Epistemic Game Theory: Incomplete Information Part I: Common Belief in Rationality

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Introduction

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### Games with Uncertainty About Payoffs

- In games with incomplete information, players face uncertainty about their opponents' payoffs.
- More formally speaking, player *i* might be unsure whether the utility function of his opponent *j* is u<sub>j</sub> or u'<sub>j</sub>.
- There are various occasions in the real-world where incomplete information applies: firms unsure about competitors' profits, employers doubful about their employees' qualities, etc.

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# Interactive Reasoning about Rationality in the Presence of Incomplete Information

- A choice of player *i* is said to be **rational**, if it is optimal for some conjecture β<sub>i</sub> AND some utility function u<sub>i</sub>.
- Infusing such a notion of rationality into the interactive reasoning of a player enables the formulation of common belief in rationality also under incomplete information in the usual way:
  - player i believes his opponents to choose rationally,
  - player i believes his respective opponents to believe their respective opponents to choose rationally,
  - player i believes his respective opponents to believe their respective opponents to believe their respective opponents to choose rationally,

etc.

#### Story:

- Barbara and you are going together to another party.
- *You* wonder what colour you should wear.
- You prefer *blue* (4) to *green*, *green* (3) to *red*, *red* (2) to *yellow* (1), and dislike most to wear the same colour (0) as *Barbara*.
- However, you drank so much at the last party, that you forgot Barbara's colour preferences.
- You are still certain about Barbara also disliking most to wear the same colour (0) as you.
- Also, you remember that Barbara either prefers red (4) to yellow, yellow (3) to blue, blue (2) to green (1); or blue (4) to yellow, yellow (3) to green, green (2) to red (1).
- Question: Which colours can you rationally choose for tonight's party under common belief in rationality?

- *Yellow* is not optimal for any conjecture about *Barbara*'s choice.
- Thus, if Barbara believes in your rationality, then she assigns probability 0 to <u>yellow</u> for you.
- Barbara's 1<sup>st</sup> preferences: blue and green are not optimal for any conjecture about your choice that exludes yellow.
- Barbara's 2<sup>nd</sup> preferences: green and red are not optimal for any conjecture about your choice that exludes yellow.
- Suppose that *you* believe in *Barbara*'s **rationality** and that she believes in your **rationality**.
- Then, you believe Barbara not to choose green independent of her preferences and hence green is better than red for you.

- The following belief hierarchy expresses common belief in rationality and supports your choice of blue:
  - You believe Barbara to choose red and entertain "red-peaked"-preferences.
  - You believe Barbara to believe you to choose blue and entertain your preferences.
  - You believe Barbara to believe you to believe Barbara to choose red and entertain "red-peaked"-preferences.

etc.

Hence, you can rationally choose blue under common belief in rationality.

- The following belief hierarchy expresses common belief in rationality and supports your choice of green:
  - You believe Barbara to choose blue and entertain "blue-peaked"-preferences.
  - You believe Barbara to believe you to choose green and entertain your preferences.
  - You believe Barbara to believe you to believe Barbara to choose blue and entertain "blue-peaked"-preferences.

etc.

Hence, you can rationally choose green under common belief in rationality.

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### Outline

One-Person Perspective Form

Epistemic Model

Common Belief in Rationality

Generalized Iterated Strict Dominance

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## ONE-PERSON PERSPECTIVE FORM

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## Static Games with Incomplete Information

#### **Definition 1**

A game with incomplete information is a tuple

$$\Gamma = (I, (C_i, U_i)_{i \in I})$$

where

- I denotes a finite set of players,
- C<sub>i</sub> denotes a finite set of choices of player i,
- U<sub>i</sub> denotes a finite set of *utility functions* of player i, with  $u_i : \times_{i \in I} C_i \to \mathbb{R}$  for all  $u_i \in U_i$ .

- Two possible sources for uncertainty for a player *i*:
  - opponents' choice combinations c<sub>−i</sub> ∈ C<sub>−i</sub> ("strategic uncertainty")
  - opponents' utility function combinations  $u_{-i} \in U_{-i}$  ("payoff uncertainty")
- Complete information as a special case, if  $|U_i| = 1$  for all  $i \in I$
- FRAMEWORK: belief-free games with private values

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## **Decision Problems**

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**One-Person Perspective Form** 

Introduction

Given a game with incomplete information  $\Gamma = (I, (C_i, U_i)_{i \in I})$ , a decision problem for player *i* is a tuple

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$$\Gamma_i(u_i) = (D_i, D_{-i}, u_i|_{D_i \times D_{-i}})$$

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where  $D_i \subseteq C_i$  is a subset of *i*'s choice set,  $D_{-i} \subseteq C_{-i}$  is a subset of the set of opponents' choice combinations, and  $u_i \in U_i$  is a concrete utility function of player *i*.

