Additive Context-Dependent Preferences

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EPICENTER Working Paper No. 31 (2024)

"Money equals utility" remains the benchmark across a vast range of economics- and game theory-experiments. Subjecting this assumption to experimental testing requires an empirically tractable theory of context-dependent preferences. This paper presents novel foundations for expected utility based on context-dependent preferences in games and other state-dependent decision problems. Crucially, these behavioral foundations do not require empirically implausible comparisons of alternatives across different states. Moreover, they substantially relax previous diversity-type assumptions, significantly expanding the range of problems that can be accommodated and paving the way for a general state-dependent expected utility representation for any multi-act, multi-state context-dependent preference system. A central application is to direct utility measurement in games, enabling a causal understanding of how e.g. risk- and social-preferences affect strategic choice.

Experimental economics demonstrates that real-life agents systematically deviate from standard predictions based on individual payoff maximization. Examples include cooperation in Prisoner's dilemmas (Cooper et al. 1996), equal splits in ultimatum games (Güth et al. 1982), and the frequent incidence of payoff-dominant, non-risk-dominant coordination (Jagau 2024).

Importantly, these findings do not necessarily contradict game-theory's central tenet of rational strategic behavior. Since it is common practice in experiments to assume that individual monetary payoffs approximate utility, most studies implicitly test a "money equals utility"-assumption in conjunction with strategic rationality. This is problematic, not least since a long-standing line of research on social preferences (e.g. Andreoni and Miller 2002) suggests that experimental subjects *do* tend to maximize a fixed preference over monetary allocations — just not one that depends on their individual payoffs alone.

Abandoning the 'axiom' that individual monetary payoff equal utility is not straightforward. The classical 'foundation' for utility in games (von Neumann and Morgenstern 1953) simply assumes that utility values in game matrices can be derived from preferences over lotteries implementing a game's monetary outcomes with known probabilities. If one wants to move beyond correlating behavior in games with exogenously measured utility functions, a different approach is needed: To *directly* measure utility over outcomes of strategic interactions, it must be derived from *context-dependent preferences*, that is preference rankings conditional on different game situations.

This extended abstract develops an axiomatic theory of additive context-dependent preferences. To define context-dependent preferences, let (A, X) be a decision problem with $A = \{a_1, \ldots, a_n\}$ a finite set of acts and $X = \{x^1, \ldots, x^m\}$ a finite set of states. Then, letting $\Delta(X)$ denote the set of probability measures on X, a **system of context-dependent preferences** is a mapping $\succsim \in \mathscr{P}(A \times A)^{\Delta(X)}$. I.e., at each $p = (p^1, \ldots, p^m) \in \Delta(X)$, $\succsim$ induces a local preference $\succsim_p \subseteq A \times A$. As usual, $\sim_p$ and $\succ_p$ will denote symmetric and asymmetric parts of the local preference.

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In the context of **game theory** (the leading application considered in this paper), A represents pure strategies of a player in a simultaneous-move game, X represents combinations of opponents' pure strategies, and $\Delta(X)$ represents all possible beliefs the considered player could have regarding his opponents' pure-strategy combinations.[1]

We seek conditions on $\succsim$ that characterize an **expected utility representation**.

Definition 0.1. *(Expected Utility Representation)*
The system $\succsim$ has an expected-utility representation if there exists a function $u : A \times X \to \mathbb{R}$ such that $a_i \succsim_p a_j \Leftrightarrow \sum_{s=1}^{m} p^s[u(a_i, x^s) - u(a_j, x^s)] \geq 0$ for all $a_i, a_j \in A$ and all $p \in \Delta(X)$.

In **game theory**, expected utility as above allows to elicit utility over game outcomes based on belief-conditional rankings over strategies. Without assuming specific preferences over game outcomes (e.g. "money = utility"), one can then empirically determine whether a player acts like a rational expected-utility maximizer and, if yes, what utility function is maximized.[2]

Two previous papers study additive context-dependent preferences. Gilboa and Schmeidler (2003) consider expected utility in games. For two acts, they prove a representation theorem like Theorem 2.6 below. For $n > 2$ acts, they impose a *diversity restriction*, demanding that, for each triple and each quadruple of acts, every permutation of transitive rankings among the acts is observed at some $p \in \Delta(X)$. By contrast, this article is working towards a globally applicable representation theorem without any constraints on admissible datasets. Perea (2020) proves a general representation theorem for additive context-dependent preferences as considered in this paper. His conditions rely on consistency of observed preferences with a second, hypothetical preference system that *uniformly increases* the preference for one act compared to all other acts. Similarly, Perea (2023) provides an alternative representation imposing conditions on *extended* preference systems for any *signed* belief. By contrast, the conditions here rely exclusively on the decision-maker's observed behavior at standard beliefs in $\Delta(X)$.

The rest of this extended abstract proceeds as follows. Section 1 introduces a simplified representation of additive context-dependent preferences in terms of *utility differences* that will be used throughout the investigation. Section 2 concerns pairwise comparisons, with Theorem 2.6 giving a representation of preferences over a pair of acts in terms of a vector of utility differences. Section 3 considers multi-act settings. Standard transitivity at all beliefs p together with two new conditions, called *scaling transitivity* and *transitive preference sensitivity*, is shown to be necessary and sufficient for expected utility in any three-act problem and expected utility in any multi-act problem, such that $\succsim$ exhibits preference reversals for every pair of acts. In addition, a *decomposition of payoff matrices* is developed to show that representations for any two sets of acts that are *not* connected by preference reversals can be combined into a global representation without further assumptions. Already these findings consider a significant improvement over previous results.

[1]Context-dependent preferences also apply to other settings: E.g. for **choice under uncertainty**, one gets state-dependent preferences. A represents alternatives for a decision maker (DM), X represents possible states of the world, and $\Delta(X)$ represents all possible beliefs the DM could have regarding the probabilities of different states. And for **social choice**, one gets a social contract theory setting behind a veil of ignorance. I.e., A represents distributional policies considered by a decision-maker, X represents possible resource allocations across a population of m individuals, and $\Delta(X)$ represents all possible beliefs the decision-maker could attach to their chance of ending up in the shoes of each individual.

[2]For **decision under uncertainty**, expected utility as above is reminiscent of state-dependent expected utility in, e.g., Karni et al. (1987). Different from that paper, the model considered here does not require the DM to form empirically implausible *hypothetical preferences* regarding act-state tuples. Instead, the DM need only form preferences over acts conditional on each state. For **social choice**, expected utility yields a version of Harsanyi's (1975) social contract theory were the decision-maker assigns subjective ex-post preferences to each individual in society before committing to a belief regarding their chance of ending up in each individual's role.

1 Difference Representations

Throughout, identify the states in X with m-dimensional standard unit vectors. The set of probability distributions $\Delta(X)$ then corresponds to the standard simplex in $\mathbb{R}^m$. In this view, $\Delta(X)$ lies in the hyperplane of signed distributions $H^1(\mathbf{1}) = \{v \in \mathbb{R}^m | \, v \cdot \mathbf{1} = 1\}$ where $\mathbf{1}$ is the m-vector of ones. Every pair $v, w \in H^1(\mathbf{1})$ is related by a unique displacement vector $v - w \in H^0(\mathbf{1}) = \{v \in \mathbb{R}^m | \, v \cdot \mathbf{1} = 0\}$.

Furthermore, it will be easier to work with utility differences $d^s_{ij} = u(a_i, x^s) - u(a_j, x^s)$, $a_i, a_j \in A$, $x^s \in X$ rather than with utilities $u(a_i, x^s)$, $a_i \in A$, $x^s \in X$.

Definition 1.1. *(Difference Representation)*
$\succsim$ has a **difference representation** *on (A, X) if, for each each pair $a_i, a_j \in A$, there are vectors $d_{ij} \in \mathbb{R}^m$ such that*

1. *$a_i \succsim_p a_j \Leftrightarrow \sum_{s=1}^m p^s d^s_{ij} = p \cdot d_{ij} \geq 0$, for all $a_i, a_j \in A$, $p \in \Delta(X)$ (Pairwise Representation)*
2. *$d_{ij} + d_{jk} = d_{ik}$, for all $a_i, a_j, a_k \in A$ (Strong Transitivity)*

Note that the pairwise representation together with $a_i \succsim_p a_j \Leftrightarrow a_j \precsim_p a_i$ automatically implies $d_{ij} = -d_{ji}$ for all $a_i, a_j \in A$ and (in particular) $d_{ii} = 0$ for all $a_i \in A$. So any given difference representation is summarized by a matrix $\mathcal{D} = [d_{12}, d_{13}, \ldots, d_{1n}, d_{23}, \ldots, d_{n-1,n}] \in \mathbb{R}^{m \times \binom{n}{2}}$.

