EPICENTER Summer Course on Epistemic Game Theory 2024

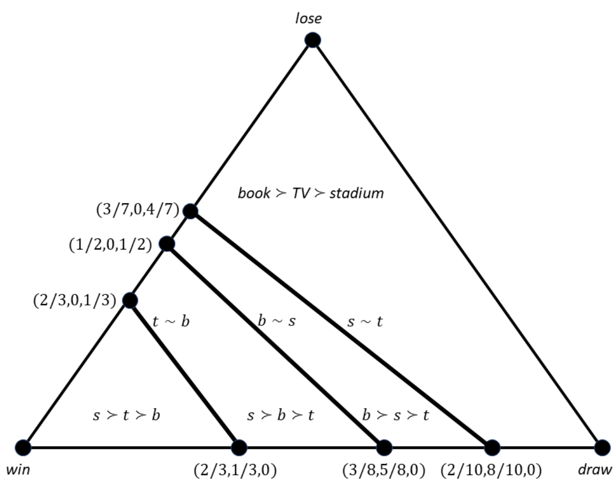
Incomplete Information, Unawareness and Psychological Games

Practice Exam

July 1–12, 2024

Problem 1. The football match.

Your local football team, with Barbara and Chris as big stars, is playing this evening. You have to decide whether to go to the *stadium*, to watch the match on TV, or to read a *book.* The relative pleasure you enjoy from these three options heavily depends on whether the team will *win, draw* or *lose.* Your conditional preference relation is given by the figure below.



(a) Are there any choices that are strictly, or weakly, dominated by other choices? If so, which one(s)?

(b) Which choices are rational and which are not?

(c) It turns out that this conditional preference relation has an expected utility representation. Explain why the relative utility differences are unique across all expected utility representations.

(d) Compute the unique expected utility representation u where the utility of choosing *stadium* is always 5, and $u(TV,\ win) = 1$. Which procedure do you use?

(e) For every irrational choice found in (b), find a randomized choice that strictly dominates it.

Problem 2. The karaoke competition.

Barbara, Chris and you are on a holiday in Japan. Next week Chris will organize a karaoke competition between Barbara and you in some fancy bar in Tokyo. The question is: How many hours should you practice for the competition this week? You can spend at most 20 hours, and for simplicity let us assume that you can only choose between practicing for 0, 10 or 20 hours. Barbara has more time, and she could in principle practice for 30 hours. Again, for simplicity, assume that Barbara can choose between practicing for 0, 10, 20 or 30 hours.

You are both cautious, and think that you will only win if you practice for *more* hours than the other person. Hence, if you both practice the same number of hours, then you think you will lose, and similarly for Barbara. If you win the competition, then your utility will increase by 40, and the same holds for Barbara.

You have never liked karaoke very much, and you hate practicing karaoke even more. More precisely, every hour you practice will decrease your utility by 1, independent of whether you win or lose, and Barbara knows this. The problem is that you don't know whether Barbara *hates* or *likes* practicing karaoke. If she *hates* practicing then, similarly to you, every hour she practices will decrease her utility by 1. If, on the other hand, she *likes* practicing, then every hour she practices will *increase* her utility by 1.

(a) Model this situation as a game with *incomplete information* between you and Barbara, by specifying the decision problems for you and Barbara for every possible utility function.

(b) Find the number of hours that you and Barbara can rationally practice under common belief in rationality for each of the possible utility functions. Which procedure do you use?

(c) Use the output of the procedure to generate a beliefs diagram in which all belief hierarchies express common belief in rationality, and every choice for you found in (b) is supported by some belief hierarchy. Which belief hierarchies for you are simple? Which are symmetric?

(d) Translate the beliefs diagram into an epistemic model.

(e) Show that each of your choices found in (b) can rationally be made by you under common belief in rationality with a *simple* belief hierarchy. Which concept do you use here?