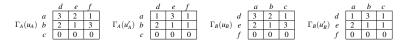
- A decision problem can be viewed as a one-person perspective model of a game-theoretic choice problem.
- A decision problem is called full, if  $D_i = C_i$  and  $D_{-i} = C_{-i}$ , and reduced, if it is not the case that  $D_i = C_i$  and  $D_{-i} = C_{-i}$ .

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#### A game with incomplete information:



■ All full decision problems of the game:



Example of a reduced decision problem for Alice:

$$\hat{\Gamma}_A(u'_A) \begin{array}{c} a \\ b \end{array} \left[ \begin{array}{c} d \\ 1 \\ 2 \\ 1 \end{array} \right]$$

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# Representation of a Game in terms of Full Decision Problems

#### **Definition 2**

Let  $\Gamma$  be a game with incomplete information. The tuple

$$\mathcal{O}^{\Gamma} := \left( \cup_{u_i \in U_i} \left\{ \Gamma_i(u_i) \right\} \right)_{i \in I}$$

is called the *one-person perspective form* of  $\Gamma$ , where

$$\Gamma_i(u_i) := (C_i, C_{-i}, u_i)$$

for all  $u_i \in U_i$  and for all  $i \in I$ .

#### $\blacksquare \mathcal{O}^{\Gamma} \text{ assembles all full decision problems per player.}$

The subsequent analysis of games with incomplete information makes use of O<sup>Γ</sup> – to formulate a solution concept generalizing ISD but also for the purposes of lucidity in exposition.

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## Illustration

#### **A** game with incomplete information $\Gamma$ :

• One-person perspective form  $\mathcal{O}^{\Gamma}$  of  $\Gamma$ :

$$\Gamma_{A}(u_{A}) \begin{array}{c} a \\ b \end{array} \left[ \begin{array}{c} c \\ 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} c \\ 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} a \\ r_{B}(u_{B}) \end{array} \right] \left[ \begin{array}{c} a \\ 1 \\ c \\ d \end{array} \right] \left[ \begin{array}{c} a \\ 1 \\ 0 \end{array} \right] \left[ \begin{array}{c} a \\ r_{B}(u_{B}') \end{array} \right] \left[ \begin{array}{c} c \\ 0 \\ c \\ d \end{array} \right] \left[ \begin{array}{c} a \\ 0 \\ 2 \end{array} \right] \left[ \begin{array}{c} a \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} a \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} a \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} a \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} a \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} a \\ 0 \\ 0 \end{array} \right] \left[ \begin{array}{c} a \\ 0 \end{array} \right] \left$$

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## **EPISTEMIC MODEL**

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## **Reasoning About the Game: Belief Hierarchies**

- With incomplete information a belief hierarchy for player i consists of
  - what i believes about his opponents' choices and payoffs (FIRST-ORDER BELIEF),
  - what i believes about what his opponents believe about their opponents' choices and payoffs (SECOND-ORDER BELIEF),
  - what i believes about what his opponents believe about what their opponents believe about their opponents' choices and payoffs (THIRD-ORDER BELIEF),
  - etc.
- Thus with incomplete information in a belief hierarchy player i holds a belief
  - about his opponents' choices and payoffs,
  - as well as about his opponents' belief hierarchies.

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## **Epistemic Model**

#### **Definition 3**

Let  $\Gamma$  be a game with incomplete information. An *epistemic model* 

$$\mathcal{M}^{\Gamma} = (T_i, b_i)_{i \in I}$$

of  $\Gamma$  provides for every player  $i \in I$ ,

- a finite set T<sub>i</sub> of types,
- a description map

$$b_i: T_i \to \Delta\left((C_j \times T_j \times U_j)_{j \in I \setminus \{i\}}\right)$$

that assigns to every type  $t_i \in T_i$  a probability distribution  $b_i[t_i]$  over the opponents' choice type utility function combinations.