It easily shown that difference representations and expected-utility representations are equivalent. (The proof is left to the reader.)

Observation 1.2. *(Equivalence of Representations)*
Let (A, X) be a decision problem and let $\succsim$ be a system of context-dependent preference relations for (A, X). Then $\succsim$ admits an expected-utility representation iff it admits a difference representation.

Definition 1.1 reveals a central structural requirement enabling expected utility — strong transitivity of difference vectors. Strong transitivity refines standard transitivity (Definition 3.1 below) in ways reminiscent of distances on a line where, e.g., the distance from locality i to locality j plus the distance from j to k must equal the distance from i to k. Saari (2014), (2021) shows that refining transitivity in this way is crucial in order to avoid cyclical outcomes in multi-alternative settings whenever systems of rankings are aggregated like in Definitions 0.1 and 1.1. Conditions ensuring strong transitivity are explored in Sections 3. Before, Section 2 recaps difference representations for pairwise comparisons as in (e.g.) Gilboa and Schmeidler (2003).

2 Pairwise Comparisons

It is useful to introduce some additional notation around pairwise comparisons. First, for any two acts $a_i, a_j \in A$, let $P^0_{ij} = \{p \in \Delta(X) | a_i \sim a_j\}$, $P^+_{ij} = \{p \in \Delta(X) | a_i \succ a_j\}$, and $P^-_{ij} = \{p \in \Delta(X) | a_i \prec a_j\}$. Analogously, define $X^0_{ij} = \{x \in X | a_i \sim_x a_j\}$, $X^+_{ij} = \{x \in X | a_i \succ_x a_j\}$, and $X^-_{ij} = \{x \in X | a_i \prec_x a_j\}$.

Throughout, $\succsim$ is assumed to be complete at all $p \in \Delta(X)$.

Property 2.1. *(Completeness)*
$\succsim$ is complete if $a_i \succsim_p a_j$ or $a_j \succsim_p a_i$ for every $a_i, a_j \in A$ and $p \in \Delta(X)$.

By completeness, we must have $\Delta(X) = P^0_{ij} \cup P^+_{ij} \cup P^-_{ij}$ and $X = X^0_{ij} \cup X^+_{ij} \cup X^-_{ij}$ for all $a_i, a_j \in A$. Using the sets $P^0_{ij}, P^+_{ij}, P^-_{ij}$, we can distinguish different types of pairwise comparisons in context-dependent preference systems. They are familiar from game theory:

Definition 2.2. *(Types of Pairwise Comparisons)*
Let (A, X) be a decision problem, and let $\succsim$ be a system of context-dependent preference relations $\succsim_p$ for (A, X). Consider any two acts $a_i, a_j \in A$. We can distinguish the following configurations:

- *a_i, a_j are* **observationally equivalent** *given $\succsim$ if $P^0_{ij} = \Delta(X)$.*
- *$\succsim$ exhibits* **preference reversals** *for a_i, a_j if both P^+_{ij} and P^-_{ij} are non-empty.*
- *a_i* **weakly dominates** *a_j if $\Delta(X) = P^0_{ij} \cup P^+_{ij}$ and P^+_{ij} is non-empty.*

Observational equivalence, preference reversals, and weak dominance are mutually exclusive and jointly exhaustive of all possible pairwise comparisons for a_i and a_j given a *complete* preference $\succsim$. Note that observational equivalence includes the case of *ii*-reflexive comparisons. Also note that *strict* dominance of a_i over a_j would correspond to the special case of weak dominance where P^0_{ij} is empty.

Three more properties are needed for pairwise comparisons to have a difference representation:

Property 2.3. *(Continuity)*
$\succsim$ is continuous if $a_i \succ_p a_j$ and $a_j \succ_q a_i$ for $p, q \in \Delta(X)$ implies that there exists $\lambda \in (0,1)$ such that $a_i \sim_r a_j$ for $r = \lambda p + (1-\lambda)q$.

Property 2.4. *(Preservation of Indifference)*[3]
$\succsim$ preserves indifference if $a_i \sim_p a_j$ and $a_i \sim_q a_j$ for $p, q \in \Delta(X)$ implies that $a_i \sim_r a_j$ for every $r = \lambda p + (1-\lambda)q,\ \lambda \in (0,1)$.

Property 2.5. *(Preservation of Strict Preference)*
$\succsim$ preserves strict preference if $a_i \succ_p a_j$ and $a_i \succsim_q a_j$ for $p, q \in \Delta(X)$ implies that $a_i \succ_r a_j$ for every $r = \lambda p + (1-\lambda)q,\ \lambda \in (0,1)$.

Call a preference system $\succsim$ satisfying Properties 2.1 and 2.3-2.5 **pairwise linear**. Now prove:

Theorem 2.6. Pairwise Representation
For every $a_i, a_j \in A$, $\succsim$ admits a difference representation on $(\{a_i, a_j\}, X)$ iff it is pairwise linear.

For the proof, we need additional notation: For any set of vectors $V \subseteq \mathbb{R}^m$, let $\langle V \rangle$ denote the linear span of V. Whenever $V = \{v\}$, slightly abuse notation and write $\langle v \rangle$ for the linear span of V. Furthermore, for any $a_i, a_j \in A$ we will write $\Pi_{ij} := \langle P^0_{ij} \rangle$ for the span of the ij-indifference set.

Next, for any set of vectors $U \subseteq \mathbb{R}^m$, $\mathcal{O}(\langle U \rangle) = \{v \in \mathbb{R}^m |\ v \cdot u = 0,\ \forall u \in U\}$ will denote the orthogonal complement of $\langle U \rangle$. Lastly, for any vector $v \in \mathbb{R}^m$, $H^0(v) = \mathcal{O}(\langle v \rangle)$ denotes the unique hyperplane that is orthogonal to v and that passes through the origin $\mathbf{0} \in \mathbb{R}^m$. Every hyperplane $H^0(v)$ separates two open half spaces $H^+(v) = \{u \in \mathbb{R}^m |\ v \cdot u > 0\}$ and $H^-(v) = \{u \in \mathbb{R}^m |\ v \cdot u < 0\}$.

The following Lemma is an important preparatory result.

Lemma 2.7. (*Properties of* Π_{ij})
Let $\succsim$ be pairwise linear. Then

[3] An implication of Property 2.4 is that $r \in \mathrm{Conv}(P)$ implies $a_i \sim_r a_j$, whenever P is a finite subset of P^0_{ij}. Here $\mathrm{Conv}(P)$ denotes the convex hull of P. The straightforward inductive proof is left to the reader.

1. $\Pi_{ij} \cap \Delta(X) = P^0_{ij}$ *for all* $a_i, a_j \in A$,

2. if $\succsim$ *exhibits reversals for* $a_i, a_j \in A$*, then* Π_{ij} *is a hyperplane,*

3. if $\succsim$ *exhibits no reversals for* $a_i, a_j \in A$*, then* Π_{ij} *is a subspace spanned by* X^0_{ij}.

Proof. (*Lemma 2.7*)
Part 1: The direction $P^0_{ij} \subseteq \Pi_{ij} \cap \Delta(X)$ is clear. For $\Pi_{ij} \cap \Delta(X) \subseteq P^0_{ij}$, take any $p \in \Pi_{ij} \cap \Delta(X)$. There then exist $p_k \in P^0_{ij}$, $1 \leq k \leq m$ and scalars λ_k, $1 \leq k \leq m$ such that $p = \sum_{k=1}^m \lambda_k p_k$, where (using $p, p_k \in H^1(\mathbf{1})$, $\forall k$) one has $\sum_{k=1}^m \lambda_k = 1$.

If $\lambda_k \geq 0$, $\forall k$, then $p \in P^0_{ij}$ with preservation of indifference.

Otherwise, define

$$p_+ := \frac{1}{\sum_k \max\{\lambda_k, 0\}} \left(\sum_k \max\{\lambda_k, 0\} p_k \right), \; p_- := \frac{1}{\sum_k \min\{\lambda_k, 0\}} \left(\sum_k \min\{\lambda_k, 0\} p_k \right)$$

and let $\lambda^+ := \sum_k \max\{\lambda_k, 0\}$. Note that $\lambda^+ > 1$, $\sum_k \min\{\lambda_k, 0\} = 1 - \lambda^+$, $p_+, p_- \in H^1(\mathbf{1})$, $p^s_+, p^s_- \geq 0$, $\forall x^s \in X$, and $p_+ = \frac{1}{\lambda^+} p + \left(1 - \frac{1}{\lambda^+}\right) p^-$ by construction. Furthermore, using preservation of indifference, it follows that $p_+, p_- \in P^0_{ij}$.

