ and for every cell $P_j \in \mathcal{I}_j$ of player j, let $\hat{\sigma}_j(P_j) := \hat{\sigma}_j(\omega')$ for some $\omega' \in P_j$ denote j's choice across all worlds in the cell, which is well-defined by $\mathcal{I}_j$-measurability of the choice function $\hat{\sigma}_j$. Construct an epistemic model $\mathcal{M}^{\Gamma}$ of Γ, with type sets $T_j := \{t_j^{P_j} : P_j \in \mathcal{I}_j\}$ for every player $j \in I$, a probability measure $\varphi \in \Delta\big(\times_{j \in I} (C_j \times T_j)\big)$ such that

$$\varphi\big((c_j, t_j^{P_j})_{j \in I}\big) := \begin{cases} \pi(\cap_{j \in I} P_j), & \text{if } c_j = \hat{\sigma}_j(P_j) \text{ for all } j \in I, \\ 0, & \text{otherwise,} \end{cases}$$

and for every player $j \in I$ induced belief functions $b_j : T_j \to \Delta(C_{-j} \times T_{-j})$ such that for every type $t_j^{P_j} \in T_j$

$$b_j[t_j^{P_j}](c_{-j}, t_{-j}^{P_{-j}}) := \begin{cases} \frac{\pi(\cap_{k \in I} P_k)}{\pi(P_j)}, & \text{if } c_k = \hat{\sigma}_k(P_k) \text{ for all } k \neq j, \\ 0, & \text{otherwise,} \end{cases}$$

for all $(c_{-j}, t_{-j}^{P_{-j}}) \in C_{-j} \times T_{-j}$.

Next it is shown that $\mathcal{M}^{\Gamma}$ satisfies the common prior assumption, by establishing that for all $j \in I$ and $t_j^{P_j} \in T_j$, it is the case that

$$b_j[t_j^{P_j}](c_{-j}, t_{-j}^{P_{-j}}) = \frac{\varphi\big(c_j, t_j^{P_j}, c_{-j}, t_{-j}^{P_{-j}}\big)}{\varphi\big(c_j, t_j^{P_j}\big)}$$

for all $c_j \in C_j$ with $\varphi(c_j, t_j^{P_j}) > 0$, and for all $(c_{-j}, t_{-j}^{P_{-j}}) \in C_{-j} \times T_{-j}$. Note that $\varphi(c_j, t_j^{P_j}) > 0$ only holds if $c_j = \hat{\sigma}_j(P_j)$. It thus has to be established that

$$b_j[t_j^{P_j}](c_{-j}, t_{-j}^{P_j}) = \frac{\varphi\Big(\big(\hat{\sigma}_j(P_j), t_j^{P_j}\big), (c_{-j}, t_{-j}^{P_j})\Big)}{\varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}\big)}$$

for all $(c_{-j}, t_{-j}^{P_{-j}}) \in C_{-j} \times T_{-j}$ and for all $t_j^{P_j} \in T_j$. Consider some $P_j \in \mathcal{I}_j$ and distinguish two cases (I) and (II).

Case (I). Suppose that $P_j \cap (\cap_{k \in I \setminus \{j\}} P_k) \neq \emptyset$ and $c_k = \hat{\sigma}_k(P_k)$ for all $k \in I \setminus \{j\}$. Observe that

$$b_j[t_j^{P_j}](c_{-j}, t_{-j}^{P_{-j}}) = b_j[t_j^{P_j}](\hat{\sigma}_{-j}(P_{-j}), t_{-j}^{P_{-j}}) = \frac{\pi(\cap_{l \in I} P_l)}{\pi(P_j)}$$

$$= \frac{\varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}, \hat{\sigma}_{-j}(P_{-j}), t_{-j}^{P_{-j}}\big)}{\sum_{\hat{P}_{-j} \in \mathcal{I}_{-j}} \pi\big(P_j \cap (\cap_{k \in I \setminus \{j\}} \hat{P}_k)\big)}$$

$$= \frac{\varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}, \hat{\sigma}_{-j}(P_{-j}), t_{-j}^{P_{-j}}\big)}{\sum_{\hat{P}_{-j} \in \mathcal{I}_{-j}} \varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}, \hat{\sigma}_{-j}(\hat{P}_{-j}), t_{-j}^{\hat{P}_{-j}}\big)}$$

$$= \frac{\varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}, \hat{\sigma}_{-j}(P_{-j}), t_{-j}^{P_{-j}}\big)}{\sum_{(c_{-j}, t_{-j}) \in C_{-j} \times T_{-j}} \varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}, c_{-j}, t_{-j}\big)}$$

$$= \frac{\varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}, \hat{\sigma}_{-j}(P_{-j}), t_{-j}^{P_{-j}}\big)}{\varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}\big)}.$$

Case (II). Suppose that $P_j \cap (\cap_{k \in I \setminus \{j\}} P_k) = \emptyset$ or $c_k \neq \hat{\sigma}_k(P_k)$ for some $k \in I \setminus \{j\}$. Then,

$$b_j[t_j^{P_j}](c_{-j}, t_{-j}^{P_{-j}}) = 0 = \frac{\varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}, c_{-j}, t_{-j}^{P_{-j}}\big)}{\varphi\big(\hat{\sigma}_j(P_j), t_j^{P_j}\big)}$$

holds by definition. Hence, $\mathcal{M}^{\Gamma}$ satisfies the common prior assumption.

Fix some world $\omega \in \Omega$, some player $j \in J$, and some type $t_j^{P_j} \in T_j$ of player j such that $\omega \in P_j$. Then, $\hat{v}_j\big(\hat{\sigma}_j(\omega), \hat{\sigma}_{-j}, \omega\big) \geq \hat{v}_j(c_j, \hat{\sigma}_{-j}, \omega)$ holds for all $c_j \in C_j$

by correlated equilibrium. For every world $\omega \in P_j$ and for every choice $c_j \in C_j$ observe that

$$u_j(c_j, t_j^{P_j}) = \sum_{(c_{-j}, t_{-j}^{P_{-j}}) \in C_{-j} \times T_{-j}} b_j[t_j^{P_j}](c_{-j}, t_{-j}^{P_{-j}}) \cdot u_j(c_j, c_{-j})$$

$$= \sum_{P_{-j} \in \mathcal{I}_{-j}} \frac{\pi\big(P_j \cap (\cap_{k \in I \setminus \{j\}} P_k)\big)}{\pi(P_j)} \cdot u_j\big(c_j, \hat{\sigma}_{-j}(P_{-j})\big)$$

$$= \sum_{\omega' \in P_j} \frac{\pi(\omega')}{\pi(P_j)} \cdot u_j\big(c_j, \hat{\sigma}_{-j}(\omega')\big)$$

$$= \hat{v}_j(c_j, \hat{\sigma}_{-j}, \omega).$$

Since $\hat{v}_j\big(\hat{\sigma}_j(P_j), \hat{\sigma}_{-j}, \omega\big) \geq \hat{v}_j(c_j, \hat{\sigma}_{-j}, \omega)$ for all $c_j \in C_j$, it follows that $u_j\big(\hat{\sigma}_j(P_j), t_j^{P_j}\big) \geq u_j(c_j, t_j^{P_j})$ for all $c_j \in C_j$, and hence $\hat{\sigma}_j(P_j)$ is optimal for $t_j^{P_j}$. As every type in $\mathcal{M}^\Gamma$ only assigns positive probability to such choice type pairs $\big(\hat{\sigma}_k(P_k), t_k^{P_k}\big) \in C_k \times T_k$, where the choice is optimal for the type, every type in $\mathcal{M}^\Gamma$ expresses common belief in rationality.

Now consider the choice c_i^* of player i. Because c_i^* is optimal in the correlated equilibrium $(\hat{\sigma}_j)_{j \in I}$, there exists a world $\omega \in \Omega$ such that $\hat{v}_i(c_i^*, \hat{\sigma}_{-i}, \omega) \geq \hat{v}_i(c_i, \hat{\sigma}_{-i}, \omega)$ for all $c_i \in C_i$. Let $P_i \in \mathcal{I}_i$ such that $\omega \in P_i$. Then, $u_i(c_i^*, t_i^{P_i}) \geq u_i(c_i, t_i^{P_i})$ for all $c_i \in C_i$. Since $\mathcal{M}^\Gamma$ satisfies the common prior assumption and $t_i^{P_i}$ expresses common belief in rationality, c_i^* is rational under common belief in rationality with a common prior.

For the *if* direction of the theorem, suppose that c_i^* is rational under common belief in rationality with a common prior. Let $\mathcal{M}^\Gamma$ be an epistemic model satisfying the common prior assumption with probability measure $\varphi \in \Delta\big(\times_{j \in I} (C_j \times T_j)\big)$, $t_i^* \in T_i$ be a type of player i in $\mathcal{M}^\Gamma$ such that c_i^* is optimal for t_i^* and t_i^* expresses common belief in rationality. Moreover, let $\hat{T}(t_i^*)$ be the set of types reachable from t_i^*. Construct an Aumann model $\mathcal{A}^\Gamma$ of Γ, with set of all possible worlds

$$\Omega := \{\omega^{(c_j, t_j)_{j \in I}} : t_j \in \hat{T}(t_i^*) \text{ for all } j \in I \text{ and } \varphi\big((c_j, t_j)_{j \in I}\big) > 0\},$$

common prior $\pi \in \Delta(\Omega)$ such that

$$\pi(\omega^{(c_j, t_j)_{j \in I}}) := \frac{\varphi\big((c_j, t_j)_{j \in I}\big)}{\varphi\big(\{(c_j', t_j')_{j \in I} \mid t_j' \in \hat{T}(t_i^*) \text{ for all } j\}\big)}$$

for all $\omega^{(c_j, t_j)_{j \in I}} \in \Omega$, information partition

$$\mathcal{I}_j(\omega^{(c_j, t_j, c_{-j}, t_{-j})})$$

$$= \{\omega^{(c_j, t_j, c_{-j}', t_{-j}')} : (c_{-j}', t_{-j}') \in C_{-j} \times T_{-j}, t_k' \in \hat{T}(t_i^*) \text{ for all } k \in I \setminus \{j\}, \text{ and } \varphi(c_j, t_j, c_{-j}', t_{-j}') > 0\}$$

for all $\omega^{(c_j,t_j,c_{-j},t_{-j})} \in \Omega$ and for all $j \in I$, as well as choice function $\hat{\sigma}_j : \Omega \to C_j$ such that

$$\hat{\sigma}_j\big(\omega^{(c_j,t_j)_{j\in I}}\big) := c_j$$

for all $\omega^{(c_j,t_j)_{j\in I}} \in \Omega$ and for all $j \in I$.

Consider some world $\omega^{(c_j,t_j)_{j\in I}} \in \Omega$ and some choice $c'_j \in C_j$. Then,

$$\hat{v}_j\big(c'_j, \hat{\sigma}_{-j}, \omega^{(c_j,t_j)_{j\in I}}\big)$$

$$= \sum_{\omega' \in \mathcal{I}_j\left(\omega^{(c_j,t_j)_{j\in I}}\right)} \frac{\pi(\omega')}{\pi\Big(\mathcal{I}_j\big(\omega^{(c_j,t_j)_{j\in I}}\big)\Big)} \cdot u_j\big(c'_j, \hat{\sigma}_{-j}(\omega')\big)$$

$$= \sum_{(c'_{-j},t'_{-j}) \in C_{-j}\times T_{-j} : \varphi(c_j,t_j,c'_{-j},t'_{-j})>0 \text{ and } t'_k \in \hat{T}(t_i^*) \text{ for all } k \in I\setminus\{j\}} \frac{\varphi(c_j,c'_{-j},t_j,t'_{-j})}{\varphi(c_j,t_j)} \cdot u_j(c'_j,c'_{-j})$$

$$= \sum_{(c'_{-j},t'_{-j}) \in C_{-j}\times T_{-j} : b_j[t_j](c'_{-j},t'_{-j})>0} b_j[t_j](c'_{-j},t'_{-j}) \cdot u_j(c'_j,c'_{-j})$$

$$= u_j(c'_j,t_j),$$

where the third equality follows from the fact that $\mathcal{M}^{\Gamma}$ satisfies the common prior assumption. By construction of Ω, it is the case that $t_j \in \hat{T}(t_i^*)$. Since $\varphi(c_j,t_j) > 0$, there exists a type $t_k \in \hat{T}(t_i^*)$ such that $b_k[t_k](c_j,t_j) > 0$. As t_i^* expresses common belief in rationality and thus, by Lemma 1, t_k expresses common belief in rationality, t_k believes in j's rationality. Hence

$$u_j(c_j,t_j) \geq u_j(c'_j,t_j)$$

for all $c'_j \in C_j$. As $u_j(c'_j,t_j) = \hat{v}_j\big(c'_j,\hat{\sigma}_{-j},\omega^{(c_j,t_j)_{j\in I}}\big)$ for all $c'_j \in C_j$, it follows that

$$\hat{v}_j\Big(\hat{\sigma}_j\big(\omega^{(c_j,t_j)_{j\in I}}\big), \hat{\sigma}_{-j}, \omega^{(c_j,t_j)_{j\in I}}\Big) = \hat{v}_j\big(c_j,\hat{\sigma}_{-j},\omega^{(c_k,t_k)_{k\in I}}\big) = u_j(c_j,t_j)$$

$$\geq u_j(c'_j,t_j) = \hat{v}_j\big(c'_j,\hat{\sigma}_{-j},\omega^{(c_j,t_j)_{j\in I}}\big)$$

holds for all $c'_j \in C_j$, and thus $(\hat{\sigma}_j)_{j\in I}$ constitutes a correlated equilibrium.