Note that – similarly to the case of games with complete information – the description map  $b_i$  of player *i* provides for every type  $t_i \in T_i$ 

- a conjecture (i.e. belief about the opponents' choice combinations),
- a **belief** about the opponents' **types** (i.e. implicit belief hierarchies).

#### Consider the following epistemic model of the game:

 $(u_v \text{ represents your preferences } / u_B Barbara's "red-peaked" preferences / u'_B her "blue-peaked" preferences)$ 

- Type Sets:  $T_{vou} = \{t_v, t'_v\}$  $T_{Barbara} = \{t_B, t'_B\},\$ 
  - Beliefs for you:

$$b_{you}[t_y] = (blue, t_B, u_B)$$
  
$$b_{you}[t'_y] = \frac{1}{2} \cdot (red, t_B, u_B) + \frac{1}{4} \cdot (red, t_B, u'_B) + \frac{1}{4} \cdot (yellow, t'_B, u'_B)$$

#### Beliefs for Barbara:

 $b_{Barbara}[t_B] = (blue, t_v, u_v)$  $b_{Barbara}[t'_{B}] = (green, t_{v}, u_{v})$ 

 
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## Some Basic Epistemic Notions

For a given type  $t_i \in T_i$ , and a utility function  $u_i \in U_i$ , the expression

$$\mathbb{E}u_i(c_i,t_i) := \sum_{c_{-i}\in C_{-i}} b_i[t_i](c_{-i}) \cdot u_i(c_i,c_{-i})$$

denotes the *expected utility* for choosing  $c_i \in C_i$ .

A choice  $c_i \in C_i$  is *optimal* for the type utility function pair  $(t_i, u_i)$ , if

$$\mathbb{E}u_i(c_i, t_i) \geq \mathbb{E}u_i(c'_i, t_i)$$

holds for all  $c'_i \in C_i$ .

■ A choice  $c_i \in C_i$  is *rational* given some utility function  $u_i \in U_i$ , if there exists an epistemic model of the game with a type  $t_i \in T_i$  such that  $c_i$  is optimal for  $(t_i, u_i)$ .

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## COMMON BELIEF IN RATIONALITY

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## **Iterating Belief in Rationality**

- Infusing the idea of rationality into higher-order thinking of players in the presence of incomplete information imposes restrictions on belief hierarchies with extended basic uncertainty (choices and utility functions).
- A player can be said to believe in rationality, if he only assigns positive probability to CHOICE & TYPE & UTILITY FUNCTION triples s.t. the CHOICE is optimal for the TYPE (induced conjecture) & UTILITY FUNCTION.
- Full-fledged rationality reasoning then again gives rise to the key epistemic notion of

common belief in rationality:

believing in rationality,

believing your opponents to believe in rationality,

believing your opponents to believe their opponents to believe in rationality,

etc.

These ideas are now laid out formally within the framework of epistemic models.

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## **Belief in Rationality**

#### **Definition 4**

Let  $\Gamma$  be a game with incomplete information,  $\mathcal{M}^{\Gamma}$  an epistemic model of  $\Gamma$ ,  $i \in I$  some player, and  $t_i \in T_i$  some type of player *i*. The type  $t_i$  believes in rationality, if  $t_i$  only assigns positive probability to choice type utility function combinations

$$((c_1, t_1, u_1), \ldots, (c_{i-1}, t_{i-1}, u_{i-1}), (c_{i+1}, t_{i+1}, u_{i+1}), \ldots, (c_n, t_n, u_n))$$

such that  $c_j$  is optimal for  $(t_j, u_j)$  for all  $j \in I \setminus \{i\}$ .

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## Higher-order Beliefs in Rationality

#### **Definition 5**

Let  $\Gamma$  be a game with incomplete information,  $\mathcal{M}^{\Gamma}$  an epistemic model of  $\Gamma$ ,  $i \in I$  some player, and  $t_i \in T_i$  some type of player *i*.

- The type  $t_i$  expresses 1-fold belief in rationality, if  $t_i$  believes in rationality.
- Let k > 1. The type  $t_i$  expresses k-fold belief in rationality, if  $t_i$ only assigns positive probability to opponents' types that express (k-1)-fold belief in rationality.

Let l > 1. The type  $t_i$  expresses up to *l*-fold belief in rationality, if  $t_i$ expresses k-fold belief in rationality for all k < l.