Now toward a contradiction, let $p \notin P^0_{ij}$. Then, $p \in P^+_{ij} \cup P^-_{ij}$ using completeness. Assume $p \in P^+_{ij}$ ($p \in P^-_{ij}$). Since $0 < \frac{1}{\lambda^+} < 1$, preservation of strict preference implies $p_+ \in P^+_{ij}$ ($p^+ \in P^-_{ij}$) – a contradiction. So $p \in P^0_{ij}$, and it follows that $\Pi_{ij} \cap \Delta(X) \subseteq P^0_{ij}$.
Parts 2 and 3: Start by constructing a basis for Π_{ij}. Consider any pair of states $x \in X^+_{ij}$, $y \in X^-_{ij}$. By continuity and preservation of strict preference, there exists a unique number $w_{xy} \in (0,1)$ such that $a_i \sim_{p_{xy}} a_j$ for $p_{xy} = w_{xy} x + (1 - w_{xy}) y \in \Delta(\{x, y\})$. Now fix $x \in X^+_{ij}$, $y \in X^-_{ij}$, and consider the set

$$B_{ij} := \{p_{xy}\} \cup \bigcup_{z \in X^-_{ij} \setminus \{y\}} \{p_{xz}\} \cup \bigcup_{z \in X^+_{ij} \setminus \{x\}} \{p_{zy}\} \cup X^0_{ij}.$$

Observe that $|B_{ij}| = m - 1$ if X^+_{ij}, X^-_{ij} are non-empty and $|B_{ij}| = |X^0_{ij}| \leq m - 1$ otherwise.

Furthermore, note that all vectors in B_{ij} are linearly independent. To see this, consider the linear combination

$$a_{xy} p_{xy} + \sum_{z \in X^-_{ij} \setminus \{y\}} \alpha_z p_{xz} + \sum_{z \in X^+_{ij} \setminus \{x\}} \alpha_z p_{zy} + \sum_{z \in X^0_{ij}} a_z z$$

with scalars $a_{xy}, a_z \in \mathbb{R}$. By construction of the p_{xy}, one may write

$$\begin{aligned} a_{xy} p_{xy} + \sum_{z \in X^-_{ij} \setminus \{y\}} \alpha_z p_{xz} + \sum_{z \in X^+_{ij} \setminus \{x\}} \alpha_z p_{zy} + \sum_{z \in X^0_{ij}} a_z z = & \sum_{z \in X^-_{ij} \setminus \{y\}} \alpha_z (1 - w_{xz}) z + \sum_{z \in X^+_{ij} \setminus \{x\}} \alpha_z w_{zy} z + \sum_{z \in X^0_{ij}} a_z z \\ & + \left[a_{xy} w_{xy} + \sum_{z \in X^-_{ij} \setminus \{y\}} \alpha_z w_{xz} \right] x \\ & + \left[a_{xy} (1 - w_{xy}) + \sum_{z \in X^+_{ij} \setminus \{x\}} \alpha_z (1 - w_{zy}) \right] y \end{aligned}$$

Since $X = X^+_{ij} \cup X^-_{ij} \cup X^0_{ij}$ forms the standard basis of $\mathbb{R}^m$ and since $w_{xy} \in (0,1)$ for all $(x, y) \in X^+_{ij} \times X^-_{ij}$, it follows that $a_{xy} = 0$ and $a_z = 0$ for all $z \in X \setminus \{x, y\}$. Hence the vectors in B_{ij} are linearly independent.

Now, for **Part 2**, suppose $\succsim$ exhibits reversals for a_i, a_j. Then $B_{ij} \subset P^0_{ij}$ contains exactly $m-1$ linearly independent vectors. And since $\Pi_{ij} \cap \Delta(X) \neq \Delta(X)$, also $\dim(\Pi_{ij}) \leq m-1$, and it follows that B_{ij} spans Π_{ij}. So, in particular, Π_{ij} is a hyperplane.

Next, for **Part 3**, suppose $\succsim$ does not exhibit reversals for a_i, a_j and note that $B_{ij} = X^0_{ij}$ in that case. Wlog, assume that X^-_{ij} is empty (if not, relabel a_i, a_j). Since every $p \in \Delta(X)$ can then be written as a convex combination of vectors in $X^+_{ij} \cup X^0_{ij}$, preservation of indifference and preservation of strict preference imply that $p \in P^0_{ij}$ iff $p \in \Delta(X^0_{ij})$. Using $B_{ij} = X^0_{ij}$, this immediately implies $\Pi_{ij} = \langle X^0_{ij} \rangle$. □

Proof. (*Theorem 2.6*)
The direction $\Rightarrow$ is easy to check and left to the reader.

$\Leftarrow$: To start, let a_i and a_j be observationally equivalent. Then $d_{ij} = \mathbf{0}$ represents $\succsim$. Henceforth assume that a_i, a_j are not observationally equivalent.

In case $\succsim$ exhibits reversals for a_i, a_j, Lemma 2.7 implies that Π_{ij} is a hyperplane s.th. $\Pi_{ij} \cap \Delta(X) = P^0_{ij}$. Moreover, Π_{ij} separates P^+_{ij} and P^-_{ij}. (If this were not the case, wlog there would be $p, p' \in P^+_{ij}$ and $\lambda \in (0,1)$ such that $\lambda p + (1-\lambda)p' \in P^0_{ij}$ – contradicting preservation of strict preference.) In particular, there must then exist $d_{ij} \in \mathcal{O}(\Pi_{ij})$ such that $d^s_{ij} > 0$ for all $x^s \in X^+_{ij}$ and $d^s < 0$ for all $x^s \in X^-_{ij}$. By construction, one now has $\langle P^0_{ij} \rangle = H^0(d_{ij})$, $P^+_{ij} \subset H^+(d_{ij})$, and $P^-_{ij} \subset H^-(d_{ij})$.

Otherwise, Lemma 2.7 implies $\mathcal{O}(\Pi_{ij}) = \{d \in \mathbb{R}^m | \, d^s = 0, \ \forall x^s \in X^0_{ij}\}$. Now wlog assume that P^-_{ij} is empty, and take any d_{ij} such that $d^s_{ij} = 0$ for all $x^s \in X^0_{ij}$ and $d^s_{ij} > 0$ for all $x^s \in X^+_{ij}$. Then, since $d_{ij} \in \mathcal{O}(\Pi_{ij})$, $H^0(d_{ij})$ is a hyperplane such that $\Pi_{ij} \subset H^0(d_{ij})$. And, recalling that P^-_{ij} is empty, we have $P^+_{ij} = \Delta(X) \backslash \Delta(X^0_{ij}) \subset H^+(d_{ij})$.

In both cases, it follows that d_{ij} is a difference representation for $\succsim$ on $(\{a_i, a_j\}, X)$. □

Uniqueness properties of pairwise representations directly follow from the proof of Theorem 2.6.

Observation 2.8. *(Uniqueness of Pairwise Difference Representation)*
Let $\succsim$ be a pairwise-linear system of context-dependent preference relations for (A, X). If $d_{ij} \in \mathbb{R}^m$ is a difference representation for $\succsim$ on $(\{a_i, a_j\}, X)$, then for any $\alpha > 0$, αd_{ij} is a difference representation for $\succsim$ on $(\{a_i, a_j\}, X)$. Furthermore, if a_i weakly dominates a_j, then any $\tilde{d}_{ij}$ with $\operatorname{sgn}\left(\tilde{d}^s_{ij}\right) = \operatorname{sgn}(d^s_{ij})$ for all $x^s \in X$ is a difference representation for $\succsim$ on $(\{a_i, a_j\}, X)$.

Theorem 2.6 is analogous to results in Gilboa and Schmeidler (2003) and Perea (2023). It ensures that preferences over a pair of acts a_i, a_j describe a hyperplane in $\mathbb{R}^m$ that intersects $\Delta(X)$ at P^0_{ij}. The difference representation d_{ij} is then an oriented normal vector to this hyperplane.

3 Multi-Act Representation

Theorem 2.6 ensures that pairwise linear preferences $\succsim$ over a pair of acts a_i, a_j are represented by a difference vector d_{ij}. For a full difference representation over n acts $a_1, \ldots, a_n$, it remains to determine when differences can be chosen to satisfy strong transitivity $d_{ij} + d_{jk} = d_{ik}$ for all triples of acts a_i, a_j, a_k. For $n > 2$ acts, strong transitivity implies that reversals between some pairs of alternatives may indirectly constrain whether and where reversals among other pairs may be observed. Such indirect constraints can then more fully determine the representation $\mathcal{D}$ for $\succsim$ on (A, X) or preclude its existence — even where pairs of acts exhibit weak (or strict) dominance.