As c_i^* is optimal for t_i^*, it is the case that $u_i(c_i^*,t_i^*) \geq u_i(c_i,t_i^*)$ for all $c_i \in C_i$. Consider the world $\omega^{(c_i,t_i^*,c_{-i},t_{-i})} \in \Omega$ for some $c_i \in C_i$ and for some $(c_{-i},t_{-i}) \in C_{-i} \times T_{-i}$. Then,

$$\hat{v}_i\big(c_i^*,\hat{\sigma}_{-i},\omega^{(c_i,t_i^*,c_{-i},t_{-i})}\big) = u_i(c_i^*,t_i^*) \geq u_i(c_i,t_i^*) = \hat{v}_i(c_i,\hat{\sigma}_{-i},\omega^{(c_i,t_i^*,c_{-i},t_{-i})})$$

holds for all $c_i \in C_i$. Therefore, c_i^* is optimal in the correlated equilibrium $(\hat{\sigma}_j)_{j\in I}$. ■

From an epistemic perspective correlated equilibrium is thus behaviourally equivalent to rational choice under common belief in rationality with a common prior. Note that Theorem 1 provides an epistemic characterization – both only if and if directions – of correlated equilibrium, whereas Aumann (1987, Main Theorem) constitutes an epistemic foundation – if direction – for correlated equilibrium. In fact, the if direction of our characterization is weaker than Aumann's (1987, Main Theorem), which gives an epistemic foundation for correlated equilibrium in terms of universal rationality and Bayesian conditionalization on a common prior. More precisely, Aumann (1987) assumes that players are rational at all possible worlds, which is significantly stronger than common belief in rationality. Intuitively, in Aumann's (1987) model no irrationality in the system is admitted at all. Besides, in comparison to the reasoning assumptions underlying Nash's (1950) and (1951) notion of equilibrium, correlated equilibrium crucially differs. Indeed, according to the epistemic analysis of Nash equilibrium (cf. Aumann and Brandenburger, 1995; Polak, 1999; Perea, 2007; Barelli, 2009; Bach and Tsakas, 2014; Bonanno, 2017) the decisive epistemic property of Nash equilibrium is some correctness of beliefs assumption. In contrast, no correctness of beliefs assumption whatsoever is needed – neither in Aumann (1987, Main Theorem) nor in Theorem 1 – for correlated equilibrium.

4 Bayesian Equilibrium

4.1 Definition of Bayesian Equilibrium

The standard solution concept for static games with incomplete information is due to Harsanyi (1967-68). Before formally defining the so-called Bayesian equilibrium, Harsanyi's framework for incomplete information games is presented.

Definition 10. *Let Γ be a game with incomplete information. A* Harsanyi model *of Γ is a tuple $\mathcal{H}^\Gamma = \big((H_i)_{i\in I}, \pi, (\tilde{u}_i)_{i\in I}, (\tilde{\sigma}_i)_{i\in I}\big)$, where*

- *H_i is a finite set of Harsanyi types for every player $i \in I$,*
- *$\pi \in \Delta(\times_{j\in I} H_j)$ is a common prior probability measure on the set of all Harsanyi type combinations $\times_{j\in I} H_j$,*
- *$\tilde{u}_i : H_i \to U_i$ assigns a utility function to every Harsanyi type $h_i \in H_i$ for every player $i \in I$,*
- *$\tilde{\sigma}_i : H_i \to \Delta(C_i)$ is a mixed choice function for every player $i \in I$.*

Note that mixed choices are considered in Harsanyi models. For the Harsanyi types of the players the utility functions from the underlying game can then be extended in the usual way from pure to mixed choices. Indeed, given a Harsanyi type $h_i \in H_i$ and an opponents' mixed choice combination $\sigma_{-i} \in \times_{j\in I\setminus\{i\}} \Delta(C_j)$, the utility of i's mixed choice $\tilde{\sigma}_i[h_i]$ is denoted by

$$\tilde{w}_i(\tilde{\sigma}_i[h_i], \sigma_{-i}, h_i) := \sum_{c_i\in C_i} \sum_{c_{-i}\in C_{-i}} \tilde{\sigma}_i[h_i](c_i) \cdot \sigma_{-i}(c_{-i}) \cdot \tilde{u}_i[h_i](c_i, c_{-i}).$$

In the case of pure choices – which are degenerate mixed choices – the utility for player i simplifies to $\tilde{w}_i\big(c_i, \sigma_{-i}, h_i\big) := \sum_{c_{-i} \in C_{-i}} \sigma_{-i}(c_{-i}) \cdot \tilde{u}_i[h_i](c_i, c_{-i})$ for every pure choice $c_i \in C_i$. Moreover, players are assumed to be Bayesian and form their beliefs by conditioning the common prior on their respective Harsanyi types. Thus the belief of a given Harsanyi type h_i on his opponents' Harsanyi type combinations is given by $\pi(h_{-i} \mid h_i) = \frac{\pi(h_i, h_{-i})}{\pi(h_i)}$ for all $h_{-i} \in H_{-i}$, if $\pi(h_i) > 0$. Consequently, given a Harsanyi type h_i, the expected utility of a mixed choice $\tilde{\sigma}_i[h_i]$ is obtained as $\tilde{v}_i(\tilde{\sigma}_i[h_i], \tilde{\sigma}_{-i}, h_i) = \sum_{h_{-i} \in H_{-i}} \pi(h_{-i} \mid h_i) \cdot \tilde{w}_i(\tilde{\sigma}_i[h_i], \tilde{\sigma}_{-i}[h_{-i}], h_i)$. Also, given a Harsanyi type $h_i \in H_i$, a utility function $u_i \in U_i$, and a choice $c_i \in C_i$, the utility of c_i is defined by $\tilde{v}_i(c_i, \tilde{\sigma}_{-i}, h_i, u_i) := \sum_{h_{-i} \in H_{-i}} \pi(h_{-i} \mid h_i) \cdot u_i(c_i, \tilde{\sigma}_{-i}[h_{-i}])$, where u_i may be different from $\tilde{u}_i[h_i]$. Besides, given a utility function $u_i \in U_i$, the set $H_i[u_i] := \{h_i \in H_i : \tilde{u}_i[h_i] = u_i\}$ contains those Harsanyi types of player i that induce u_i.

Within the formal structure provided by Definition 10 the notion of Bayesian equilibrium can be defined as follows.

Definition 11. *Let Γ be a game with incomplete information, and $\mathcal{H}^{\Gamma}$ a Harsanyi model of it with mixed choice function $\tilde{\sigma}_i : H_i \to \Delta(C_i)$ for every player $i \in I$. The tuple $(\tilde{\sigma}_i)_{i \in I}$ of mixed choice functions constitutes a* Bayesian equilibrium*, if for every player $i \in I$ and for every Harsanyi type $h_i \in H_i$ of player i such that $\pi(h_i) > 0$ it is the case that*

$$\tilde{v}_i(\tilde{\sigma}_i[h_i], \tilde{\sigma}_{-i}, h_i) \geq \tilde{v}_i(c_i, \tilde{\sigma}_{-i}, h_i)$$

for all $c_i \in C_i$.

Intuitively, a profile of mixed choice functions is called Bayesian equilibrium, if every choice is a best response to the Bayesian beliefs induced by the underlying Harsanyi types. Note that the universal quantifier in the defining condition for Bayesian equilibrium only runs over all pure choices of the respective player. It can be dispensed with mixed choices here without loss of generality, as mixed choices can never obtain a higher payoff than the best pure choice.

In order to behaviourally relate decision-making in line with the solution concept of Bayesian equilibrium to decision-making in line with the reasoning concept of common belief in rationality, optimal choice in a Bayesian equilibrium is defined next.

Definition 12. *Let Γ be a game with incomplete information, $i \in I$ a player, and $u_i \in U_i$ some utility function of player i. A choice $c_i \in C_i$ of player i is* optimal for the utility function u_i in a Bayesian equilibrium*, if there exists a Harsanyi model $\mathcal{H}^{\Gamma}$ of Γ such that the tuple $(\tilde{\sigma}_i)_{i \in I}$ constitutes a Bayesian equilibrium and with some Harsanyi type h_i of player i such that $\pi(h_i) > 0$ and $\tilde{v}_i(c_i, \tilde{\sigma}_{-i}, h_i, u_i) \geq \tilde{v}_i(c'_i, \tilde{\sigma}_{-i}, h_i, u_i)$ for all $c'_i \in C_i$.*

4.2 Epistemic Characterization of Bayesian Equilibrium

Harsanyi's solution concept for incomplete information games can be analyzed in terms of the players' reasoning. In fact, it turns out that Bayesian equilibrium can be epistemically characterized in terms of common belief in rationality and the common prior assumption.

Theorem 2. *Let Γ be a game with incomplete information, $i \in I$ a player, $u_i^* \in U_i$ a utility function of player i, and $c_i^* \in C_i$ a choice of player i. The choice c_i^* is optimal for the utility function u_i^* in a Bayesian equilibrium, if and only if, c_i^* is rational for the utility function u_i^* under common belief in rationality with a common prior.*

Proof. For the *only if* direction of the theorem, consider a Bayesian equilibrium $(\tilde{\sigma}_i)_{i\in I}$ in some Harsanyi model $\mathcal{H}^\Gamma = \big((H_i)_{i\in I}, \pi, (\tilde{u}_i)_{i\in I}, (\tilde{\sigma}_i)_{i\in I}\big)$ of Γ. Construct an epistemic model $\mathcal{M}^\Gamma = \big((T_i)_{i\in I}, (b_i)_{i\in I}\big)$ of Γ such that $T_i := \{t_i^{h_i} : h_i \in H_i \text{ and } \pi(h_i) > 0\}$ for every player $i \in I$, and

$$b_i[t_i^{h_i}](c_{-i}, t_{-i}^{h_{-i}}, u_{-i}) := \begin{cases} \pi(h_{-i} \mid h_i) \cdot \mathit{\Pi}_{j\in I\setminus\{i\}} \tilde{\sigma}_j[h_j](c_j), & \text{if } u_j = \tilde{u}_j[h_j] \text{ for all } j \in I \setminus \{i\}, \\ 0, & \text{otherwise,} \end{cases}$$

for every type $t_i^{h_i} \in T_i$ and for every player $i \in I$.