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## **Common Belief in Rationality**

#### **Definition 6**

Let  $\Gamma$  be a game with incomplete information,  $\mathcal{M}^{\Gamma}$  an epistemic model of  $\Gamma$ ,  $i \in I$  some player, and  $t_i \in T_i$  some type of player *i*. The type  $t_i$  expresses *common belief in rationality*, if  $t_i$  expresses *k*-fold belief in rationality for all  $k \geq 1$ .

#### Remark:

If all types in an epistemic model express belief in rationality, then all types express common belief in rationality.

This also holds under incomplete information!

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## **Rational Choice under CBR**

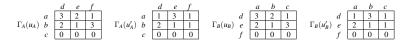
#### **Definition 7**

Let  $\Gamma$  be a game with incomplete information,  $i \in I$  some player,  $c_i \in C_i$  some choice of player *i*, and  $u_i \in U_i$  some utility function of player *i*. The choice  $c_i$  is *rational under common belief in rationality* given  $u_i$ , if there exists an epistemic model  $\mathcal{M}^{\Gamma}$  of  $\Gamma$  with some type  $t_i \in T_i$  of player *i* such that

*t<sub>i</sub>* expresses common belief in rationality,

•  $c_i$  is optimal for  $(t_i, u_i)$ .

Consider the following game in one-person perspective form:



Suppose the following epistemic model of this game:

• 
$$T_{Alice} = \{t_A, t'_A\}$$
 and  $T_{Bob} = \{t_B, t'_B\}$ ,

• 
$$b_{Alice}[t_A] = (d, t_B, u_B)$$
 and  $b_{Alice}[t'_A] = (e, t'_B, u'_B)$ ,

•  $b_{Bob}[t_B] = (a, t_A, u_A)$  and  $b_{Bob}[t'_B] = \frac{1}{2}(b, t_A, u'_A) + \frac{1}{2}(c, t'_A, u_A).$ 

**Types**  $t_A$  and  $t_B$  believe in rationality.

Types  $t_A$  and  $t_B$  also express common belief in rationality.

#### The game in one-person perspective form:



Suppose the following epistemic model of this game:

- All types believe in rationality, and thus by the "SHORTCUT" also express common belief in rationality.
- Consequently, you can rationally choose blue as well as green under common belief in rationality given utility function uy.

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## GENERALIZED ITERATED STRICT DOMINANCE

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## Strict Dominance and Incomplete Information

- The notion of strict dominance is first adapted to decision problems.
- Via decision problems it is then carried into the setting of games with incomplete information in one-person perspective form.
- Subsequently, an iteration of strict dominance in the one-person perspective form gives rise to:

GENERALIZED ITERATED STRICT DOMINANCE (GISD)

GISD constitutes a **solution concept** for incomplete information.

### Strict Dominance In Decision Problems

#### **Definition 8**

Let  $\Gamma$  be a game with incomplete information,  $\mathcal{O}^{\Gamma}$  the one-person perspective form of  $\Gamma$ ,  $i \in I$  some player,  $\Gamma_i(u_i)$  some decision problem of player *i*, and  $c_i \in D_i$  some choice of player *i*. The choice  $c_i$ is *strictly dominated*, if there exists some mixed choice  $r_i \in \Delta(D_i)$  of player *i* such that

$$u_i(c_i, c_{-i}) < \sum_{c'_i \in D_i} r_i(c'_i) \cdot u_i(c'_i, c_{-i})$$

for all  $c_{-i} \in D_{-i}$ .

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## **Generalized Iterated Strict Dominance**

#### **Definition 9**

Let  $\Gamma$  be a game with incomplete information and  $\mathcal{O}^{\Gamma}$  the one-person perspective form of  $\Gamma$ .

**Round 1:** For all  $i \in I$  and for all  $u_i \in U_i$  consider the initial decision problem  $\Gamma_i^0(u_i) := (C_i^0(u_i), C_{-i}^0(u_i), u_i)$  from  $\mathcal{O}$  where  $C_i^0(u_i) := C_i$  and  $C_{-i}^0(u_i) := C_{-i}$ .