In line with qualitatively different constraints that arise, the multi-act measurement problem is subdivided into two parts. The first part concerns representations for preferences over three-act decision problems. Two additional properties, *transitivity* and *scaling transitivity*, are needed to deliver a representation in that case. Next, representations for preferences over $n > 3$-act decision problems are considered. One additional property, *transitive preference sensitivity*, delivers a multi-act representation whenever all pairs of acts exhibit preference reversals.

3.i Transitivity

Clearly, a necessary requirement for a difference representation with more than two acts will be that $\succsim$ is transitive at every $p \in \Delta(X)$.

Property 3.1. *(Transitivity)*
$\succsim$ is transitive if $a_i \succsim_p a_j$ and $a_j \succsim_p a_k$ implies $a_i \succsim_p a_k$ for all $p \in \Delta(X)$ and all $a_i, a_j, a_k \in A$.

As a first implication of transitivity, we may treat the representation of observationally equivalent acts separately.

Observation 3.2. *(Representation for Equivalent Acts)*
Let (A, X) be a decision problem and let $\succsim$ be a pairwise linear and transitive system of context-dependent preference relations for (A, X). Let $a_i, a_j \in A$ be observationally equivalent, and let $\hat{\mathcal{D}}$ be a difference representation for $\succsim$ on $(A \backslash \{a_j\}, X)$. Then $\mathcal{D}$ with $d_{ij} = 0$, $d_{ik} = \hat{d}_{jk}$ for all $k \neq i, j$, and $d_{k\ell} = \hat{d}_{k\ell}$ for all $\ell, k \neq i, j$ represents $\succsim$ on (A, X).

Given Observation 3.2, the remainder of the treatment will disregard comparisons between observationally equivalent acts (including ii-reflexive comparisons).

A second, less obvious, implication considered next is that transitivity together with pairwise linearity structures the measurement problem in terms of finitely many **connected components**.

Definition 3.3. *Let (A, X) be a decision problem with context-dependent preference $\succsim$. A non-empty set $C \subseteq A$ is a* **(connected) component** *(on (A, X) wrt $\succsim$) if the following properties hold:*

1. *Take any $a_0, b \in C$. Then defining $b := a_k$, there are acts $a_1, \dots, a_{k-1}$ s.th. $\succsim$ exhibits reversals for every a_i, a_{i+1}, $0 \leq i \leq k$.*
2. *$\succsim$ exhibits no reversals for any a_i, a_j s.th. $a_i \in C$ and $a_j \in A \backslash C$.*

In words, within each component, any act is connected to any other act by a sequence of acts exhibiting pairwise reversals. Furthermore, every act outside a component must either weakly dominate or be weakly dominated by all acts within.

For future reference, the special case of a component C s.th. $\succsim$ exhibits reversals for every distinct pair $a_i, a_j \in C$ will be called a **fully connected** component.

Once can now prove:

Observation 3.4. *Let $\succsim$ on (A, X) be pairwise linear and transitive and let $C \subseteq A$ be a component. Then there exists a partition C^+, C^-, C of A and s.th. a_i weakly dominates a_j and a_j weakly dominates a_k for every $a_i \in C^+$, $a_j \in C$, $a_k \in C^-$ and every $p \in \Delta(X)$.*

Proof. Take any $b \notin C$. We show that either b weakly dominates all $a \in C$ or is weakly dominated by every $a \in C$. The statement then follows with completeness and transitivity.

So, toward a contradiction, take any two distinct $a_i, a_j \in C$ and wlog assume b is weakly dominated by a_i and weakly dominates a_j. Then, by transitivity, a_i weakly dominates a_j.

Next, since C is a component, there must exist $a_{i+1}, \dots, a_{i+(n-1)}$ connecting a_i and $a_j := a_{i+n}$ via pairwise reversals. Furthermore, since $b \in A \backslash C$, it must weakly dominate or be weakly dominated by each $a_{i+1}, \dots, a_{i+(n-1)}$.

In particular, since $\succsim$ exhibits reversals between a_i, a_{i+1}, it must be that b is weakly dominated by a_{i+1}. (Otherwise, a_i would weakly dominate a_{i+1} using transitivity.) Then, by the same logic, a_{i+2} must weakly dominate b since $\succsim$ exhibits reversals between a_i, a_{i+1} and a_{i+1} weakly dominates b. But continuing in this fashion $a_{i+(n-1)}$ must weakly dominate b and, since $\succsim$ exhibits reversals between $a_{i+(n-1)}, a_{i+n}$, $a_{i+n} = a_j$ must weakly dominate b – a contradiction.

It follows that b either weakly dominates or is weakly dominated by every $a \in C$. □

Observation 3.4 shows that any transitive and pairwise linear $\succsim$ partitions the related decision problem (A, X) into a finite number of sub-problems $(C_1, X), \dots, (C_k, X)$, where every (C_i, X), $1 \le i \le k$ is a connected component. Furthermore, these components are in a *weak dominance hierarchy.* I.e., for any two $C, D \subset A$, let $C \trianglerighteq D$ denote weak dominance of each act in C over each act in D, then wlog $C_1 \trianglerighteq \cdots \trianglerighteq C_k$.

The following observation further structures the dominance hierarchy across sets.

Observation 3.5. *Let $B \trianglerighteq C$ and let $x^s \in X$, $a_i \in B$, $a_\ell \in C$ satisfy $a_i \sim_s a_\ell$. Then*

1. *$a_j \succsim_s a_i$ and $a_\ell \succsim_s a_k$ for all $a_j \in B$ and $a_k \in C$.*
2. *$a_j \sim_s a_k$ implies $a_k \sim_s a_\ell$ for any $a_j \in B$ and $a_k \in C$.*

Proof. **1)** Suppose $a_i \succ_s a_j$ ($a_k \succ_s a_\ell$). Then $a_\ell \succ_s a_j$ ($a_k \succ_s a_i$) by transitivity, contradicting $B \trianglerighteq C$.
2) Assume $a_k \succ_s a_\ell$ ($a_k \prec_s a_\ell$). Then $a_k \succ_s a_i$ ($a_\ell \succ_s a_j$) with transitivity, contradicting $B \trianglerighteq C$. □

Observation 3.5.2 shows that all pairwise indifferences at any state x^s across $\trianglerighteq$-ranked sets must lie in a total-indifference set $I^s_{B,C}$ s.th. $a \sim_s b$ for all $a, b \in I^s_{B,C}$. By implication, we must have $a \succ_s b$ for any $a \in C \backslash I^s_{B,C}$ and $b \in D \backslash I^s_{B,C}$. Finally, Observation 3.5.1 then implies that the acts in $I^s_{B,C} \cap B$ are strictly bottom-ranked in B at x^s, whereas the acts in $I^s_{B,C} \cap C$ are strictly top-ranked in C at x^s.

We are now ready to prove that, for any decision problem (A, X) and any pairwise linear and transitive preference $\succsim$ on it, the existence of difference representations for each component of A implies the existence of a global representation.

Lemma 3.6. *(Component-wise construction of representation)*
Let $\succsim$ on (A, X) be pairwise linear and transitive and let $B, C \subseteq A$ satisfy $B \trianglerighteq C$. Then $\succsim$ has a difference representation on $(B \cup C, X)$ if it has difference representations on (B, X) and (C, X).

Proof. Let $\mathcal{D}_B$ and $\mathcal{D}_C$ be difference representations for $\succsim$ on, respectively, (B, X) and (C, X). Take any $a_j \in B$ and $a_k \in C$ and select a pairwise representation d_{jk} in four steps.

1. For any state $x^s \in X^+_{jk}$ s.th. $a_j \in I^s_{B,C}$ and $a_k \notin I^s_{B,C}$, take any $a_\ell \in I^s_{B,C} \cap C$ and fix $d^s_{jk} = -d^s_{k\ell}$ (noting that $x^s \in X^-_{kl}$ by transitivity).

2. For any state $x^s \in X^+_{jk}$ s.th. $a_j \notin I^s_{B,C}$ and $a_k \in I^s_{B,C}$, take any $a_i \in I^s_{B,C} \cap B$ and fix $d^s_{jk} = -d^s_{ij}$ (noting that $x^s \in X^-_{ij}$ by transitivity).