It is first shown that $\mathcal{M}^\Gamma$ satisfies the common prior assumption. Define the probability measure $\varphi \in \Delta\big(\times_{j\in I} (C_j \times T_j \times U_j)\big)$ by

$$\varphi\big((c_j, t_j^{h_j}, u_j)_{j\in I}\big) := \begin{cases} \pi\big((h_j)_{j\in I}\big) \cdot \mathit{\Pi}_{j\in I} \tilde{\sigma}_j[h_j](c_j), & \text{if } u_j = \tilde{u}_j[h_j] \text{ for all } j \in I, \\ 0, & \text{otherwise,} \end{cases}$$

for all $(c_j, t_j^{h_j}, u_j)_{j\in I} \in \times_{j\in I}(C_j \times T_j \times U_j)$. It thus has to be established that

$$b_j[t_j^{h_j}](c_{-j}, t_{-j}^{h_{-j}}, u_{-j}) = \frac{\varphi(c_j, c_{-j}, t_j^{h_j}, t_{-j}^{h_{-j}}, u_j, u_{-j})}{\varphi(c_j, t_j^{h_j}, u_j)}$$

for all $t_j^{h_j} \in T_j$ and for all $(c_j, u_j) \in C_j \times U_j$ with $\varphi(c_j, t_j^{h_j}, u_j) > 0$. Consider some $t_j^{h_j} \in T_j$ and distinguish two cases (I) and (II).

Case (I). Suppose that $u_k = \tilde{u}_k[h_k]$ for all $k \in I \setminus \{j\}$. Observe that

$$b_j[t_j^{h_j}](c_{-j}, t_{-j}^{h_{-j}}, u_{-j}) = \pi(h_{-j} \mid h_j) \cdot \mathit{\Pi}_{k\in I\setminus\{j\}} \tilde{\sigma}_k[h_k](c_k)$$

$$= \frac{\pi(h_j, h_{-j})}{\pi(h_j)} \cdot \mathit{\Pi}_{k\in I\setminus\{j\}} \tilde{\sigma}_k[h_k](c_k)$$

for all $(c_{-j}, t_{-j}^{h_{-j}}, u_{-j}) \in C_{-j} \times T_{-j} \times U_{-j}$. Take some $(c_j, u_j) \in C_j \times U_j$ such that $\varphi(c_j, t_j^{h_j}, u_j) > 0$. Then, $\tilde{\sigma}_j[h_j](c_j) > 0$ and $u_j = \tilde{u}_j[h_j]$. It follows that

$$\frac{\pi(h_j, h_{-j})}{\pi(h_j)} \cdot \mathit{\Pi}_{k\in I\setminus\{j\}} \tilde{\sigma}_k[h_k](c_k)$$

$$= \frac{\pi(h_j, h_{-j}) \cdot \tilde{\sigma}_j[h_j](c_j) \cdot \mathit{\Pi}_{k\in I\setminus\{j\}} \tilde{\sigma}_k[h_k](c_k))}{\pi(h_j) \cdot \tilde{\sigma}_j[h_j](c_j)}$$

$$= \frac{\varphi(c_j, c_{-j}, t_j^{h_j}, t_{-j}^{h_{-j}}, u_j, u_{-j})}{\varphi(c_j, t_j^{h_j}, u_j)}$$

for all $h_{-j} \in H_{-j}$. Hence,

$$b_j[t_j^{h_j}](c_{-j}, t_{-j}^{h_{-j}}, u_{-j}) = \frac{\varphi(c_j, c_{-j}, t_j^{h_j}, t_{-j}^{h_{-j}}, u_j, u_{-j})}{\varphi(c_j, t_j^{h_j}, u_j)}.$$

Case (II). Suppose that $u_k \neq \tilde{u}_k[h_k]$ for some $k \in I \setminus \{j\}$. Then,

$$b_j[t_j^{h_j}](c_{-j}, t_{-j}^{h_{-j}}, u_{-j}) = 0 = \frac{\varphi(c_j, c_{-j}, t_j^{h_j}, t_{-j}^{h_{-j}}, u_j, u_{-j})}{\varphi(c_j, t_j^{h_j}, u_j)}$$

holds for all $(c_{-j}, t_{-j}^{h_{-j}}, u_{-j}) \in C_{-j} \times T_{-j} \times U_{-j}$, for all $t_j^{h_j} \in T_j$, and for all $(c_j, u_j) \in C_j \times U_j$ with $\varphi(c_j, t_j^{h_j}, u_j) > 0$ by definition. Therefore, $\mathcal{M}^{\Gamma}$ satisfies the common prior assumption.

Next, it is shown that every type in the epistemic model $\mathcal{M}^{\Gamma}$ believes in the opponents' rationality. For every player $j \in I$, for every type $t_j^{h_j} \in T_j$, and for every choice combination $c_{-j} \in C_{-j}$, it is the case that

$$b_j[t_j^{h_j}](c_{-j}) = \sum_{(t_{-j}^{h_{-j}}, u_{-j}) \in T_{-j} \times U_{-j}} b_j[t_j^{h_j}](c_{-j}, t_{-j}^{h_{-j}}, u_{-j})$$

$$= \sum_{h_{-j} \in H_{-j}} \pi(h_{-j} \mid h_j) \cdot \Pi_{k \in I \setminus \{j\}} \tilde{\sigma}_k[h_k](c_k) = \pi(c_{-j} \mid h_j).$$

Hence for every choice $c_j \in C_j$ and for every utility function $u_j \in U_j$,

$$\tilde{v}_j(c_j, \tilde{\sigma}_{-j}, h_j, u_j) = \sum_{c_{-j} \in C_{-j}} \pi(c_{-j} \mid h_j) \cdot u_j(c_j, c_{-j})$$

$$= \sum_{c_{-j} \in C_{-j}} b_j[t_j^{h_j}](c_{-j}) \cdot u_j(c_j, c_{-j}) = v_j(c_j, t_j^{h_j}, u_j)$$

holds. Let $j \in I$ be some player and $t_j^{h_j} \in T_j$ some type of player j. Consider some opponent $k \in I \setminus \{j\}$ of player j, and suppose some choice type utility function triple $(c_k, t_k^{h_k}, u_k) \in C_k \times T_k \times U_k$ of player k such that $b_j[t_j^{h_j}](c_k, t_k^{h_k}, u_k) > 0$. Then, $\pi(h_k \mid h_j) > 0$, $\tilde{\sigma}_k[h_k](c_k) > 0$, and $\tilde{u}_k[h_k] = u_k$. As $\tilde{\sigma}_k$ is part of a Bayesian equilibrium with Harsanyi type h_k, the choice c_k is optimal for h_k and $\tilde{u}_k[h_k] = u_k$. Hence, $\tilde{v}_k(c_k, \tilde{\sigma}_{-k}, h_k, u_k) \geq \tilde{v}_k(c'_k, \tilde{\sigma}_{-k}, h_k, u_k)$ for all $c'_k \in C_k$. Consequently, $v_k(c_k, t_k^{h_k}, u_k) \geq v_k(c'_k, t_k^{h_k}, u_k)$ for all $c'_k \in C_k$, and c_k is optimal for $(t_k^{h_k}, u_k)$. Thus, t_j believes in k's rationality. Hence, all types in the epistemic model $\mathcal{M}^{\Gamma}$ believe in the opponents' rationality, and therefore all types also express common belief in rationality.

Now, consider the choice c_i^*, which is optimal for the utility function u_i^* in the Bayesian equilibrium $(\tilde{\sigma}_j)_{j\in I}$. Then, there exists $h_i \in H_i$ with $\pi(h_i) > 0$ such that $\tilde{v}_i(c_i^*, \tilde{\sigma}_{-i}, h_i, u_i^*) \geq \tilde{v}_i(c_i, \tilde{\sigma}_{-i}, h_i, u_i^*)$ for all $c_i \in C_i$. Consequently, $v_i(c_i^*, t_i^{h_i}, u_i^*) \geq v_i(c_i, t_i^{h_i}, u_i^*)$ for all $c_i \in C_i$, and c_i^* is optimal for $(t_i^{h_i}, u_i^*)$. Since $t_i^{h_i}$ expresses common belief in rationality and $\mathcal{M}^\Gamma$ satisfies the common prior assumption, c_i^* is rational for u_i^* under common belief in rationality with a common prior.

For the *if* direction of the theorem, let $\mathcal{M}^\Gamma$ be an epistemic model of Γ with a common prior $\varphi \in \Delta\big(\times_{j\in I}(C_j \times T_j \times U_j)\big)$ and a type $t_i^* \in T_i$ of player i such that c_i^* is optimal for t_i^* and u_i^* and t_i^* expresses common belief in rationality. Construct a Harsanyi model $\mathcal{H}^\Gamma = \big((H_j)_{j\in I}, \pi, (\tilde{u}_j)_{j\in I}, (\tilde{\sigma}_j)_{j\in I}\big)$ of Γ such that $H_j := \{h_j^{(c_j,t_j,u_j)} : t_j \in \hat{T}(t_i^*) \text{ and } \varphi(c_j,t_j,u_j) > 0\}$ for all $j \in I$, where

$$\pi\Big(\big(h_j^{(c_j,t_j,u_j)}\big)_{j\in I}\Big) := \frac{\varphi\big((c_j,t_j,u_j)_{j\in I}\big)}{\varphi\Big(\{(c_j,t_j,u_j)_{j\in I} : t_j \in \hat{T}(t_i^*) \text{ for all } j \in I\}\Big)}$$

for all $\big(h_j^{(c_j,t_j,u_j)}\big)_{j\in I} \in \times_{j\in I} H_j$, and $\tilde{u}_j\big(h_j^{(c_j,t_j,u_j)}\big) := u_j$ as well as $\tilde{\sigma}_j\big(h_j^{(c_j,t_j,u_j)}\big) := c_j$ for all $h_j^{(c_j,t_j,u_j)} \in H_j$ and for all $j \in I$.

Let $j \in I$ be some player and $h_j^{(c_j,t_j,u_j)} \in H_j$ some Harsanyi type of player j such that $\pi\big(h_j^{(c_j,t_j,u_j)}\big) > 0$. Hence, $\varphi(c_j,t_j,u_j) > 0$ and there exists a type $t_k \in \hat{T}(t_i^*)$ such that $b_k[t_k](c_j,t_j,u_j) > 0$. As t_i^* expresses common belief in rationality and thus, by Lemma 1, t_k expresses common belief in rationality, t_k believes in j's rationality. Hence c_j is optimal for (t_j, u_j).

For all $c_j' \in C_j$ and for all $u_j' \in U_j$ it is the case that

$$\begin{aligned}
&\tilde{v}_j\big(c_j', \tilde{\sigma}_{-j}, h_j^{(c_j,t_j,u_j)}, u_j'\big) \\
&= \sum_{h_{-j}^{(c_{-j},t_{-j},u_{-j})} \in H_{-j}} \pi\big(h_{-j}^{(c_{-j},t_{-j},u_{-j})} \mid h_j^{(c_j,t_j,u_j)}\big) \cdot u_j'\big(c_j', \tilde{\sigma}_{-j}[h^{(c_{-j},t_{-j},u_{-j})}]\big) \\
&= \sum_{h_{-j}^{(c_{-j},t_{-j},u_{-j})} \in H_{-j}} \frac{\pi\big(h_j^{(c_j,t_j,u_j)}, h_{-j}^{(c_{-j},t_{-j},u_{-j})}\big)}{\pi\big(h_j^{(c_j,t_j,u_j)}\big)} \cdot u_j'(c_j', c_{-j}) \\
&= \sum_{(c_{-j},t_{-j},u_{-j}) \in C_{-j}\times T_{-j}\times U_{-j}} \frac{\varphi(c_j,c_{-j},t_j,t_{-j},u_j,u_{-j})}{\varphi(c_j,t_j,u_j)} \cdot u_j'(c_j', c_{-j}) \\
&= \sum_{(c_{-j},t_{-j},u_{-j}) \in C_{-j}\times T_{-j}\times U_{-j}} b_j[t_j](c_{-j},t_{-j},u_{-j}) \cdot u_j'(c_j', c_{-j}) \\
&= v_j(c_j', t_j, u_j'),
\end{aligned}$$

where the fourth equality follows from the fact that $\mathcal{M}^\Gamma$ satisfies the common prior assumption.

As c_j is optimal for (t_j, u_j), it follows that $v_j(c_j, t_j, u_j) \geq v_j(c'_j, t_j, u_j)$ for all $c'_j \in C_j$. Consequently, $\tilde{v}_j\big(c_j, \tilde{\sigma}_{-j}, h_j^{(c_j,t_j,u_j)}, u_j\big) \geq \tilde{v}_j\big(c'_j, \tilde{\sigma}_{-j}, h_j^{(c_j,t_j,u_j)}, u_j\big)$ for all $c'_j \in C_j$ and for all $h_j^{(c_j,t_j,u_j)} \in H_j$ with $\pi\big(h_j^{(c_j,t_j,u_j)}\big) > 0$. Hence, $(\tilde{\sigma}_j)_{j \in I}$ constitutes a Bayesian equilibrium.