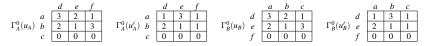
- Step 1.1: Set  $C^1_{-i}(u_i) := C^0_{-i}(u_i)$ .
- Step 1.2: Form the reduced decision problem  $\Gamma_i^1(u_i) := (C_i^1(u_i), C_{-i}^1(u_i), u_i)$ , where  $C_i^1(u_i) \subseteq C_i^0(u_i)$  only contains choices of *i* that are not stric. dom. in  $(C_i^0(u_i), C_{-i}^1(u_i), u_i)$ .
- **Round** k > 1: For all  $i \in I$  and for all  $u_i \in U_i$  consider the reduced decision problem  $\Gamma_i^{k-1}(u_i) := (C_i^{k-1}(u_i), C_{-1}^{k-1}(u_i), u_i)$  from the previous round k 1.
  - Step k.1: Form the set  $C_{-i}^k(u_i)$  by eliminating from  $C_{-i}^{k-1}(u_i)$  every opponents' choice combination that contains for some opponent  $j \in I \setminus \{i\}$  a choice which is strictly dominated for all  $u_j \in U_j$  in  $\Gamma_i^{k-1}(u_i)$  of the previous round k-1.
  - Step k.2: Form the reduced decision problem Γ<sup>k</sup><sub>i</sub>(u<sub>i</sub>) := (C<sup>k</sup><sub>i</sub>(u<sub>i</sub>), C<sup>k</sup><sub>-i</sub>(u<sub>i</sub>), u<sub>i</sub>), where C<sup>k</sup><sub>i</sub>(u<sub>i</sub>) ⊆ C<sup>k-1</sup><sub>i</sub>(u<sub>i</sub>) only contains choices of *i* not strict. dom. in (C<sup>k-1</sup><sub>i</sub>(u<sub>i</sub>), C<sup>k</sup><sub>-i</sub>(u<sub>i</sub>), u<sub>i</sub>).
- Output: The set

$$GISD := \times_{i \in I} GISD_i \subseteq \times_{i \in I} (C_i \times U_i)$$

is called *Generalized Iterated Strict Dominance*, where for every player  $i \in I$  the set  $GISD_i \subseteq C_i \times U_i$  only contains choice utility function pairs  $(c_i, u_i) \in C_i \times U_i$  such that  $c_i \in C_i^k(u_i)$  for all  $k \ge 0$ .



One-person perspective form = initial decision problems of GENERALIZED ITERATED STRICT DOMINANCE:



#### Round 1:

For *Alice*, choice *c* is strictly dominated in the initial decision problems  $\Gamma_A^0(u_A)$  and  $\Gamma_A^0(u'_A)$ .

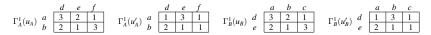
**Eliminate** choice *c* from  $\Gamma_A^0(u_A)$  and from  $\Gamma_A^0(u'_A)$ .

Similarly, eliminate choice *f* from *Bob*'s initial decision problems  $\Gamma_B^0(u_B)$  and  $\Gamma_B^0(u'_B)$ .

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#### Output from **Round 1** = 1-fold reduced decision problems:



#### Round 2:

- For *Bob*, choice *f* is not in any of his decision problems  $\Gamma_B^1(u_B)$  and  $\Gamma_B^1(u'_B)$ .
- **Eliminate** choice *f* from  $\Gamma_A^1(u_A)$  and from  $\Gamma_A^1(u'_A)$ .
- Similarly, eliminate choice *c* from *Bob*'s 1-fold reduced decision problems  $\Gamma_B^1(u_B)$  and  $\Gamma_B^1(u'_B)$ .

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$$\begin{pmatrix} C_{A}^{1}(u_{A}), C_{B}^{2}(u_{A}), u_{A} \end{pmatrix} \stackrel{a}{=} \frac{d}{2} \frac{e}{2} \\ \frac{3}{2} \frac{2}{1} \\ \frac{2}{1} \\ \frac{1}{2} \\ \frac{1$$

#### Round 2 (continued):

- Then, choice *b* becomes strictly dominated for *Alice* in the decision problem  $(C_A^1(u_A), C_B^2(u_A), u_A)$ .
- Eliminate choice *b* from the decision problem  $(C_A^1(u_A), C_B^2(u_A), u_A).$
- Similarly, eliminate choice *e* from *Bob*'s decision problem  $(C_B^1(u_B), C_A^2(u_B), u_B)$ .

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Output from **Round 2** = 2-fold reduced decision problems:

#### Round 3:

The algorithm stops.