(f) Consider the common prior $\hat{\pi}$ on choice-utility combinations given by

$$\hat{\pi}((0, u_1), (10, u_2^h)) = 0.3, \ \hat{\pi}((0, u_1), (30, u_2^l)) = 0.5 \text{ and } \hat{\pi}((20, u_1), (10, u_2^h)) = 0.2,$$

where u_1 is your (unique) utility function, u_2^h is Barbara's utility function if she *hates* practicing, and u_2^l is Barbara's utility function if she *likes* practicing. Show that $\hat{\pi}$ is a *canonical Bayesian equilibrium.*

(g) In view of (f), which choices can you rationally make under common belief in rationality with a *symmetric* belief hierarchy that uses *one theory per choice-utility pair*?

During the week of practicing you have heard Barbara singing all the time in the shower, and you therefore suspect that, very probably, she likes practicing. More precisely, let $p = (p_1, p_2)$ be the combination of beliefs on utility functions where p_1 assigns probability 1 to your unique utility function u_1, and p_2 assigns probability 0.2 to Barbara's utility function u_2^h and probability 0.8 to her utility function u_2^l.

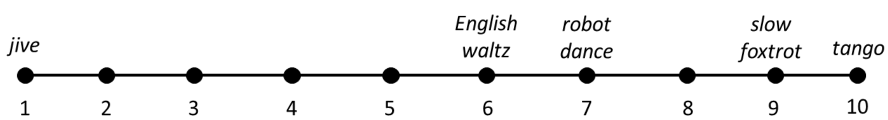
(h) For how many hours can you rationally practice under common belief in rationality and common belief in p? Which procedure do you use?

Problem 3. Dancing at Chris' birthday.

Chris will celebrate his birthday this evening, and he has asked Barbara and you last minute to give a dance performance together. You are both very surprised by this, but of course you cannot refuse a request on somebody's birthday. The problem is that Barbara and you must both do some shopping today and do not have time to practice together. You do not even have time to talk to each other on the phone to coordinate on the dance. Moreover, you only know how to dance the *jive* and the *tango*, whereas Barbara only masters the *English waltz* and the *slow foxtrot*, and you both know this. However, there is a brand new dance that has become very popular on Tik-Tok recently, which is the *robot dance*. It is very easy to dance, even you know how to do it after having watched the movie on Tik-Tok yesterday. Barbara, on the other hand, does not have Tik-Tok on her phone, and therefore you don't know whether Barbara is aware of the robot dance or not. But if she is aware, she would certainly know how to perform the robot dance.

The question for you and Barbara is: Which dance will you prepare for Chris' birthday? Remember that you must perform together, but it is very well possible that you will both perform different dances, which would be a strange spectacle. Only if Barbara is aware of the robot dance, it is possible, but not necessary, to perform the same dance – the robot dance.

You like uniformity, and prefer that Barbara and you perform dances that are similar to each other, whereas Barbara would find it funny if you both perform dances that are very different. More precisely, the figure below shows how different the various dances are by identifying every dance with an integer number on the line between 1 and 10.

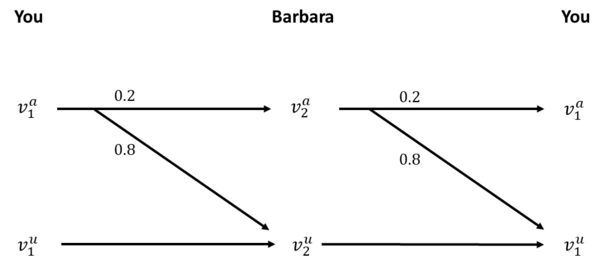


The difference between two dances is then the distance between the respective numbers. Your utility is 10 minus the distance between the dances that you and Barbara perform, whereas the utility for Barbara is simply the distance between these two dances.

(a) Model this situation as a game with *unawareness*, by specifying the decision problems for you and Barbara for each of the possible views. Make sure that there are two views for you and two views for Barbara.

(b) Find the dances that you can rationally prepare at each of your views under common belief in rationality. Which procedure do you use?