Now consider some Harsanyi type $h_i^{(c_i,t_i^*,u_i)} \in H_i$ of player i with $\pi\big(h_i^{(c_i,t_i^*,u_i)}\big) > 0$. As c_i^* is optimal for t_i^* and u_i^*, observe that

$$\tilde{v}_i\big(c_i^*, \tilde{\sigma}_{-i}, h_i^{(c_i,t_i^*,u_i)}, u_i^*\big) = v_i(c_i^*, t_i^*, u_i^*)$$

$$\geq v_i(c'_i, t_i^*, u_i^*) = \tilde{v}_i\big(c'_i, \tilde{\sigma}_{-i}, h_i^{(c_i,t_i^*,u_i)}, u_i^*\big)$$

for all $c'_i \in C_i$, where the inequality follows from the fact that c_i^* is optimal for (t_i^*, u_i^*). Hence, c_i^* is optimal for the utility function u_i^* in the Bayesian equilibrium $(\tilde{\sigma}_j)_{j \in I}$. ■

Note that the epistemic conditions – common belief in rationality with a common prior – characterizing Bayesian equilibrium in Theorem 2 become precisely the epistemic conditions characterizing correlated equilibrium in Theorem 1, if attention is restricted to the class of complete information games. Consequently, the solution concept of Bayesian equilibrium is behaviourally equivalent to the solution concept of correlated equilibrium for games with complete information, and the following corollary obtains directly.

Corollary 1. *Let Γ be a game with complete information, $i \in I$ a player, and $c_i \in C_i$ a choice of player i. The choice c_i is optimal in a Bayesian equilibrium, if and only if, c_i is optimal in a correlated equilibrium.*

Conceptually, two consequences ensue from Theorem 2 and Corollary 1. Firstly, no correctness of beliefs assumption whatsoever underlies the solution concept of Bayesian equilibrium in terms of reasoning. This is in stark contrast to the solution concept of Nash equilibrium, which crucially requires some correctness of beliefs assumption, as the epistemic analysis of Nash equilibrium has revealed (Aumann and Brandenburger, 1995; Polak, 1999; Perea, 2007; Barelli, 2009; Bach and Tsakas, 2014). Secondly, Bayesian equilibrium constitutes a generalization of correlated equilibrium from complete to incomplete information. Notably, Bayesian equilibrium is *not* an incomplete information analogue of Nash equilibrium.

4.3 Simplified Bayesian Equilibrium

In order to characterize the probability measures on choice utility combinations induced by the solution concept of Bayesian equilibrium, the notion of simplified Bayesian equilibrium is introduced. In particular, this simple tool can be used to compute Bayesian equilibria in specific games.

Definition 13. *Let Γ be a game with incomplete information, and $\rho \in \Delta\big(\times_{i\in I} (C_i \times U_i)\big)$ a probability measure on the players' choice utility function combinations. The probability measure ρ constitutes a* simplified Bayesian equilibrium, *if for all $i \in I$ and for all $(c_i, u_i) \in C_i \times U_i$ such that $\rho(c_i, u_i) > 0$ it is the case that*

$$\sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i)\cdot u_i(c_i, c_{-i}) \geq \sum_{(c_{-i},u_{-i})\in C_{-i}\times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i)\cdot u_i(c_i', c_{-i})$$

holds for all $c_i' \in C_i$.

In order to relate simplified Bayesian equilibrium to Harsanyi's solution concept of Bayesian equilibrium, the notion of induced first-order measure needs to be defined. Recall that for every player $i \in I$ and for every utility function $u_i \in U_i$ of player i, the set $H_[u_i]$ contains those Harsanyi types of player i that induce u_i.

Definition 14. *Let Γ be a game with incomplete information, and $\mathcal{H}^\Gamma$ a Harsanyi model of it. The* induced first-order measure *of $\mathcal{H}^\Gamma$ is the probability measure $\rho \in \Delta\big(\times_{i\in I} (C_i \times U_i)\big)$, where*

$$\rho\big((c_i, u_i)_{i\in I}\big) := \sum_{(h_i)_{i\in I}\in\times_{i\in I} H_i[u_i]} \pi\big((h_i)_{i\in I}\big) \cdot \mathit{\Pi}_{i\in I}\tilde{\sigma}_i[h_i](c_i)$$

for all $(c_i, u_i)_{i\in I} \in \times_{i\in I}(C_i \times U_i)$.

The following result provides a direct characterization of the probability measures on choice utility combinations that can arise in a Bayesian equilibrium, in terms of simplified Bayesian equilibrium.

Theorem 3. *Let Γ be a game with incomplete information, and $\rho \in \Delta\big(\times_{j\in I} (C_j \times U_j)\big)$ a probability measure on the players' choice utility function combinations. There exists a Harsanyi model $\mathcal{H}^\Gamma$ of Γ with induced first-order product measure ρ such that $(\tilde{\sigma}_i)_{i\in I}$ constitutes a Bayesian equilibrium, if and only if, ρ constitutes a simplified Bayesian equilibrium.*

Proof. For the *only if* direction of the theorem, let $(c_i, u_i) \in C_i \times U_i$ be a choice utility function pair of some player $i \in I$ such that $\rho(c_i, u_i) > 0$. Then,

$$\rho(c_{-i}, u_{-i} \mid c_i, u_i) = \frac{\rho\big((c_i, u_i), (c_{-i}, u_{-i})\big)}{\rho(c_i, u_i)}$$

$$= \frac{\sum_{(h_j)_{j\in I}\in\times_{j\in I} H_j[u_j]} \pi\big((h_j)_{j\in I}\big) \cdot \mathit{\Pi}_{j\in I}\tilde{\sigma}_j[h_j](c_j)}{\sum_{h_i\in H_i[u_i]} \pi(h_i) \cdot \tilde{\sigma}_i[h_i](c_i)}$$

$$= \sum_{h_i\in H_i[u_i]} \frac{\sum_{h_{-i}\in H_{-i}[u_{-i}]} \pi(h_i, h_{-i}) \cdot \tilde{\sigma}_i[h_i](c_i) \cdot \mathit{\Pi}_{j\in I\setminus\{i\}}\tilde{\sigma}_j[h_j](c_j)}{\sum_{h_i'\in H_i[u_i]} \pi(h_i') \cdot \tilde{\sigma}_i[h_i'](c_i)}$$

$$= \sum_{h_i \in H_i[u_i]} \Big(\sum_{h_{-i} \in H_{-i}[u_{-i}]} \frac{\pi(h_i, h_{-i})}{\pi(h_i)} \cdot \mathit{\Pi}_{j \in I \setminus \{i\}} \tilde{\sigma}_j[h_j](c_j) \Big) \cdot \Big(\frac{\pi(h_i) \cdot \tilde{\sigma}_i[h_i](c_i)}{\sum_{h'_i \in H_i[u_i]} \pi(h'_i) \cdot \tilde{\sigma}_i[h'_i](c_i)} \Big)$$

$$= \sum_{h_i \in H_i[u_i]} \alpha(c_i, h_i) \cdot \sum_{h_{-i} \in H_{-i}[u_{-i}]} \pi(h_{-i} \mid h_i) \cdot \mathit{\Pi}_{j \in I \setminus \{i\}} \tilde{\sigma}_j[h_j](c_j)$$

for all $(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}$, where $\alpha(c_i, h_i) := \frac{\pi(h_i) \cdot \tilde{\sigma}_i[h_i](c_i)}{\sum_{h'_i \in H_i[u_i]} \pi(h'_i) \cdot \tilde{\sigma}_i[h'_i](c_i)}$ for all $h_i \in H_i[u_i]$.

Hence,

$$\sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i) \cdot u'_i(c'_i, c_{-i})$$

$$= \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \sum_{h_i \in H_i[u_i]} \alpha(c_i, h_i) \sum_{h_{-i} \in H_{-i}[u_{-i}]} \pi(h_{-i} \mid h_i) \cdot \mathit{\Pi}_{j \in I \setminus \{i\}} \tilde{\sigma}_j[h_j](c_j) \cdot u'_i(c'_i, c_{-i})$$

$$= \sum_{h_i \in H_i[u_i]} \sum_{c_{-i} \in C_{-i}} \sum_{u_{-i} \in U_{-i}} \sum_{h_{-i} \in H_{-i}[u_{-i}]} \alpha(c_i, h_i) \cdot \pi(h_{-i} \mid h_i) \cdot \mathit{\Pi}_{j \in I \setminus \{i\}} \tilde{\sigma}_j[h_j](c_j) \cdot u'_i(c'_i, c_{-i})$$

$$= \sum_{h_i \in H_i[u_i]} \sum_{c_{-i} \in C_{-i}} \sum_{h_{-i} \in H_{-i}} \alpha(c_i, h_i) \cdot \pi(h_{-i} \mid h_i) \cdot \mathit{\Pi}_{j \in I \setminus \{i\}} \tilde{\sigma}_j[h_j](c_j) \cdot u'_i(c'_i, c_{-i})$$

$$= \sum_{h_i \in H_i[u_i]} \alpha(c_i, h_i) \cdot \tilde{v}_i(c'_i, \tilde{\sigma}_{-i}, h_i, u'_i)$$

for all $u'_i \in U_i$ and for all $c'_i \in C_i$. As $(\tilde{\sigma}_i)_{i \in I}$ constitutes a Bayesian equilibrium, $\tilde{v}_i(c_i, \tilde{\sigma}_{-i}, h_i) \geq \tilde{v}_i(c'_i, \tilde{\sigma}_{-i}, h_i)$ holds for all $c'_i \in C_i$, if $\alpha(c_i, h_i) > 0$. It follows that

$$\sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i) \cdot u_i(c_i, c_{-i})$$

$$= \sum_{h_i \in H_i[u_i]} \alpha(c_i, h_i) \cdot \tilde{v}_i(c_i, \tilde{\sigma}_{-i}, h_i)$$

$$\geq \sum_{h_i \in H_i[u_i]} \alpha(c_i, h_i) \cdot \tilde{v}_i(c'_i, \tilde{\sigma}_{-i}, h_i)$$

$$= \sum_{(c_{-i}, u_{-i}) \in C_{-i} \times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i) \cdot u_i(c'_i, c_{-i})$$

for all $(c_i, u_i) \in C_i \times U_i$ with $\rho(c_i, u_i) > 0$ and for all $c'_i \in C_i$. Therefore, ρ constitutes a simplified Bayesian equilibrium.

For the *if* direction of the theorem, construct a Harsanyi model $\mathcal{H}^{\Gamma} = \big((H_i)_{i \in I}, \pi, (\tilde{u}_i)_{i \in I}, (\tilde{\sigma}_i)_{i \in I}\big)$ of Γ, where

$$H_i := \{h_i^{(c_i, u_i)} : \rho(c_i, u_i) > 0\}$$

for all $i \in I$,

$$\pi\Big(\big(h_i^{(c_i, u_i)}\big)_{i \in I}\Big) := \rho\big((c_i, u_i)_{i \in I}\big),$$

$$\tilde{u}_i[h_i^{(c_i,u_i)}] := u_i$$

for all $h_i^{(c_i,u_i)} \in H_i$ and for all $i \in I$, as well as $\tilde{\sigma}_i : H_i \to \Delta(C_i)$ such that

$$\tilde{\sigma}_i[h_i^{(c_i,u_i)}](c_i) := 1$$

for all $c_i \in C_i$, for all $h_i^{(c_i,u_i)} \in H_i$, and for all $i \in I$.

By the definition of the common prior π it follows that $\pi\big(h_{-i}^{(c_{-i},u_{-i})} \mid h_i^{(c_i,u_i)}\big) = \rho(c_{-i}, u_{-i} \mid c_i, u_i)$ for all $h_i^{(c_i,u_i)} \in H_i$, and for all $i \in I$. Consider some Harsanyi type $h_i^{(c_i,u_i)} \in H_i$. Since ρ is a simplified Bayesian equilibrium and $\rho(c_i, u_i) > 0$, it is the case that

$$\sum_{(c_{-i},u_{-i}) \in C_{-i} \times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i) \cdot u_i(c_i, c_{-i})$$

$$\geq \sum_{(c_{-i},u_{-i}) \in C_{-i} \times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i) \cdot u_i(c_i', c_{-i})$$

for all $c_i' \in C_i$ and for all $i \in I$.