Output:

$$GISD_{Alice} = \{(a, u_A), (a, u'_A), (b, u'_A)\}$$

and

$$GISD_{Bob} = \{(d, u_B), (d, u'_B), (e, u'_B)\}.$$

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One-person perspective form = initial decision problems of GENERALIZED ITERATED STRICT DOMINANCE:



#### Round 1:

- For you, choice <u>yellow</u> is strictly dominated in the initial decision problem  $\Gamma_{y}^{0}(u_{y})$  by  $\frac{1}{2}$  blue  $+\frac{1}{2}$  green.
- Eliminate choice yellow from  $\Gamma_{y}^{0}(u_{y})$ .
- For Barbara, choice green is strictly dominated in the initial decision problem  $\Gamma_B^0(u_B)$  by  $\frac{1}{2}$  red  $+\frac{1}{2}$  yellow.
- Eliminate choice green from  $\Gamma_B^0(u_B)$ .
- For Barbara, choice red is strictly dominated in the initial decision problem  $\Gamma_B^0(u'_B)$  by  $\frac{1}{2}$  green  $+\frac{1}{2}$  yellow.
- Eliminate choice *red* from  $\Gamma_B^0(u_B)$ .

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#### Output from **Round 1** = 1-fold reduced decision problems:

	blue	green	red			blue	green	red			blue	green	red	
blue	0	4	4	4	blue	0	2	2	2	blue	0	4	4	4
$\Gamma_y^1(u_y)$ green	3	0	3	3	$\Gamma_B^1(u_B)$ red	4	4	0	4	$\Gamma_B^1(u'_B)$ green	2	2	0	2
red	2	2	0	2	yellow	3	3	3	0	yellow	3	3	3	0

#### Round 2:

- For *you*, choice *yellow* is not in any of your decision problems as *you* only have one decision problem  $\Gamma_V^1(u_Y)$  and it is not in there.
- Eliminate choice *yellow* from  $\Gamma_B^1(u_B)$  and from  $\Gamma_B^1(u'_B)$ .
- For every choice of *Barbara* there exists a 1-fold reduced decision problem that contains it.
- Consequently, nothing can be eliminated in terms of *Barbara*'s choices from  $\Gamma_{v}^{1}(u_{v})$ .

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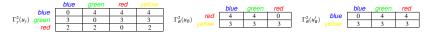
#### Round 2 (continued):

- Choice *blue* becomes strictly dominated by *yellow* for *Barbara* in the decision problem  $(C_B^1(u_B), C_v^2(u_B), u_B)$ .
- Eliminate choice *blue* from  $(C_B^1(u_B), C_y^2(u_B), u_B)$ .
- Choice green becomes strictly dominated by yellow for Barbara in the decision problem  $(C_B^1(u'_B), C_y^2(u'_B), u'_B)$ .
- Eliminate choice green from  $(C_B^1(u'_B), C_v^2(u'_B), u'_B)$ .

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#### Output from **Round 2** = 2-fold reduced decision problems:



#### Round 3:

- For *you*, the choices *blue*, *green*, and *red* are in your only decision problem  $\Gamma_y^2(u_y)$ .
- For *Barbara*, choice *green* is not in any of her decision problems  $\Gamma_B^2(u_B)$  and  $\Gamma_B^2(u'_B)$ .
- Eliminate choice green from  $\Gamma_y^2(u_y)$ .

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#### Round 3 (continued):

- Choice *red* becomes strictly dominated by *green* for *you* in the decision problem  $(C_v^2(u_v), C_B^3(u_v), u_v)$ .
- Eliminate choice *red* from  $(C_y^2(u_y), C_B^3(u_y), u_y)$ .

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#### Output from **Round 3** = 3-fold reduced decision problems:

		blue	red	yellow		blue	green	red		blue	green	red
$\Gamma_{y}^{3}(u_{y})$	blue	0	4	4	$\Gamma_B^3(u_B)$ red	4	4	0	$\Gamma_B^3(u'_B)$ blue	0	4	4
$1_y(u_y)$	green	3	3	3	<sup>1</sup> B( <sup>IIB</sup> ) yellow	3	3	3	<sup>1</sup> <sup>B(uB)</sup> yellow	3	3	3

#### Round 4:

- For you, choice red is no longer in any of your decision problems, i.e. it is not in  $\Gamma_{y}^{3}(u_{y})$ .
- Eliminate choice *red* from  $\Gamma_B^3(u_B)$  and from  $\Gamma_B^3(u'_B)$ .
- For Barbara, the choices red, blue, and yellow are in some 3-fold decision problem of hers.