3. For any state $x^s \in X^+_{jk}$ s.th. $a_j \notin I^s_{B,C}$, $a_k \notin I^s_{B,C}$, and $I^s_{B,C}$ is non-empty, take any $a_i \in I^s_{B,C} \cap B$, $a_\ell \in I^s_{B,C} \cap C$ and fix $d^s_{jk} = -(d^s_{ij} + d^s_{k\ell})$ (noting that $x^s \in X^-_{ij} \cap X^-_{k\ell}$ by Observation 3.5.1 combined with $a_j \notin I^s_{B,C}$, $a_k \notin I^s_{B,C}$).

4. For any state s.th. $I^s_{B,C}$ is empty, choose $d^s_{jk} > 0$ large enough that $d^s_{jk} > -(d^s_{ij} + d^s_{k\ell})$ for all $a_i \in B$ and all $a_\ell \in C$.

Finally, for any $a_i \in B$ and $a_\ell \in C$ s.th. $(a_i, a_\ell) \neq (a_j, a_k)$, fix $d_{i\ell} = d_{ij} + d_{jk} + d_{k\ell}$.

Let $\mathcal{D}$ denote the matrix of difference vectors resulting from this construction. Note that, by design, the vectors in $\mathcal{D}$ satisfy strong transitivity and induce representations for $\succsim$ on (B, X) and (C, X), and $(\{a_j, a_k\}, X)$. So it remains to show that $\mathcal{D}$ represents $\succsim$ for any $(\{a_i, a_\ell\}, X)$ s.th. $a_i \in B$, $a_\ell \in C$, and $(a_i, a_\ell) \neq (a_j, a_k)$.

To see this, proceed by states $x^s \in X$ and distinguish three cases:

1. If $a_i, a_\ell \in I^s_{B,C}$, then $d^s_{i\ell} = d^s_{ij} + d^s_{jk} + d^s_{k\ell} = d^s_{jk} - d^s_{jk} = 0$, using steps (1)–(3) of the construction above and noting $d^s_{ij} = 0$ and $d^s_{k\ell} = 0$ in, respectively, steps (1) and (2).

2. If $a_i, a_\ell \notin I^s_{B,C}$ and $a_j \sim_s a_k$, then we have $a_i \succ_s a_j \sim_s a_k \succ_s a_\ell$ using Observation 3.5.1. And hence $d_{i\ell} = d^s_{ij} + d^s_{jk} + d_{k\ell} = d^s_{ij} + d_{k\ell} > 0$ as needed.

3. Finally let $a_i, a_\ell \notin I^s_{B,C}$ and $a_j \succ_s a_k$. If $I^s_{B,C}$ is empty, then $d^s_{i\ell} > 0$ follows from case (4) above. Otherwise, there exist $a_h \in B$ and $a_m \in C$ s.th. $a_i \succ_s a_h \sim_s a_m \succ_s a_\ell$. We then have $d^s_{hm} = 0$ using case (1), and it follows that $d^s_{i\ell} = d^s_{ih} + d^s_{hm} + d^s_{m\ell} = d^s_{ih} + d^s_{m\ell} > 0$ as needed.

□

Following Lemma 3.6, we may henceforth concentrate on preferences $\succsim$ over decision problems (A, X) such that A forms a connected component in the sense of Definition 3.3.

The proof of Lemma 3.6 also pins down the degrees of freedom left after one fixes representations for each component wrt $\succsim$. I.e., let $\succsim$ induce the dominance hierarchy $C_1 \trianglerighteq \cdots \trianglerighteq C_k$ on (A, X). Then for each component C_i, $1 \leq i \leq k$, we may choose any admissible difference representation. In addition, for every pair of components C_i, C_j that are neighboring each other in the weak dominance hierarchy (i.e. $C_i \trianglerighteq C_j$ and $C_k \trianglerighteq C_i$ or $C_j \trianglerighteq C_k$ for any $C_k \neq C_i, C_j$) we may choose exactly one pairwise difference representation d_{jk}, $a_j \in C_i$, $a_k \in C_j$ connecting the two components.

A special case of Lemma 3.6 arises *absent reversals* between any two acts in (A, X). Then, each component is singleton, and pairwise linearity and transitivity suffice for a difference representation.

Observation 3.7. *(Difference Representation absent Reversals)*
Let $\succsim$ be pairwise linear, transitive, and exhibit no reversals on (A, X). Then $\succsim$ admits a difference representation on (A, X).

Furthermore, given the constant weak dominance ranking among choices $a_1 \succsim_p \cdots \succsim_p a_n$, $\forall p \in \Delta(X)$, Lemma 3.6 implies that one may choose any pairwise representations (subject to the constraints from Observation 2.8) for each neighboring pair (a_i, a_{i+1}), $1 \leq i \leq n$. The full representation $\mathcal{D}$ is then uniquely determined via $d_{ij} + d_{jk} = d_{ik}$, $a_i, a_j, a_k \in A$.

3.ii Scaling Transitivity

A third implication of transitivity is stated in the next observation: For triples of difference representations (Theorem 2.6) over three acts, transitivity causes the inner products of difference vectors at any $p \in \Delta(X)$ to be transitively related as well.

Observation 3.8. *(Transitivity and Difference Representations)*
$\succsim$ is pairwise linear and transitive iff it admits difference representations d_{ij} on $(\{a_i, a_j\}, X)$ for every $a_i, a_j \in A$ such that $d_{ij} \cdot p \geq 0$ and $d_{jk} \cdot p \geq 0$ implies $d_{ik} \cdot p \geq 0$ for every $a_i, a_j, a_k \in A$ and $p \in \Delta(X)$.

Generally (allowing for preference reversals) this is insufficient for a difference representation. Even with transitivity, utility differences d_{ij}, d_{jk}, d_{ik} need not satisfy *strong* transitivity $d_{ij}+d_{jk} = d_{ik}, \ \forall a_i, a_j, a_k$. To illustrate, consider the preference in Figure 1 for a decision problem $(A, X) = (\{a_1, a_2, a_3\}, \{x^1, x^2, x^3\})$.

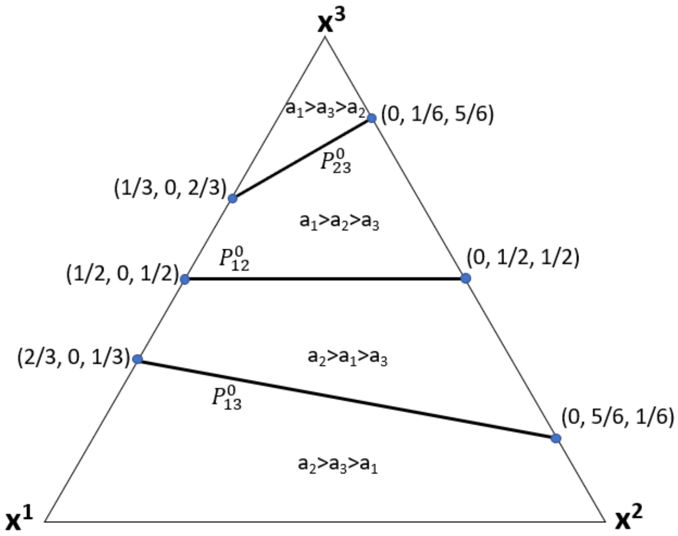


Figure 1: A pairwise linear and transitive preference that admits no difference representation.

This preference is pairwise linear and transitive, but it does not admit a difference representation. To see this, first use Theorem 2.6 to find pairwise representations. This yields

$$d_{12} \in \{k_{12}(-1,-1,1)^T,\ k_{12} > 0\},\ d_{23} \in \{k_{23}(2,5,-1)^T,\ k_{23} > 0\},\ d_{13} \in \{k_{13}(-5,-2,10)^T,\ k_{13} > 0\}.$$

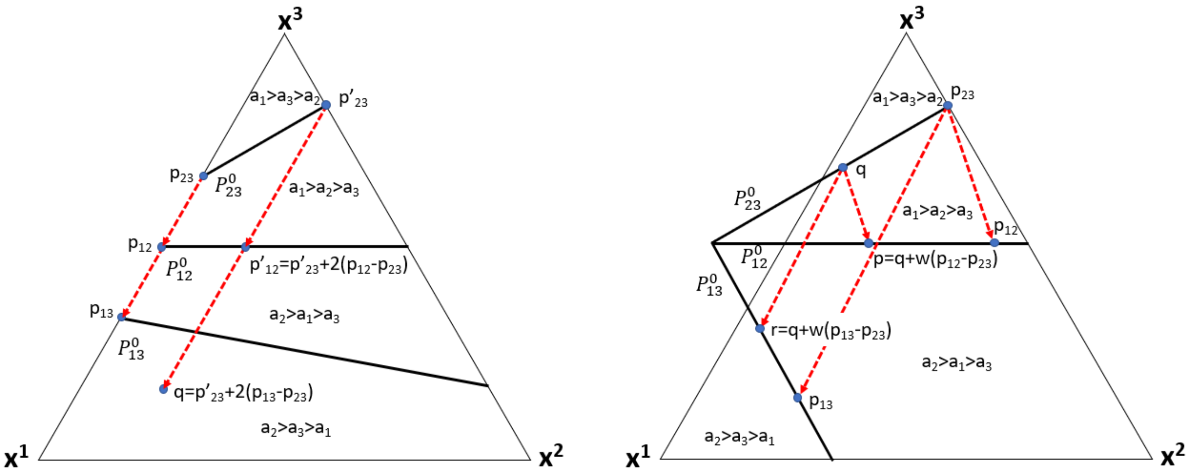
One may now check that the vectors $(-1,-1,1)^T$, $(2,5,-1)^T$, and $(-5,-2,10)^T$ are linearly independent. But then, no combination of weights k_{12}, k_{23}, k_{13} can ever satisfy strong transitivity $d_{12}+d_{23} = d_{13}$! Hence there is no difference representation for this preference system.