For all $i \in I$, for all $h_i^{(c_i,u_i)} \in H_i$, for all $c_i' \in C_i$, and for all $u_i' \in U_i$, note that

$$\tilde{v}_i\big(c_i', \tilde{\sigma}_{-i}, h_i^{(c_i,u_i)}, u_i'\big)$$

$$= \sum_{(c_{-i},u_{-i}) \in C_{-i} \times U_{-i} : \rho(c_{-i},u_{-i}) > 0} \pi\big(h_{-i}^{(c_{-i},u_{-i})} \mid h_i^{(c_i,u_i)}\big) \cdot u_i'\Big(c_i', \tilde{\sigma}_{-i}\big(h_{-i}^{(c_{-i},u_{-i})}\big)\Big)$$

$$= \sum_{(c_{-i},u_{-i}) \in C_{-i} \times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i) \cdot u_i'(c_i', c_{-i}).$$

Hence, for every Harsanyi type $h_i^{(c_i,u_i)} \in H_i$ it follows that

$$\tilde{v}_i\Big(\tilde{\sigma}_i\big(h_i^{(c_i,u_i)}\big), \tilde{\sigma}_{-i}, h_i^{(c_i,u_i)}\Big)$$

$$= \sum_{(c_{-i},u_{-i}) \in C_{-i} \times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i) \cdot u_i(c_i, c_{-i})$$

$$\geq \sum_{(c_{-i},u_{-i}) \in C_{-i} \times U_{-i}} \rho(c_{-i}, u_{-i} \mid c_i, u_i) \cdot u_i(c_i', c_{-i})$$

$$= \tilde{v}_i\big(c_i', \tilde{\sigma}_{-i}, h_i^{(c_i,u_i)}\big)$$

for all $c_i' \in C_i$. Consequently, $(\tilde{\sigma}_i)_{i \in I}$ constitutes a Bayesian equilibrium. ■

Due to Theorem 3, the notion of simplified Bayesian equilibrium can be of practical virtue, whenever Bayesian equilibria need to be determined. Indeed, simplified Bayesian equilibria are easier to compute than Bayesian equilibria, and can thus be used to identify the probability measures on choice utility combinations that are possible in Bayesian equilibria. In particular, for applications in incomplete information games simplified Bayesian equilibrium could be a useful tool. However, it follows from Bach and Perea (2017b) that simplified Bayesian equilibrium and Bayesian equilibrium are not behaviourally equivalent. Besides, note that for the special case of complete information the notion of simplified Bayesian equilibrium is actually often employed as the definition of correlated equilibrium.

The incomplete information notions of Bayesian equilibrium and simplified Bayesian equilibrium, as well as the epistemic characterization of Bayesian equilibrium are illustrated next.

Example 1. Consider a two player game with incomplete information between *Alice* and *Bob*, where the choices sets are $C_{Alice} = \{a, b, c\}$ as well as $C_{Bob} = \{d, e, f\}$, respectively, and the set of utility functions are given by $U_{Alice} = \{u_A, u'_A\}$ as well as $U_{Bob} = \{u_B, u'_B\}$, respectively. In Figure 2, the utility functions are spelled out in detail.

u_A	d	e	f
a	3	2	1
b	2	1	3
c	0	0	0

u'_A	d	e	f
a	1	3	1
b	2	1	1
c	0	0	0

u_B	a	b	c
d	3	2	1
e	2	1	3
f	0	0	0

u'_B	a	b	c
d	1	3	1
e	2	1	1
f	0	0	0

Fig. 2. Utility functions of *Alice* and *Bob*.

An interactive representation of the game is provided in Figure 3.

Alice (u_A) \ *Bob* (u_B)	d	e	f
a	3, 3	2, 2	1, 0
b	2, 2	1, 1	3, 0
c	0, 1	0, 3	0, 0

Alice (u_A) \ *Bob* (u'_B)	d	e	f
a	3, 1	2, 2	1, 0
b	2, 3	1, 1	3, 0
c	0, 1	0, 1	0, 0

Alice (u'_A) \ *Bob* (u_B)	d	e	f
a	1, 3	3, 2	1, 0
b	2, 2	1, 1	1, 0
c	0, 1	0, 3	0, 0

Alice (u'_A) \ *Bob* (u'_B)	d	e	f
a	1, 1	3, 2	1, 0
b	2, 3	1, 1	1, 0
c	0, 1	0, 1	0, 0

Fig. 3. Interactive representation of the two-player game with incomplete information and utility functions as specified in Figure 2.

Suppose the Harsanyi model $\big((H_i)_{i \in I}, \pi, (\tilde{u}_i)_{i \in I}, (\tilde{\sigma}_i)_{i \in I}\big)$ of the game, where

- $H_{Alice} = \{h_A, h'_A\}$ and $H_{Bob} = \{h_B, h'_B\}$,

- $\pi(h_A, h_B) = \pi(h_A, h'_B) = \pi(h'_A, h'_B) = \frac{1}{3}$,
- $\tilde{u}_{Alice}[h_A] = u_A$ and $\tilde{u}_{Alice}[h'_A] = u'_A$,
- $\tilde{u}_{Bob}[h_B] = u_B$ and $\tilde{u}_{Bob}[h'_B] = u'_B$,
- $\tilde{\sigma}_{Alice}[h_A] = a$ and $\tilde{\sigma}_{Alice}[h'_A] = a$,
- $\tilde{\sigma}_{Bob}[h_B] = d$ and $\tilde{\sigma}_{Bob}[h'_B] = e$.

Observe that all four Harsanyi types induce choices that are optimal given the respective Harsanyi type's posterior belief and the type's utility function, thus the tuple of choice functions $(\tilde{\sigma}_{Alice}, \tilde{\sigma}_{Bob})$ constitute a Bayesian equilibrium. Consequently, the choice a is optimal for the utility function u_A and u'_A in a Bayesian equilibrium, the choice d is optimal for the utility function u_B in a Bayesian equilibrium, and the choice e is optimal for the utility function u'_B in a Bayesian equilibrium.

The induced first-order measure is $\rho \in \Delta(C_{Alice} \times C_{Bob} \times U_{Alice} \times U_{Bob})$ such that $\rho(a, d, u_A, u_B) = \rho(a, e, u_A, u'_B) = \rho(a, e, u'_A, u'_B) = \frac{1}{3}$. Observe that a is optimal for $\rho(\cdot \mid a, u_A)$ and u_A as well as for $\rho(\cdot \mid a, u'_A)$ and u'_A. Moreover, d is is optimal for $\rho(\cdot \mid d, u_B)$ and u_B, as well as e is optimal for $\rho(\cdot \mid e, u'_B)$ and u'_B. The induced first-order measure ρ thus constitutes a simplified Bayesian equilibrium.

For the epistemic characterization of $(\tilde{\sigma}_{Alice}, \tilde{\sigma}_{Bob})$, consider the epistemic model $\big((T_i)_{i \in I}, (b_i)_{i \in I}\big)$ of the game with a common prior $\varphi \in \Delta(C_{Alice} \times C_{Bob} \times T_{Alice} \times T_{Bob} \times U_{Alice} \times U_{Bob})$, where

- $T_{Alice} = \{t_A, t'_A\}$,
- $T_{Bob} = \{t_B, t'_B\}$,
- $\varphi\big((a, d), (t_A, t_B), (u_A, u_B)\big) = \varphi\big((a, e), (t_A, t'_B), (u_A, u'_B)\big) = \varphi\big((a, e), (t'_A, t'_B), (u'_A, u'_B)\big) = \frac{1}{3}$,
- $b_{Alice}[t_A] = \varphi(\cdot \mid a, t_A, u_A) = \frac{1}{2} \cdot (d, t_B, u_B) + \frac{1}{2} \cdot (e, t'_B, u'_B)$,
- $b_{Alice}[t'_A] = \varphi(\cdot \mid a, t'_A, u'_A) = (e, t'_B, u'_B)$,
- $b_{Bob}[t_B] = \varphi(\cdot \mid d, t_B, u_B) = (a, t_A, u_A)$,
- $b_{Bob}[t'_B] = \varphi(\cdot \mid e, t'_B, u'_B) = \frac{1}{2} \cdot (a, t_A, u_A) + \frac{1}{2} \cdot (a, t'_A, u'_A)$.

Observe that *each* of the four types in this epistemic model believes in the opponents' rationality, and thus each of them also expresses common belief in rationality. Besides, a is optimal for (t_A, u_A) as well as for (t'_A, u'_A), d is optimal for (t_B, u_B), and e is optimal for (t'_B, u'_B). ♣

4.4 Relationship between Generalized Iterated Strict Dominance and Bayesian Equilibrium

Arguably, the most basic solution concepts for games with complete information are iterated strict dominance as well as rationalizability (Bernheim, 1984 and Pearce, 1984), which are behaviourally equivalent for two-player games. Due to Brandenburger and Dekel (1987) as well as Tan and Werlang (1988) the corresponding epistemic conditions only require common belief in rationality. Analogous solution concepts for games with incomplete information include Δ-rationalizability (Battigalli, 2003; Battigalli and Siniscalchi, 2003 and 2007;

Battigalli et al., 2011, Battigalli and Prestipino, 2013; Dekel and Siniscalchi, 2015), interim rationalizability (Ely and Pęski, 2006), interim correlated rationalizability (Dekel et al., 2007), as well as generalized iterated strict dominance (Bach and Perea, 2016). Common belief in rationality has been formalized and used in different forms for epistemic characterizations of the Δ-rationalizability variants by Battigalli and Siniscalchi (1999), (2002), and (2007), Battigalli et al. (2011), as well as Battigalli and Prestipino (2013), for interim correlated rationalizability by Battigalli et al. (2011), and for generalized iterated strict dominance by Bach and Perea (2016). Actually, the solution concepts of Δ-rationalizability and generalized iterated strict dominance are more elementary than interim rationalizability and interim correlated rationalizability in the sense that the latter fix the players' belief hierarchies on utilities while the former do not. Besides note that Δ-rationalizability and generalized iterated strict dominance are behaviourally equivalent, if Δ-rationalizability does not contain any exogenous restrictions. Since the basic epistemic framework – i.e. the notions of an epistemic model and common belief in rationality – used here is identical to Bach and Perea's (2016) epistemic framework for the solution concept of generalized iterated strict dominance, Bayesian equilibrium is now related to generalized iterated strict dominance.

The algorithm of generalized iterated strict dominance is built on the notion of a decision problem. Given a game Γ, a player $i \in I$, and a utility function $u_i \in U_i$, a decision problem $\Gamma_i(u_i) = (D_i, D_{-i}, u_i)$ for player i consists of choices $D_i \subseteq C_i$ for i, choice combinations $D_{-i} \subseteq C_{-i}$ for i's opponents, as well as the utility function u_i restricted to $D_i \times D_{-i}$. Given a utility function $u_i \in U_i$ for player i and his corresponding decision problem $\Gamma_i(u_i) = (D_i, D_{-i}, u_i)$, a choice $c_i \in D_i$ is called strictly dominated, if there exists a probability measure $r_i \in \Delta(D_i)$ such that $u_i(c_i, c_{-i}) < \sum_{c_i' \in D_i} r_i(c_i') \cdot u_i(c_i', c_{-i})$ for all $c_{-i} \in D_{-i}$. Generalized iterated strict dominance is then formally defined as follows.