#### Round 4 (continued):

- Choice *yellow* becomes strictly dominated by *red* for *Barbara* in the decision problem  $(C_B^3(u_B), C_y^4(u_B), u_B)$ .
- Eliminate choice yellow from  $(C_B^3(u_B), C_v^4(u_B), u_B)$ .

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Output from **Round 4** = 4-fold reduced decision problems:



#### Round 5:

The algorithm stops.

Output:

$$GISD_{you} = \{(blue, u_y), (green, u_y)\}$$

and

$$GISD_{Barbara} = \{(red, u_B), (blue, u'_B), (yellow, u'_B)\}$$

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Introduction	One-Person Perspective Form	Epistemic Model	CBR	GISD	Characterization
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## **CHARACTERIZATION**

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- - The epistemic condition of common belief in rationality and the solution concept of Generalized Iterated Strict Dominance are now related to each other in general.
  - The two notions gave rise to the same choices in the two specific games considered.
  - Indeed this relationship turns out to be true in general: CBR and GISD are equivalent in terms of their results.
    - Thus, the meaning of GISD in terms of interactive thinking is captured by CBR.

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 Alternatively phrased, CBR can be procedurally characterized by GISD.

## A Generalization of Pearce's Lemma

#### Theorem 10

Let  $\Gamma$  be a game with incomplete information,  $\mathcal{O}$  the one-person perspective form of  $\Gamma$ , *i* some player,  $\Gamma_i(u_i)$  some decision problem of player *i*, and  $c_i \in D_i$  some choice of player *i*. The choice  $c_i$  is strictly dominated in  $\Gamma_i(u_i)$ , if and only if, there exists some conjecture  $\beta_i \in \Delta(D_{-i})$  such that  $c_i$  is optimal for  $(\beta_i, u_i)$ .

Introduction	One-Person Perspective Form	Epistemic Model	GISD 000000000000000000000000000000000000	Characterization ○○○●○○

## Proof

- Define a two player game  $\Gamma' = ((i,j), \{C'_i, C'_i\}, \{u'_i, u'_i\})$  with complete information, where
  - $C'_i := D_i$ ,
  - $\bullet \quad C_j' := D_{-i},$
  - $u'_i(d_i, d_{-i}) := u_i(d_i, d_{-i})$  for all  $d_i \in C'_i$  and for all  $d_{-i} \in C'_j$ ,
  - and  $u'_j(d_{-i}, d_i) := 0$  for all  $d_{-i} \in C'_j$  and for all  $d_i \in C'_i$ .
- Observe that the choice c<sub>i</sub> ∈ D<sub>i</sub> is strictly dominated in the decision problem Γ<sub>i</sub>(u<sub>i</sub>), if and only if, it is strictly dominated in the game Γ'.
- By PEARCE'S LEMMA applied to the game  $\Gamma'$ , if follows that  $c_i$  is strictly dominated in  $\Gamma'$ , if and only if, there exists no conjecture  $\beta_i \in \Delta(C'_i)$  such that  $c_i$  is optimal for  $(\beta_i, u'_i)$ .
- Since  $C'_j = D_{-i}$  and  $u'_i = u_i$ , it follows that  $c_i \in D_i$  is strictly dominated in the decision problem  $\Gamma_i(u_i)$ , if and only if, there exists no conjecture  $\beta_i \in \Delta(D_{-i})$  such that  $c_i$  is optimal for  $(\beta_i, u_i)$ .

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# Epistemic Characterization of Generalized Iterated Strict Dominance

#### Theorem 11

Let  $\Gamma$  be a game with incomplete information,  $i \in I$  some player,  $c_i \in C_i$  some choice of player i, and  $u_i \in U_i$  some utility function of player i. The choice  $c_i$  is rational under common belief in rationality given  $u_i$ , if and only if,  $(c_i, u_i) \in GISD_i$ .

## Summary

- The reasoning concept of common belief in rationality has been extended to static games with incomplete information.
- Its algorithmic characterization has brought to light a basic (non-equilibrium) solution concept:

#### **Generalized Iterated Strict Dominance (GISD)**

The ready-made algorithm of GISD could constitute a useful tool for economists when analyzing situations with payoff uncertainty.

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