To understand why the preference from Figure 1 has no difference representation, consider any three points in the three indifference sets, e.g. $p_{12} = (\frac{1}{2}, 0\frac{1}{2})^T \in P^0_{12}$, $p_{23} = (\frac{1}{3}, 0, \frac{2}{3})^T \in P^0_{23}$ and $P_{13} = (\frac{2}{3}, 0, \frac{1}{3})^T \in P^0_{13}$. If $\succsim$ had a difference representation, then clearly it should be the case that

$$d_{12} \cdot p_{12} = 0 \Leftrightarrow (d_{13} - d_{23}) \cdot p_{12} = 0 \Leftrightarrow d_{13} \cdot (p_{12} - p_{13}) = d_{23} \cdot (p_{12} - p_{23})$$

where strong transitivity was used for the first equivalence. So once we know $a_2 \sim_{p_{23}} a_3$ and $a_1 \sim_{p_{13}} a_3$, observing $a_1 \sim_{p_{12}} a_2$ reveals that the d_{23}-change associated with displacement $p_{12} - p_{23} = (\frac{1}{6}, 0, -\frac{1}{6})$ and the d_{13}-change associated with displacement $p_{12} - p_{13} = (-\frac{1}{6}, 0, \frac{1}{6})$ must be equal. And since d_{23} and d_{13} represent pairwise preferences for all $p \in \Delta(X)$, *congruent* displacements $\lambda(p_{12} - p_{23})$ and $\lambda(p_{12} - p_{13})$

for any $\lambda \in \mathbb{R}$ must imply equal d_{23}- and d_{13}-changes *anywhere* in the probability simplex. To see how this causes constraints for triples of acts, reconsider points p_{12}, p_{23}, p_{13} as introduced above and any $p'_{23} \in P^0_{23}$, $p'_{13} \in P^0_{13}$, and $p \in \Delta(X)$ such that $p = p'_{23} + \lambda(p_{12} - p_{23})$ and $p = p'_{13} + \lambda(p_{12} - p_{13})$ for some $\lambda \in \mathbb{R}$. To reach p, one travels $\lambda(p_{12} - p_{23})$ from p'_{23} and $\lambda(p_{12} - p_{13})$ from p'_{13} – displacements that are known to cause equal 2, 3-and 1, 3-difference changes away from 0. It follows that the 2, 3-and 1, 3-differences at p are equal and, hence, with strong transitivity $d_{12} = d_{13} - d_{23}$ of utility differences, p must be an 1, 2-indifference point! In this sense, expected utility requires that triples of indifference points across different acts are linearly *scalable*. Figure 2a below illustrates that the Figure 1-preference fails the scalability test described above.



(a) A violation of scaling transitivity.

(b) A preference satisfying scaling transitivity.

Figure 2: Scaling transitivity.

To check for existence of a difference representation in a three-act problem, consider all paths between triples of indifference points for distinct pairs. Whenever two different triples of indifference points are related by parallel displacements, the corresponding displacement vectors across the two triples must scale linearly. This leads to the following property.

Definition 3.9. *(Scaling Transitivity)*
$\succsim$ is scaling-transitive if, for any $p_{ij}, p'_{ij} \in P^0_{ij}$, $t_{ik}, t_{jk} \in H^0(\mathbf{1})$, and $\lambda \in \mathbb{R}$ s.th. $p_{ij} + t_{ik} \in P^0_{jk}$, $p_{ij} + t_{jk} \in P^0_{ik}$, and $p'_{ij} + \lambda t_{ik} \in P^0_{jk}$, one has $p'_{ij} + \lambda t_{jk} \in P^0_{ik}$ whenever $p'_{ij} + \lambda t_{jk} \in \Delta(X)$.

Figure 2b above shows a preference that is scaling-transitive. As seen in the figure, p, q, and r are related by scaled-down versions of the displacement vectors $p_{12} - p_{23}$ and $p_{13} - p_{23}$. Once it is known that $p_{23} \in P^0_{23}$, $p_{12} \in P^0_{12}$ and $p_{13} \in P^0_{13}$, observing $q \in P^0_{23}$ and $p \in P^0_{12}$ implies that $r \in P^0_{13}$.[4]

The following Theorem proves that pairwise linearity, transitivity, and scaling transitivity are necessary and sufficient for a difference representation with three acts.

Theorem 3.10. Difference Representation for 3 Acts
For every pairwise distinct $a_1, a_2, a_3 \in A$, $\succsim$ admits a difference representation on $(\{a_1, a_2, a_3\}, X)$ iff it satisfies pairwise linearity, transitivity, and scaling transitivity.

[4]Given pairwise linearity and $|X| = 3$, the reader may check that one such test among six points is enough to conclude that the Figure 2b-preference is globally consistent with a one-step additive preference intensity. This is true because two distinct indifference points for each pairwise comparison uniquely pin down indifference hyperplanes in the three-act, three-state setting. Analogously, in an m-state setting, verifying scaling transitivity among three acts would require $m - 2$ six-point checks.

In what follows, for any $Y \subseteq X$, let $\Delta^+(Y) = \{p \in \Delta(Y) | \ p^s > 0, \ x^s \in Y \text{ and } p^s = 0, \ p^s \in X \backslash Y\}$. Thus, $\Delta^+(Y)$ denotes the relative interior of the probability simplex spanned by the states in Y (with $\Delta^+(X)$ the interior of the entire simplex). For any distinct (not observationally equivalent) a_i, a_j, note that $P^0_{ij} \cap \Delta^+(X)$ is non-empty iff $\succsim$ exhibits reversals for that pair, and that $P^0_{ij} \cap \Delta^+(Y)$ is empty for all $Y \subset X$ iff P^0_{ij} is either empty or $P^0_{ij} = \{x^s\}$ for some $x^s \in X$.

Next, for a finite set of vectors $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^m$, $\langle V \rangle_a = \{u \in \mathbb{R}^m | \ u = \sum_{k=1}^n \lambda_k v_k, \ \sum_{k=1}^n \lambda_k = 1\}$ will denote the *affine* span of V.

Finally, for any $a_i, a_j, a_k \in A$, define the space $\Pi_{ijk} = \Pi_{ij} \cap \Pi_{ik} + \Pi_{ij} \cap \Pi_{jk} + \Pi_{ik} \cap \Pi_{jk}$. Intuitively, for each pair a_i, a_j, Π_{ijk} contains all information regarding pairwise preferences that is indirectly revealed from a_j, a_k- and a_i, a_k-comparisons. It turns out that such implicit information on a triple precisely arises when intersections of pairwise-indifference subspaces create "pseudo-indifference points" between all three acts a_i, a_j, a_k outside of the probability simplex.

To prove Theorem 3.10, a core ingredient is the following Lemma, which rules inconsistencies between directly observed preference information that is revealed through pairwise comparisons and additional indirect preference information that arises for triples of choices.

Lemma 3.11. (*Properties of* Π_{ijk})
Let $\succsim$ satisfy pairwise linearity, transitivity, and scaling transitivity. Then, for all $a_i, a_j, a_k \in A$,

1. $(\Pi_{ijk} + \Pi_{ij}) \cap \Delta(X) = P^0_{ij}$,
2. *if two distinct pairs a_i, a_j and a_j, a_k exhibit reversals, then there exists a representation d_{ik} for $\succsim$ on $(\{a_i, a_k\}, X)$ such that $\Pi_{ijk} \subset H^0(d_{ik})$.*

Proof. **Part 1:** Take any distinct a_i, a_j, a_k. We show the statement for a_i, a_j, other pairs follow from relabeling. Furthermore, since $\Pi_{ij} \cap Y \subseteq \Pi_{ij}$ for any $Y \subseteq \mathbb{R}^m$, it will suffice to show $(\Pi_{jk} \cap \Pi_{ik} + \Pi_{ij}) \cap \Delta(X) = P^0_{ij}$.