Assume now that you are quite confident that Barbara is not aware of the *robot dance*. Moreover, if Barbara *is* aware of the robot dance, she is quite confident that you are not. More precisely, assume that your belief hierarchy about views is given by the fixed belief combination p on views below.



Here, v_1^a denotes the view where you are aware of the robot dance, whereas v_1^u represents the view where you are unaware of the robot dance. Similarly for Barbara's views v_2^a and v_2^u.

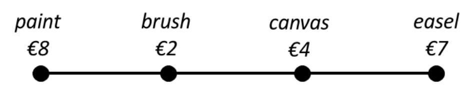
(c) Recall that you are aware of the robot dance. Which dance(s) will you rationally prepare under common belief in rationality and common belief in p? Which pair(s) of dances do you expect to see? To answer these questions, use the bottom-up version of a particular procedure. Which procedure?

(d) Use the output of the procedure to generate a beliefs diagram where all belief hierarchies express common belief in rationality and common belief in p, and where the choice(s) you found in (c) are supported by such belief hierarchies.

(e) Translate this beliefs diagram into an epistemic model.

Problem 4. A birthday present for Chris.

Before you and Barbara go to Chris' birthday you must, of course, buy him a present. Chris told you that he does not like expensive presents, and that you should therefore not spend more than 10 euros on it. Since Chris likes painting, Barbara and you consider buying him either some *paint,* a *brush,* a *canvas* or an *easel.* Hence, Barbara and you will each buy one gift out of these four. The figure below reveals the price of each of these four possible gifts, and indicates how different they are from each other by identifying them with points on a line.



Assume that the distance between every two neighbouring gifts is 1. Then, the distance between two gifts measures how different these two gifts are.

You and Barbara go shopping independently, and hence you both don't know what the other person will be buying. However, you would both like to surprise the other person by the gift you buy. More precisely, if you buy gift g and believe that Barbara believes that you buy gift g', then your utility will be

$$(\text{distance between } g \text{ and } g')^2 - \text{ price of } g,$$

and similarly for Barbara. The question is: What gift will you buy?

(a) Model this situation as a *psychological* game by specifying the decision problem for you. The decision problem for Barbara is similar. Do the utilities only depend on the first-order belief, or only on the second-order belief, or on both?

(b) Find the gifts that you can rationally buy under common belief in rationality. Which procedure do you use, and why? Do you expect to surprise Barbara by the gift(s) you buy?

Barbara has just called you with a new idea for buying the gift. She finds the *paint* and the *easel* too expensive, and you agree. These two gifts are therefore out of the question. Moreover, she proposes that you both write down the gift that you want to buy (the *brush* or the *canvas*) on a piece of paper, and that you reveal these together at Barbara's house. If you both happen to write down the same gift you will buy it together, and share the price equally between the two of you. Otherwise, you will both separately buy the (different) gifts, each paying the full amount. You both still have a preference for surprising the other person with the gift you are planning to buy. That is, if you plan to buy gift g and believe that Barbara believes that you plan to buy gift g', then your utility will be

$$(\text{distance between } g \text{ and } g')^2 - \text{ price you pay for } g,$$

and similarly for Barbara. Note that this price may be half of the price in case you buy it together.

(c) Model this situation as a psychological game by specifying the decision problem for you. The decision problem for Barbara is similar. Do the utilities only depend on the first-order belief, or only on the second-order belief, or on both?

(d) Consider the pair of beliefs (σ_1, σ_2) where σ_1 assigns probability 1 to you choosing the *brush,* and σ_2 assigns probability 0.6 to Barbara choosing *brush* and probability 0.4 to Barbara choosing *canvas.* Is (σ_1, σ_2) a *psychological Nash equilibrium*? Please explain why or why not.

(e) Find the gifts that you can rationally buy under common belief in rationality. Which procedure do you use, and why? Do you expect to surprise Barbara by the gift(s) you buy?