Definition 15. *Let Γ be a game with incomplete information.*

Round 1. *For every player $i \in I$ and for every utility function $u_i \in U_i$ consider the initial decision problem $\Gamma_i^0(u_i) := \big(C_i^0(u_i), C_{-i}^0(u_i), u_i\big)$, where $C_i^0(u_i) := C_i$ and $C_{-i}^0(u_i) := C_{-i}$.*

Step 1.1 *Set $C_{-i}^1(u_i) := C_{-i}^0(u_i)$.*

Step 1.2 *Form $\Gamma_i^1(u_i) := \big(C_i^1(u_i), C_{-i}^1(u_i), u_i\big)$, where $C_i^1(u_i) \subseteq C_i^0(u_i)$ only contains choices $c_i \in C_i$ for player i that are not strictly dominated in the decision problem $\big(C_i^0(u_i), C_{-i}^1(u_i), u_i\big)$.*

Round $k > 1$. *For every player $i \in I$ and for every utility function $u_i \in U_i$ consider the reduced decision problem $\Gamma_i^{k-1}(u_i) := \big(C_i^{k-1}(u_i), C_{-i}^{k-1}(u_i), u_i\big)$.*

Step k.1 *Form $C_{-i}^k(u_i) \subseteq C_{-i}^{k-1}(u_i)$ by eliminating from $C_{-i}^{k-1}(u_i)$ every opponents' choice combination $c_{-i} \in C_{-i}^{k-1}(u_i)$ that contains for some opponent $j \in I \setminus \{i\}$ a choice $c_j \in C_j$ for which there exists no utility function $u_j \in U_j$ such that $c_j \in C_j^{k-1}(u_j)$.*

Step k.2 *Form $\Gamma_i^k(u_i) := \big(C_i^k(u_i), C_{-i}^k(u_i), u_i\big)$, where $C_i^k(u_i) \subseteq C_i^{k-1}(u_i)$ only contains choices $c_i \in C_i^{k-1}(u_i)$ for player i that are not strictly dominated in the decision problem $\big(C_i^{k-1}(u_i), C_{-i}^k(u_i), u_i\big)$.*

The set $GISD := \times_{i \in I} GISD_i \subseteq \times_{i \in I}(C_i \times U_i)$ *is the output of* generalized iterated strict dominance*, where for every player* $i \in I$ *the set* $GISD_i \subseteq C_i \times U_i$ *only contains choice utility function pairs* $(c_i, u_i) \in C_i \times U_i$ *such that* $c_i \in C_i^k(u_i)$ *for all* $k \geq 0$*.*

For the special case of complete information generalized iterated strict dominance is equivalent to iterated strict dominance. To recall the definition of iterated strict dominance, let $\Gamma = \big(I, (C_i)_{i \in I}, (\{u_i\})_{i \in I}\big)$ be a game with complete information, and consider the sets $C_i^0 := C_i$ and

$$C_i^k := C_i^{k-1} \setminus \{c_i \in C_i : \text{ there exists } r_i \in \Delta(C_i^{k-1})$$
$$\text{such that } u_i(c_i, c_{-i}) < \sum_{c_i' \in C_i} r_i(c_i') \cdot u_i(c_i', c_{-i}) \text{ for all } c_{-i} \in C_{-i}^{k-1}\}$$

for all $k > 0$ and for all $i \in I$. The output of iterated strict dominance is then defined as $ISD := \times_{i \in I} ISD_i \subseteq \times_{i \in I} C_i$, where $ISD_i := \bigcap_{k \geq 0} C_i^k$ for every player $i \in I$. By Bach and Perea (2016, Remark 2) it is the case that $\times_{i \in I} GISD_i = \times_{i \in I}(ISD_i \times \{u_i\})$.

In fact, it turns out that Bayesian equilibrium behaviourally implies generalized iterated strict dominance.

Theorem 4. *Let* Γ *be a game with incomplete information,* $i \in I$ *some player,* $u_i \in U_i$ *some utility function of player* i*, and* $c_i \in C_i$ *some choice of player* i*. If* c_i *is optimal for* u_i *in a Bayesian equilibrium, then* $(c_i, u_i) \in GISD_i$*.*

Proof. Let c_i be optimal for the utility function u_i in a Bayesian equilibrium. In particular it then follows that, by Theorem 2, c_i is rational for the utility function u_i under common belief in rationality. Hence, by Bach and Perea (2016, Theorem 1), $(c_i, u_i) \in GISD_i$. ■

However, the converse to Theorem 4 does not hold.

Remark 1. There exists a game Γ with incomplete information, $i \in I$ some player, some utility function $u_i \in U_i$ of player i, and some choice $c_i \in C_i$ of player i such that $(c_i, u_i) \in GISD_i$ but c_i is not optimal for u_i in a Bayesian equilibrium.

Since for the special case of complete information, generalized iterated strict dominance is equivalent to iterated strict dominance (Bach and Perea, 2016, Remark 2), the following complete information example suffices to establish Remark 1.

Example 2. Consider the two player game between *Alice* and *Bob* given in Figure 4.

Note that for each of the two players every choice is optimal for some belief about the opponent. Thus, no choice is strictly dominated in this game. It follows that $ISD = \{a, b, c\} \times \{d, e, f\}$. However, it is established next that only choices c and f are optimal in a Bayesian equilibrium.

		Bob		
		d	e	f
Alice	a	3, 0	0, 3	0, 2
	b	0, 3	3, 0	0, 2
	c	2, 0	2, 0	2, 2

Fig. 4. A two player game between *Alice* and *Bob*.

The notion of simplified Bayesian equilibrium, together with Theorem 3 restricted to the class of complete information games, is now used to determine the Bayesian equilibria for the game given in Figure 4. First of all, note that $\rho^* \in \Delta(\times_{i\in I} C_i)$ such that $\rho^*(c, f) = 1$ constitutes a simplified Bayesian equilibrium, as c is optimal for $\rho^*(\cdot \mid c)$ and f is optimal for $\rho^*(\cdot \mid f)$.

Next, it is shown that choice a can never receive positive probability in a simplified Bayesian equilibrium $\rho \in \Delta(\times_{i\in I} C_i)$. Towards a contradiction suppose that $\rho(a) > 0$. Then, a must be optimal for $\rho(\cdot \mid a)$ and thus $\rho(d \mid a) \geq \frac{2}{3}$, hence $\rho(a, d) > 0$. Then, d must be optimal for $\rho(\cdot \mid d)$ and thus $\rho(b \mid d) \geq \frac{2}{3}$, hence $\rho(b, d) > 0$. Then, b must be optimal for $\rho(\cdot \mid b)$ and thus $\rho(e \mid b) \geq \frac{2}{3}$, hence $\rho(b, e) > 0$. Then, e must be optimal for $\rho(\cdot \mid e)$ and thus $\rho(a \mid e) \geq \frac{2}{3}$, hence $\rho(a, e) > 0$. Let $\rho(a, d) = \alpha$, $\rho(a, e) = \beta$, $\rho(b, d) = \gamma$, and $\rho(b, e) = \delta$. It follows that $\rho(d \mid a) = \frac{\alpha}{\alpha+\beta+\rho(a,f)} \geq \frac{2}{3}$, hence $\frac{\alpha}{\alpha+\beta} \geq \frac{2}{3}$, $\rho(b \mid d) = \frac{\gamma}{\alpha+\gamma+\rho(c,d)} \geq \frac{2}{3}$, hence $\frac{\gamma}{\alpha+\gamma} \geq \frac{2}{3}$, $\rho(e \mid b) = \frac{\delta}{\gamma+\delta+\rho(b,f)} \geq \frac{2}{3}$, hence $\frac{\delta}{\gamma+\delta} \geq \frac{2}{3}$, and $\rho(a \mid e) = \frac{\beta}{\beta+\delta+\rho(c,e)} \geq \frac{2}{3}$, hence $\frac{\beta}{\beta+\delta} \geq \frac{2}{3}$. Consequently, $\alpha \geq 2\beta$, $\gamma \geq 2\alpha$, $\delta \geq 2\gamma$ as well as $\beta \geq 2\delta$ must hold, and thus $\alpha + \beta + \gamma + \delta \geq 2(\alpha + \beta + \gamma + \delta)$ obtains, which is only possible if $\alpha = \beta = \gamma = \delta = 0$, but $\alpha = \rho(a, d) > 0$, a contradiction.

Similarly, it can be established that $\rho(b) = 0$, $\rho(d) = 0$, and $\rho(e) = 0$. Hence, $\rho^* \in \Delta(\times_{i\in I} C_i)$ such that $\rho^*(c, f) = 1$ constitutes the unique simplified Bayesian equilibrium.

Let $(\tilde{\sigma}_{Alice}, \tilde{\sigma}_{Bob}) \in \Delta(C_{Alice})^{H_{Alice}} \times \Delta(C_{Bob})^{H_{Bob}}$ be some Bayesian equilibrium of the game given in Figure 4. Then, by Theorem 3, the induced first-order measure must be $\rho^* \in \Delta(C_{Alice} \times U_{Alice} \times C_{Bob} \times U_{Bob})$ with $\rho^*(c, f) = 1$. Hence, there exists a Harsanyi model of the game such that $\tilde{\sigma}_{Alice}[h_{Alice}](c) = 1$ for every Harsanyi type $h_{Alice} \in H_{Alice}$ of *Alice* and $\tilde{\sigma}_{Bob}[h_{Bob}](f) = 1$ for every Harsanyi type $h_{Bob} \in H_{Bob}$ of *Bob*. Therefore, the only optimal choice in a Bayesian equilibrium of the game is c for *Alice* and f for *Bob*.

Thus, for the game in Figure 4 the choices a and b of *Alice* survive iterated strict dominance, i.e. $a, b \in ISD$, however, neither a nor b is optimal in a Bayesian equilibrium. ♣

According to Theorem 4 and Remark 1, Bayesian equilibrium is stronger than generalized iterated strict dominance. Thus, the same relationship emerges for incomplete information games between Bayesian equilibrium and generalized iterated strict dominance, as for complete information games between the analogous solution concepts of correlated equilibrium and iterated strict dominance.

5 Generalized Nash Equilibrium

5.1 Definition of Generalized Nash Equilibrium

The preceding section has shown that Bayesian equilibrium is not a generalization of Nash equilibrium from complete to incomplete information games. In fact, a generalized notion of Nash equilibrium is now introduced as a new solution concept for incomplete information games and epistemically characterized invoking a correctness of beliefs assumption in addition to common belief in rationality. It also turns out that generalized Nash equilibrium is more restrictive than Bayesian equilibrium.

In order to keep notation simple, for every tuple $(\xi_i)_{i\in I} \in \times_{i\in I}\Delta(C_i \times U_i)$, define $\xi_{-i}(c_{-i}, u_{-i}) := \Pi_{j\in I\setminus\{i\}}\xi_j(c_j, u_j)$ and $\xi_{-i}(c_{-i}) := \Pi_{j\in I\setminus\{i\}}\xi_j(c_j)$. Before the new solution concept of generalized Nash equilibrium for games with incomplete information is defined, attention is restricted to complete information and the definition of Nash equilibrium is recalled. For a given game Γ with complete information, a tuple $(\sigma_i)_{i\in I} \in \times_{i\in I}\Delta(C_i)$ of probability measures constitutes a *Nash equilibrium*, whenever for all $i \in I$ and for all $c_i \in C_i$, if $\sigma_i(c_i) > 0$, then $\sum_{c_{-i}\in C_{-i}} \sigma_{-i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}} \sigma_{-i}(c_{-i}) \cdot u_i(c_i', c_{-i})$ for all $c_i' \in C_i$. Moreover, a choice $c_i \in C_i$ is called *optimal in a Nash equilibrium*, if there exists a Nash equilibrium $(\sigma_j)_{j\in I} \in \times_{j\in I}\Delta(C_j)$ such that $\sum_{c_{-i}\in C_{-i}} \sigma_{-i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}} \sigma_{-i}(c_{-i}) \cdot u_i(c_i', c_{-i})$ for all $c_i' \in C_i$.

A direct generalization of Nash equilibrium to incomplete information obtains as follows.

Definition 16. *Let Γ be a game with incomplete information, and $\xi_i \in \Delta(C_i \times U_i)$ be probability measures for every player $i \in I$. The tuple $(\xi_i)_{i\in I}$ constitutes a* generalized Nash equilibrium*, whenever for all $i \in I$ and for all $(c_i, u_i) \in C_i \times U_i$, if $\xi_i(c_i, u_i) > 0$, then*

$$\sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i(c_i', c_{-i})$$

for all $c_i' \in C_i$.

Intuitively, the best-response property of a player's support required by the complete information solution concept of Nash equilibrium is extended to the augmented uncertainty space of choices and utility functions. Note that generalized Nash equilibrium imposes the analogous condition on the – due to payoff uncertainty extended – space $\times_{i\in I}\big(\Delta(C_i \times U_i)\big)$ that Nash equilibrium imposes on the space $\times_{i\in I}\Delta(C_i)$. Indeed, in the specific case of complete information, i.e. $U_i = \{u_i\}$ for all $i \in I$, the notion of generalized Nash equilibrium reduces to Nash equilibrium.