First, consider the case that $\succsim$ exhibits no reversals for a_j, a_k and a_i, a_k. Since (with Lemma 2.7), Π_{ik} and Π_{jk} are spanned by, respectively, X^0_{ik} and X^0_{jk}, and since any two distinct $x^s, x^t \in X$ are orthogonal to each other, it follows that $\Pi_{ik} \cap \Pi_{jk} = \langle X^0_{jk} \cap X^0_{ik} \rangle$. With transitivity, $X^0_{ik} \cap X^0_{jk} \subset X^0_{ij}$ implies $\Pi_{ik} \cap \Pi_{jk} \subset \Pi_{ij}$, and the statement follows with Lemma 2.7.

Second, consider the case that $\succsim$ exhibits reversals for a_j, a_k and that $\Delta^+(Y) \cap P^0_{ik}$ is empty for all $Y \subseteq X$. Since $P^0_{ik} \subset X$ is then at most singleton, it again follows that $\Pi_{ik} \cap \Pi_{jk} = \langle X^0_{jk} \cap X^0_{ik} \rangle$, and the statement follows with transitivity and Lemma 2.7.

Finally, it remains to consider the case that $\succsim$ exhibits reversals for a_j, a_k and that $\Delta^+(Y) \cap P^0_{ik}$ is non-empty for some $Y \subseteq X$.

In case $P^0_{jk} = P^0_{ik}$, transitivity implies $P^0_{jk} = P^0_{ik} = P^0_{ij}$, and the statement follows.

Otherwise, take any $v \in (\Pi_{jk} \cap \Pi_{ik}) \backslash \Pi_{ij}$. We show that $\langle v, p_{ij} \rangle \cap \Delta(X) = \{p_{ij}\}$ for all $p_{ij} \in P^0_{ij}$. Since $v \in H^k(\mathbf{1})$ implies $\frac{v}{k} \in H^1(\mathbf{1})$ for any non-zero k, one may wlog assume that $v \in H^0(\mathbf{1}) \cup H^1(\mathbf{1})$.

So let $v \in H^1(\mathbf{1}) \cap ((\Pi_{jk} \cap \Pi_{ik}) \backslash \Pi_{ij})$ and, toward a contradiction, let $p_{ij} \in P^0_{ij}$ be such that $\langle v, p_{ij} \rangle \cap \Delta(X) \neq \{p_{ij}\}$. Then, there must exist $\lambda^* < 1$ such that $\lambda v + (1 - \lambda) p_{ij} \in \Delta(X)$ for all $\lambda \in [0, \lambda^*]$.[5] Now, recall that $\succsim$ exhibits reversals for a_j, a_k and that $P^0_{ik} \cap \Delta^+(Y)$ is non-empty for some $Y \subseteq X$. It then follows that one can choose $p_{jk} \in P^0_{jk} \cap \Delta^+(X)$, $p_{ik} \in P^0_{ik} \cap \Delta^+(Y)$ and $\lambda \in (0, \mu^*]$ such that $p' = \lambda v + (1 - \lambda) p_{ij} \in \Delta(X)$, $p'_{jk} = \lambda v + (1 - \lambda) p_{jk} \in P^0_{jk}$, and $p'_{ik} = \lambda v + (1 - \lambda) p_{ik} \in P^0_{ik}$. Rewriting now

[5]Take note that the scaling parameter might be positive or negative.

yields $p'_{ik} = p'_{jk} + \lambda(p_{ik} - p_{jk})$ and $p' = p'_{ij} + \lambda(p_{ij} - p_{jk})$, and scaling transitivity implies $p' \in P^0_{12}$. But then $v \in \Pi_{ij}$ – a contradiction.

Now let $v \in H^0(\mathbf{1}) \cap ((\Pi_{jk} \cap \Pi_{ik}) \backslash \Pi_{ij})$, and again assume $p_{ij} \in P^0_{ij}$ such that $\langle v, p_{ij} \rangle \cap \Delta(X) \neq \{p_{ij}\}$. Then there exists a scalar c^* such that $cv + p_{ij} \in \Delta(X)$ for all $c \in [0, c^*]$.[5] Again since $\succsim$ exhibits reversals for a_j, a_k and since $P^0_{ik} \cap \Delta^+(Y)$ is non-empty for some $Y \subseteq X$, one can choose $p_{jk} \in P^0_{jk} \cap \Delta^+(X)$, $p_{ik} \in P^0_{ik} \cap \Delta^+(Y)$ and $c \in (0, c^*]$ such that $p' = p_{ij} + cv \in \Delta(X)$, $p'_{jk} = p_{jk} + cv \in P^0_{jk}$, and $p'_{ik} = p_{ik} + cv \in P^0_{ik}$, and rewriting yields $p'_{ik} = p'_{jk} + p_{ik} - p_{jk}$, $p' = p'_{jk} + p_{ij} - p_{jk}$. But then again, scaling transitivity implies $p' \in P^0_{ij}$ and, hence, $v \in \langle P^0_{ij} \rangle$ – a contradiction.

It follows that $\langle v, p_{ij} \rangle \cap \Delta(X) = \{p_{ij}\}$ for all $v \in (\Pi_{jk} \cap \Pi_{ik}) \backslash \Pi_{ij}$ and $p_{ij} \in P^0_{ij}$, completing the proof.

Part 2: First consider the case that $P^0_{ij} = P^0_{jk}$ or (wlog) $P^0_{ij} = P^0_{ik}$. Then $P^0_{ij} = P^0_{jk} = P^0_{ik}$ by transitivity, and the statement is trivial.

Henceforth assume that $P^0_{ij}, P^0_{jk}, P^0_{ik}$ are pairwise distinct.

If $\langle P^0_{ik} \rangle$ is a hyperplane, $(\Pi_{ijk} + \Pi_{ik}) \cap \Delta(X) \neq \Delta(X)$ from **Part 1** immediately implies $\Pi_{ijk} \subset \Pi_{ik}$ and, hence, $H^0(d_{ik}) = \Pi_{ik}$ has the desired properties.

Otherwise, wlog let a_i weakly dominate a_k and consider the subspace $\mathcal{O}(\Pi_{ik} + \Pi_{ijk}) = \{w \in \mathbb{R}^m | \, w^s = 0, \forall x^s \in X^0_{ik}$ and $w \cdot v = 0, \; \forall v \in \Pi_{ijk}\}$.

In case $\Pi_{ik} + \Pi_{ijk}$ is a hyperplane, then we must have $\mathcal{O}(\Pi_{ik} + \Pi_{ijk}) = \langle d_{ik} \rangle$ with $d^s_{ik} > 0$ for all $x^s \in X^+_{ik}$. If not, for any $x^s, x^t \in X^+_{ik}$ such that $d^s_{ik} > 0 > d^t_{ik}$, we would have $w = -\frac{d^t_{ik}}{d^s_{ik} - d^t_{ik}} \in (0,1)$ and $p_{st} = wx^s + (1-w)x^t \in \Delta(X)$. But then $((\Pi_{ik} + \Pi_{ijk}) \cap \Delta(X)) \backslash P^0_{ik}$ would be non-empty, a contradiction with Part 1.