In order to characterize decision-making in line with generalized Nash equilibrium in terms of reasoning, the notion of optimal choice in a generalized Nash equilibrium is defined next.

Definition 17. *Let Γ be a game with incomplete information, $i \in I$ a player, and $u_i \in U_i$ some utility function of player i. A choice $c_i \in C_i$ of player i is* optimal for the utility function u_i in a generalized Nash equilibrium*, if there exists a generalized Nash equilibrium $(\xi_i)_{i\in I} \in \times_{i\in I}\big(\Delta(C_i \times U_i)\big)$ such that*

$$\sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i(c_i', c_{-i})$$

for all $c_i' \in C_i$ and for all $i \in I$.

Note that optimal choice in a generalized Nash equilibrium for incomplete information games with a single utility function for every player is equivalent to optimal choice in a Nash equilibrium for complete information games.

Remark 2. Let Γ be a game with complete information, $i \in I$ a player, and $c_i \in C_i$ a choice of player i. The choice c_i is optimal in a generalized Nash equilibrium, if and only if, c_i is optimal in a Nash equilibrium.

5.2 Epistemic Characterization of Generalized Nash Equilibrium

Before an epistemic characterization of the incomplete information solution concept generalized Nash equilibrium can be given, some additional epistemic notions need to be invoked. Given a tuple $(\xi_j)_{j\in I} \in \times_{j\in I}\big(\Delta(C_j \times U_j)\big)$ of probability measures, a belief hierarchy on choices and utility functions for player i is called *generated* by $(\xi_j)_{j\in I}$, if for every opponent $j \in I \setminus \{i\}$, player i's belief about player j's choice and utility function is ξ_j; for every opponent $j \in I \setminus \{i\}$ and for every player $k \in I \setminus \{j\}$, player i believes that j's belief about player k's choice and utility function is ξ_k; etc. A type t_i is then said to entertain a *simple belief hierarchy*, if t_i's belief hierarchy is generated by some tuple $(\xi_j)_{j\in I} \in \times_{j\in I}\big(\Delta(C_j \times U_j)\big)$ of probability measures.[1]

The solution concept of generalized Nash equilibrium can be epistemically characterized as follows.

Theorem 5. *Let Γ be a game with incomplete information, $i \in I$ some player, $u_i^* \in U_i$ some utility function of player i, and $c_i^* \in C_i$ some choice of player i. The choice c_i^* is optimal for u_i^* in a generalized Nash equilibrium, if and only if, there exists an epistemic model $\mathcal{M}^\Gamma$ of Γ with a type t_i for player i such that c_i^* is optimal for (t_i, u_i^*) and t_i entertains a simple belief hierarchy as well as expresses common belief in rationality.*

Proof. For the *only if* direction of the theorem, let $(\xi_j)_{j\in I}$ be a generalized Nash equilibrium such that

$$\sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i^*, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i, c_{-i})$$

[1] The notion of simple belief hierarchy in the context of complete information is due to Perea (2012).

for all $c_i \in C_i$. Construct an epistemic model $\mathcal{M}^\Gamma = \big((T_j)_{j \in I}, (b_j)_{j \in I}\big)$ of Γ such that $T_j := \{t_j\}$ for every player $j \in I$ and $b_j[t_j](c_{-j}, t_{-j}, u_{-j}) := \xi_{-j}(c_{-j}, u_{-j})$ for every tuple $(c_{-j}, u_{-j}) \in C_{-j} \times U_{-j}$, for every type $t_j \in T_j$, and for every player $j \in I$. Observe that for every player $j \in I$ the unique type t_j's belief hierarchy is generated by $(\xi_k)_{k \in I}$ and is thus simple. Consider the unique type t_j of some player $j \in I$, and suppose that $b_j[t_j](c_k, t_k, u_k) > 0$ for some choice type utility function triple $(c_k, t_k, u_k) \in C_k \times T_k \times U_k$ of some player $k \in I \setminus \{j\}$. It is thus the case that $\xi_k(c_k, u_k) > 0$. Since $(\xi_k)_{k \in I}$ is a generalized Nash equilibrium,

$$\begin{aligned} v_k(c_k, t_k, u_k) &= \sum_{c_{-k} \in C_{-k}} \xi_{-k}(c_{-k}) \cdot u_k(c_k, c_{-k}) \\ &\geq \sum_{c_{-k} \in C_{-k}} \xi_{-k}(c_{-k}) \cdot u_k(c'_k, c_{-k}) = v_k(c'_k, t_k, u_k) \end{aligned}$$

holds for all $c'_k \in C_k$. Hence, t_j believes in the opponents' rationality. As this holds for the unique type of every player in the epistemic model $\mathcal{M}^\Gamma$, it follows that every type in $\mathcal{M}^\Gamma$ also expresses common belief in rationality. Since

$$v_i(c_i^*, t_i, u_i^*) = \sum_{c_{-i} \in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i^*, c_{-i}) \geq \sum_{c_{-i} \in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i, c_{-i}) = v_i(c_i, t_i, u_i^*)$$

for all $c_i \in C_i$, the choice c_i^* is optimal for (t_i, u_i^*).

For the *if* direction of the theorem, let t_i's belief hierarchy be generated by the tuple $(\xi_j)_{j \in I} \in \times_{j \in I}\big(\Delta(C_j \times U_j)\big)$. Let $\xi_j(c_j, u_j) > 0$ for some choice utility function pair $(c_j, u_j) \in C_j \times U_j$ of some player $j \in I$. First of all, suppose that $j \neq i$. Then, for every type $t_j \in T_j$ such that $b_i[t_i](c_j, t_j, u_j) > 0$, it is the case that c_j is optimal for (t_j, u_j), as t_i expresses common belief in rationality. Since the belief hierarchy of t_i is generated by $(\xi_j)_{j \in I}$, the marginal belief of t_j about its opponents' choices is given by $\xi_{-j}(c_{-j})$. Hence,

$$\sum_{c_{-j} \in C_{-j}} \xi_{-j}(c_{-j}) \cdot u_j(c_j, c_{-j}) \geq \sum_{c_{-j} \in C_{-j}} \xi_{-j}(c_{-j}) \cdot u_j(c'_j, c_{-j}))$$

holds for all $c'_j \in C_j$. Now, suppose that $i = j$. Consider some player $k \in I \setminus \{i\}$ and type $t_k \in T_k$ such that $b_i[t_i](t_k) > 0$. Then, for every type $t'_i \in T_i$ such that $b_k[t_k](c_i, t'_i, u_i) > 0$, it is the case that c_i is optimal for (t'_i, u_i), as t_i expresses common belief in rationality. Since the belief hierarchy of t_i is generated by $(\xi_j)_{j \in I}$, the marginal belief of t'_i about its opponents' choices is given by $\xi_{-i}(c_{-i})$. Hence,

$$\sum_{c_{-i} \in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i(c_i, c_{-i}) \geq \sum_{c_{-i} \in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i(c'_i, c_{-i})$$

holds for all $c'_i \in C_i$. Consequently, $(\xi_j)_{j \in I}$ constitutes a generalized Nash equilibrium. Since c_i^* is optimal for (t_i, u_i^*) and t_i entertains the simple belief hierarchy generated by $(\xi_j)_{j \in I}$, it follows that

$$\sum_{c_{-i} \in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i^*, c_{-i}) \geq \sum_{c_{-i} \in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i, c_{-i})$$

for all $c_i \in C_i$. As $(\xi_j)_{j \in I}$ is a generalized Nash equilibrium, the choice c_i^* is optimal for u_i^* in a generalized Nash equilibrium. ■

The preceding theorem shows that in terms of reasoning generalized Nash equilibrium requires substantially more than common belief in rationality. Indeed, the additional simple belief hierarchy condition can be interpreted as a correctness of beliefs assumption. Hence, analogous to the complete information solution concept of Nash equilibrium, which crucially depends on some correctness of beliefs assumption in terms of reasoning (Aumann and Brandenburger, 1995; Polak, 1999; Perea, 2007; Barelli, 2009; Perea, 2012; Bach and Tsakas, 2014; Bonanno, 2017), generalized Nash equilibrium does so too. Also in terms of reasoning generalized Nash equilibrium thus naturally extends Nash equilibrium from complete to incomplete information games. Indeed, Nash equilibrium can also be characterized by common belief in rationality and a simple belief hierarchy in the case of games with complete information (Perea, 2012). However, note that the epistemic characterization of generalized Nash equilibrium can actually be significantly weakened such that common belief in rationality is not even implied (Bach and Perea, 2017a). In order to relate the solution concept of generalized Nash equilibrium to Bayesian equilibrium in terms of reasoning, the epistemic characterization of the former in Theorem 5.2 is also provided in terms of common belief in rationality. In fact, both solution concepts differ by imposing the common prior assumption and simple belief hierarchy, respectively, on the reasoning of players.

Next, the incomplete information analogue of Nash equilibrium – generalized Nash equilibrium – as well as the epistemic conditions characterizing it are illustrated.

Example 3. Consider a three player game with incomplete information between *Alice*, *Bob*, and *Claire*, where the choice sets are $C_{Alice} = \{a, b\}$, $C_{Bob} = \{c, d\}$, as well as $C_{Claire} = \{e, f\}$, respectively, and the sets of utility functions are given by $U_{Alice} = \{u_A\}$, $U_{Bob} = \{u_B\}$, as well as $U_{Claire} = \{u_C, u'_C\}$, respectively. In Figure 5, the utility functions are spelled out in detail.

u_A

	(c,e)	(d,e)	(c,f)	(d,f)
a	2	1	0	3
b	0	3	2	1

u_B

	(a,e)	(b,e)	(a,f)	(b,f)
c	2	3	0	1
d	0	1	1	3

u_C

	(a,c)	(a,d)	(b,c)	(b,d)
e	1	3	2	0
f	0	2	3	1

u'_C

	(a,c)	(a,d)	(b,c)	(b,d)
e	0	1	3	2
f	1	2	3	1

Fig. 5. Utility functions of *Alice*, *Bob* and *Claire*.

An interactive representation of the game is provided in Figure 6.

Bob (u_B)

Alice (u_A)	c	d
a	2, 2, 1	1, 0, 3
b	0, 3, 2	3, 1, 0

Claire (u_C): e

Bob (u_B)

Alice (u_A)	c	d
a	0, 0, 0	3, 1, 2
b	2, 1, 3	1, 3, 1

Claire (u_C): f

Bob (u_B)

Alice (u_A)	c	d
a	2, 2, 0	1, 0, 1
b	0, 3, 3	3, 1, 2

Claire (u'_C): e

Bob (u_B)

Alice (u_A)	c	d
a	0, 0, 1	3, 1, 2
b	2, 1, 3	1, 3, 1

Claire (u'_C): f

Fig. 6. A three player game with incomplete information.

The triple $\big((a,u_A),(c,u_B),\frac{1}{2}\cdot(e,u_C)+\frac{1}{2}\cdot(f,u'_C)\big)$ constitutes a generalized Nash equilibrium, since a is optimal for $\big((c,u_B),\frac{1}{2}\cdot(e,u_C)+\frac{1}{2}\cdot(f,u'_C)\big)$ and u_A, c is optimal for $\big((a,u_A),\frac{1}{2}\cdot(e,u_C)+\frac{1}{2}\cdot(f,u'_C)\big)$ and u_B, e is optimal for $\big((a,u_A),(c,u_B)\big)$ and u_C, as well as f is optimal for $\big((a,u_A),(c,u_B)\big)$ and u'_C. It follows directly that a, c, e, and f are all optimal for some utility function in a generalized Nash equilibrium.

For the epistemic representation of the generalized Nash equilibrium $\big((a,u_A),(c,u_B),\frac{1}{2}\cdot(e,u_C)+\frac{1}{2}\cdot(f,u'_C)\big)$, consider the epistemic model $\big((T_i)_{i\in I},(b_i)_{i\in I}\big)$ of the game, where

- $T_{Alice}=\{t_A\}$, $T_{Bob}=\{t_B\}$, and $T_{Claire}=\{t_C\}$
- $b_{Alice}[t_A]=\frac{1}{2}\cdot\big((c,t_B,u_B),(e,t_C,u_C)\big)+\frac{1}{2}\cdot\big((c,t_B,u_B),(f,t_C,u'_C)\big)$,
- $b_{Bob}[t_B]=\frac{1}{2}\cdot\big((a,t_A,u_A),(e,t_C,u_C)\big)+\frac{1}{2}\cdot\big((a,t_A,u_A),(f,t_C,u'_C)\big)$,
- $b_{Claire}[t_C]=\big((a,t_A,u_A),(c,t_B,u_B)\big)$.

Observe that *each* of the three types in this epistemic model believes in the opponents' rationality, and thus each of them also expresses common belief in rationality. Also, note that every type in the epistemic model entertains a simple belief hierarchy. Besides, a is optimal for (t_A,u_A), c is optimal for (t_B,u_B), e is optimal for (t_C,u_C), and f is optimal for (t_C,u'_C). ♣

5.3 Relationship between Bayesian Equilibrium and Generalized Nash Equilibrium

Next, the relationship between the two incomplete information solution concepts of generalized Nash equilibrium and Bayesian equilibrium is considered.

In fact, it is the case that an optimal choice in a generalized Nash equilibrium also is optimal in a Bayesian equilibrium.

Theorem 6. *Let Γ be a game with incomplete information, $i \in I$ some player, u_i^* some utility function of player i, and $c_i^* \in C_i$ some choice of player i. If c_i^* is optimal for u_i^* in a generalized Nash equilibrium, then c_i^* is also optimal for u_i^* in a Bayesian equilibrium.*

Proof. Let $(\xi_j)_{j\in I} \in \times_{j\in I}\big(\Delta(C_j \times U_j)\big)$ be a generalized Nash equilibrium such that

$$\sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i^*, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i, c_{-i})$$

for all $c_i \in C_i$.

Construct a Harsanyi model $\mathcal{H}^\Gamma = \big((H_j)_{j\in I}, \pi, (\tilde{u}_j)_{j\in I}, (\tilde{\sigma}_j)_{j\in I}\big)$ of Γ, where $H_j := \{h_j^{(c_j,u_j)} : \xi_j(c_j,u_j) > 0\}$ for all $j \in I$, $\pi\Big(\big(h_j^{(c_j,u_j)}\big)_{j\in I}\Big) := \Pi_{j\in I}\xi_j(c_j,u_j)$, $\tilde{u}_j\big(h_j^{(c_j,u_j)}\big) := u_j$ for all $h_j^{(c_j,u_j)} \in H_j$ and for all $j \in I$, as well as $\tilde{\sigma}_j[h_j^{(c_j,u_j)}](c_j) := 1$ for all $c_j \in C_j$, for all $h_j^{(c_j,u_j)} \in H_j$, and for all $j \in I$.

It is first shown that $(\tilde{\sigma}_j)_{j\in I}$ constitutes a Bayesian equilibrium. Consider some Harsanyi type $h_j^{(c_j,u_j)} \in H_j$ of some player $j \in I$. Then, for all $c_j' \in C_j$ and for all $u_j' \in U_j$ it follows that

$$\begin{aligned}
&\tilde{v}_j\big(c_j', \tilde{\sigma}_{-j}, h_j^{(c_j,u_j)}, u_j'\big) \\
&= \sum_{h_{-j}^{(c_{-j},u_{-j})}\in H_{-j}} \pi\big(h_{-j}^{(c_{-j},u_{-j})} \mid h_j^{(c_j,u_j)}\big) \cdot u_j'(c_j', c_{-j}) \\
&= \sum_{h_{-j}^{(c_{-j},u_{-j})}\in H_{-j}} \frac{\pi\big(h_j^{(c_j,u_j)}, h_{-j}^{(c_{-j},u_{-j})}\big)}{\pi\big(h_j^{(c_j,u_j)}\big)} \cdot u_j'(c_j', c_{-j}) \\
&= \sum_{(c_{-j},u_{-j})\in C_{-j}\times U_{-j}} \frac{\xi_j(c_j,u_j)\cdot \xi_{-j}(c_{-j},u_{-j})}{\xi_j(c_j,u_j)} \cdot u_j'(c_j', c_{-j}) \\
&= \sum_{c_{-j}\in C_{-j}} \xi_{-j}(c_{-j}) \cdot u_j'(c_j', c_{-j}).
\end{aligned}$$

Now, suppose that $\pi\big(h_j^{(c_j,u_j)}\big) > 0$. Thus, $\xi_j(c_j,u_j) > 0$. Consequently,

$$\tilde{v}_j\big(\tilde{\sigma}_j[h_j^{(c_j,u_j)}], \tilde{\sigma}_{-j}, h_j^{(c_j,u_j)}\big) = \tilde{v}_j\big(c_j, \tilde{\sigma}_{-j}, h_j^{(c_j,u_j)}, u_j\big) = \sum_{c_{-j}\in C_{-j}} \xi_{-j}(c_{-j})\cdot u_j(c_j, c_{-j})$$

$$\geq \sum_{c_{-j}\in C_{-j}} \xi_{-j}(c_{-j}) \cdot u_j(c_j', c_{-j})) = \tilde{v}_j\big(c_j', \tilde{\sigma}_{-j}, h_j^{(c_j,u_j)}, u_j\big)$$

for all $c_j' \in C_j$, where the inequality follows from the fact that $(\xi_j)_{j\in I}$ is a generalized Nash equilibrium. Hence, the tuple $(\tilde{\sigma}_j)_{j\in I}$ constitutes a Bayesian equilibrium.

As c_i^* is optimal for u_i^* in the generalized Nash equilibrium $(\xi_j)_{j\in I}$, it is the case that $\sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i^*, c_{-i}) \geq \sum_{c_{-i}\in C_{-i}} \xi_{-i}(c_{-i}) \cdot u_i^*(c_i', c_{-i})$ for all $c_i' \in C_i$. Consequently, $\tilde{v}_i\big(c_i^*, \tilde{\sigma}_{-i}, h_i^{(c_i,u_i)}, u_i^*\big) \geq \tilde{v}_i\big(c_i', \tilde{\sigma}_{-i}, h_i^{(c_i,u_i)}, u_i^*\big)$ for all $c_i' \in C_i$ and for all $h_i^{(c_i,u_i)} \in H_i$. Therefore, c_i^* is optimal for u_i^* in a Bayesian equilibrium. ■

However, the converse to Theorem 6 does not hold.

Remark 3. There exists a game Γ with incomplete information, $i \in I$ some player, some utility function u_i of player i, and $c_i \in C_i$ some choice of player i such that c_i is optimal for u_i in a Bayesian equilibrium, but c_i is not optimal for u_i in a generalized Nash equilibrium.

Since for the special case of complete information it is the case that generalized Nash equilibrium is behaviourally equivalent to Nash equilibrium (Remark 2) and Bayesian equilibrium is behaviourally equivalent to correlated equilibrium (Corollary 1), the following complete information example suffices to establish Remark 3.

Example 4. Consider the three player game with complete information between *Alice*, *Bob*, and *Claire* given in Figure 7.

Claire: left

Alice \ *Bob*	c	d
a	1, 1, 2	1, 0, 2
b	0, 0, 2	0, 1, 0

Claire: middle

Alice \ *Bob*	c	d
a	1, 1, 1	1, 1, 0
b	1, 1, 0	1, 1, 1

Claire: right

Alice \ *Bob*	c	d
a	1, 1, 0	1, 0, 2
b	0, 0, 2	0, 1, 2

Fig. 7. A three player game between *Alice*, *Bob* and *Claire*.

Construct an Auman model $\big(\Omega, \pi, (\mathcal{I}_i)_{i\in I}, (\hat{\sigma})_{i\in I}\big)$ of the game, where

- $\Omega = \{\omega_1, \omega_2\}$,
- $\pi(\omega_1) = \pi(\omega_2) = \frac{1}{2}$,
- $\mathcal{I}_{Alice}(\omega_1) = \{\omega_1\}$ and $\mathcal{I}_{Alice}(\omega_2) = \{\omega_2\}$,
- $\mathcal{I}_{Bob}(\omega_1) = \{\omega_1\}$ and $\mathcal{I}_{Bob}(\omega_2) = \{\omega_2\}$,
- $\mathcal{I}_{Claire}(\omega_1) = \mathcal{I}_{Claire}(\omega_2) = \{\omega_1, \omega_2\}$,
- $\hat{\sigma}_{Alice}(\omega_1) = a$ and
- $\hat{\sigma}_{Alice}(\omega_2) = b$,
- $\hat{\sigma}_{Bob}(\omega_1) = c$ and
- $\hat{\sigma}_{Bob}(\omega_2) = d$,
- $\hat{\sigma}_{Claire}(\omega_1) = \hat{\sigma}_{Claire}(\omega_2) = middle$.

At world ω_1 the choices a, c, and *middle* are optimal for *Alice*, *Bob*, and *Claire*, respectively, as well as at world ω_2 the choices b, d, and *middle* are optimal for *Alice*, *Bob*, and *Claire*, respectively. Hence, $(\hat{\sigma}_i)_{i \in I}$ constitutes a correlated equilibrium. In particular, for *Claire*, the choice *middle* is optimal in a correlated equilibrium.

Define probability measures $\sigma^*_{Alice} \in \Delta(C_{Alice})$ such that $\sigma^*_{Alice}(a) = 1$, $\sigma^*_{Bob} \in \Delta(C_{Bob})$ such that $\sigma^*_{Bob}(c) = 1$, and $\sigma^*_{Claire} \in \Delta(C_{Claire})$ such that $\sigma^*_{Claire}(left) = 1$. Note that the tuple $(\sigma^*_{Alice}, \sigma^*_{Bob}, \sigma^*_{Claire})$ constitutes a Nash equilibrium, since a is optimal against $(c, left)$, c is optimal against $(a, left)$, and $left$ is optimal against (a, c). Next, it is shown that there exists no other Nash equilibrium in this game. Suppose that $\sigma_{Claire}(middle) < 1$ in some Nash equilibrium. Then, only a is optimal for *Alice*, i.e. $\sigma_{Alice}(a) = 1$. Then, only c is optimal for *Bob*, i.e. $\sigma_{Bob}(c) = 1$. Hence, $(\sigma^*_{Alice}, \sigma^*_{Bob}, \sigma^*_{Claire}) = (a, c, left)$ constitutes the only Nash equilibrium with $\sigma_{Claire}(middle) < 1$. Towards a contradiction, suppose that $\sigma_{Claire}(middle) = 1$ in some Nash equilibrium. Note that it is the case that either $\sigma_{Alice}(a) \cdot \sigma_{Bob}(c) \leq \frac{1}{4}$ or $\big(1 - \sigma_{Alice}(a)\big) \cdot \big(1 - \sigma_{Bob}(c)\big) \leq \frac{1}{4}$. However, if $\sigma_{Alice}(a) \cdot \sigma_{Bob}(c) \leq \frac{1}{4}$, then $right$ is better than *middle* for *Claire*, and if $\big(1 - \sigma_{Alice}(a)\big) \cdot \big(1 - \sigma_{Bob}(c)\big) \leq \frac{1}{4}$, then $left$ is better than *middle* for *Claire*, a contradiction. Thus, $(\sigma^*_{Alice}, \sigma^*_{Bob}, \sigma^*_{Claire}) = (a, c, left)$ constitutes the unique Nash equilibrium. In particular, only $left$ is then optimal in a Nash equilibrium for *Claire*.

Therefore, for the game in Figure 7 the choice *middle* of *Claire* is optimal in a correlated equilibrium, but not optimal in a Nash equilibrium. ♣

According to Theorem 6 and Remark 3, generalized Nash equilibrium is more restrictive than Bayesian equilibrium. Thus, the same relation emerges for incomplete information games between generalized Nash equilibrium and Bayesian equilibrium, as for complete information games between the analogous solution concepts of Nash equilibrium and correlated equilibrium.

6 Conclusion

It has been shown that Bayesian equilibrium actually constitutes a generalization of correlated equilibrium to incomplete information. To complete the picture, the new solution concept of generalized Nash equilibrium has been proposed as an incomplete information counterpart to Nash equilibrium. Since generalized Nash equilibrium is stronger than Bayesian equilibrium, it could be of interest in terms of future research to investigate how this new solution concept fares in applications with payoff uncertainty.

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