In case $\Pi_{ik} + \Pi_{ijk}$ is not a hyperplane, take a basis B for $\Pi_{ik} + \Pi_{ijk}$ and wlog assume $B = X^0_{ik} \cup V$ with $v^s = 0$ for all $x^s \in X^0_{ik}$ and all $v \in V$. (If not, simply replace the elements of V by appropriate linear combinations with elements of X^0_{ik}.) Now furthermore, note that there cannot exist a non-zero vector $v \in \langle V \rangle$ such that $v^s \geq 0$ for all $x^s \in X^+_{ik}$. Otherwise, one would have $\frac{1}{\sum_{x^s \in X} v^s} v \in \Delta(X)$, again causing $((\Pi_{ik} + \Pi_{ijk}) \cap \Delta(X)) \backslash P^0_{ik}$ to be non-empty, and contradicting **Part 1**. Using Stiemke's Theorem,[6] it follows that there exists $d_{ik} \in \mathcal{O}(\Pi_{ik} + \Pi_{ijk})$ with $d^s_{ik} > 0$ for all $x^s \in X^+_{ik}$.[7]

For both cases, it now follows that d_{ik} represents $\succsim$ on $(\{a_i, a_k\}, \Delta(X))$. And, by construction, $\Pi_{ijk} \subset H^0(d_{ik})$, completing the proof. $\square$

One can now prove Theorem 3.10:

Proof. (*Theorem 3.10*)
$\Rightarrow$: Assuming that $\succsim$ has a difference representation on $(\{a_i, a_j, a_k\}, \Delta(X))$, transitivity follows straightforwardly using *strong* transitivity $d_{ij} + d_{jk} = d_{ik}$. Next, for scaling transitivity, take $p_{ij}, p'_{ij} \in P^0_{ij}$, $t_{ij}, t_{jk} \in H^0(\mathbf{1})$, and $\lambda \in \mathbb{R}$ such that $p_{ij} + t_{ik}, p'_{ij} + \lambda t_{ik} \in P^0_{jk}$, $p_{ij} + t_{jk} \in P^0_{ik}$, and $p'_{ij} + \lambda t_{jk} \in \Delta(X)$. One then has

$$d_{ik} \cdot (p'_{ij} + \lambda t_{jk}) = (d_{ij} + d_{jk}) \cdot (p'_{ij} + \lambda t_{jk})$$

[6] Formally, for all $v \in V$, define the vectors $\tilde{v} := (v^s)_{x^s \in X^+_{ik}}$ and collect them in the set $\tilde{V}$. Since there is then no non-zero $\tilde{v} \in \langle \tilde{V} \rangle$ such that $\tilde{v} \geq 0$, Stiemke's Theorem implies that there exists a strictly positive vector $\tilde{d} \in \mathbb{R}^{|X^+_{ik}|}$ such that $\tilde{d} \cdot \tilde{v} = 0$ for all $\tilde{v} \in \tilde{V}$. $d_{ik} \in \mathbb{R}^m$ is then constructed by setting $d^s_{ik} = 0$ for $x^s \in X^0_{ik}$ and $d^s_{ik} = \tilde{d}^s_{ik}$ for $x^s \in X^+_{ik}$.

[7] A basis for $H^0(d_{ik})$ could now be constructed by adding the (up-to-scale) unique orthogonal complement of d_{ik} in $\mathcal{O}(\Pi_{ik} + \Pi_{ijk})$ to $\Pi_{ik} + \Pi_{ijk}$.

$$= d_{ij} \cdot \lambda t_{jk} + d_{jk} \cdot \lambda(t_{jk} - t_{ik})$$
$$= d_{ij} \cdot \lambda(p_{ij} + t_{jk}) + d_{jk} \cdot \lambda(p_{jk} + t_{jk} - t_{ik})$$
$$= (d_{ij} + d_{ik}) \cdot \lambda p_{ik} = \lambda d_{ik} \cdot p_{ik} = 0$$

where strong transitivity $d_{ij} + d_{jk} = d_{ik}$ was used for the first and fourth equalities and $p'_{jk} = p'_{ij} - \lambda t_{ik}$ was used for the second one. It follows that $p'_{ij} + t_{jk} \in P^0_{ik}$.
$\Leftarrow$: Consider the case that $\succsim$ exhibits reversals for two pairs a_i, a_j and a_j, a_k. (All other cases follow from Theorem 2.6 and Lemma 3.6.)

First assume that $P^0_{ij} = P^0_{jk}$. Then $P^0_{ij} = P^0_{jk} = P^0_{ik} =: P$ by transitivity. Now consider any $p \in \Delta(X) \backslash P$. By transitivity, there is a strict local preference $a_i \succ_p a_j \succ_p a_k$ at p. Now take any difference representation d_{ik} for $\succsim$ on $(\{a_i, a_k\}, X)$, and take any scalar $\alpha_{ij} \in (0,1)$. Using Theorem 2.6 and noting strong transitivity, $[d_{ij}, d_{ik}, d_{jk}] = [\alpha_{ij} d_{ik}, d_{ik}, (1-\alpha_{ij}) d_{ik}]$ is then a difference representation for $\succsim$ on $(\{a_i, a_j, a_k\}, X)$.

Henceforth, let $P^0_{ij} \neq P^0_{jk}$ and take any representations $\hat{d}_{ij}, \hat{d}_{jk}$ for $\succsim$ on, respectively, $(\{a_i, a_j\}, X)$, and $(\{a_j, a_k\}, X)$. Furthermore, using Lemma 3.11, Part 2, fix a representation d_{ik} for $\succsim$ on $(\{a_i, a_k\}, X)$ such that $\Pi_{ijk} \subset H^0(d_{ik})$. Since $P^0_{ij}, P^0_{jk}, H^0(d_{ik})$ are hyperplanes, $H^0(d_{ik}) \cap \Pi_{ijk}$ contains $m-2$ independent vectors. And hence, the null space of the matrix $\hat{\mathcal{D}}^T = \left[\hat{d}_{ij}, d_{ik}, \hat{d}_{jk}\right]^T \in \mathbb{R}^{3 \times m}$ (i.e. $\mathcal{N}(\hat{\mathcal{D}}^T) = \{v \in \mathbb{R}^m |\ \hat{d}_{ij} \cdot v = d_{ik} \cdot v = \hat{d}_{jk} \cdot v = 0\}$) is $m-2$-dimensional. Using the rank-nullity theorem, this implies $\text{Rank}\left(\hat{\mathcal{D}}^T\right) = 2$. Since the row-rank of a finite matrix is equal to its column rank, it then follows (again using the rank-nullity theorem) that $\mathcal{N}(\hat{\mathcal{D}}) = \{\alpha \in \mathbb{R}^3 |\ \alpha_1 d_{ij} + \alpha_2 d_{ik} + \alpha_3 d_{jk} = \mathbf{0}\}$ is one-dimensional. Hence, there are unique scalars α_{ij}, α_{jk} such that $\alpha_{ij}\hat{d}_{ij} + \alpha_{jk}\hat{d}_{jk} = d_{ik}$. Now consider any $p \in P^0_{ij} \cap P^+_{ij}$ (such a p exists since $P^0_{ij} \neq P^0_{jk}$ and a_j, a_k exhibit reversals. By transitivity, $p \cdot d_{ik} = p \cdot (d_{ij} + d_{jk}) = p \cdot (\alpha_{jk}\hat{d}_{jk}) > 0$ and hence $\alpha_{jk} > 0$. Similarly, take any $p \in P^0_{jk} \cap P^+_{ij}$ to show that $\alpha_{ij} > 0$, and it follows that $[d_{ij}, d_{ik}, d_{jk}] = \left[\alpha_{ij}\hat{d}_{ij}, d_{ik}, \alpha_{jk}\hat{d}_{jk}\right]$ is a difference representation for $\succsim$ on $(\{a_i, a_j, a_k\}, X)$. □

Together with Lemma 3.6, the proofs of Theorem 3.10 and Lemma 3.11 again give a clear-cut picture regarding the uniqueness of difference representations for triples of acts.

Observation 3.12. *(Uniqueness of Representation for 3 Acts)*
Let (A, X) be a decision problem, and let $\succsim$ be a system of context-dependent preference relations for (A, X) that satisfies pairwise linearity, transitivity, and scaling transitivity. For any pairwise distinct $a_i, a_j, a_k \in A$, let $\mathcal{D} = [d_{ij}, d_{jk}, d_{ik}]$ represent $\succsim$ on $(\{a_i, a_j, a_k\}, X)$. Then, for any $\alpha > 0$, $\alpha\mathcal{D}$ represents $\succsim$ on $(\{a_i, a_j, a_k\}, X)$. For additional degrees of freedom, distinguish three cases (all other cases follow from relabeling):

1) *Let $\succsim$ exhibit reversals for each of two distinct pairs a_i, a_j and a_j, a_k. Furthermore, let $P^0_{ik} \subseteq P^0_{ij} \cap P^0_{jk}$. Take $V \subset \mathbb{R}^n$ such that $\langle V \rangle = \langle P^0_{ij} \rangle \cap \langle P^0_{jk} \rangle$ and $w \in \mathcal{O}(V)$ such that $w^s = 0$ for all $x^s \in X^0_{ik}$ and such that neither $w \geq 0$ nor $w \leq 0$.*[8] *Then any matrix $\mathcal{D}' = \left[d_{ij}, d_{jk}, \tilde{d}_{ik}\right]$ with $H^0(\tilde{d}_{ik}) = \langle V \cup w \rangle$ represents $\succsim$ on $(\{a_i, a_j, a_k\}, X)$.*

2) *Let a_i, a_j be the unique pair such that $\succsim$ exhibits reversals. Furthermore, let a_j weakly dominate a_k. Then any matrix $\mathcal{D}' = \left[d_{ij}, \tilde{d}_{ik}, \tilde{d}_{jk}\right]$ with $\tilde{d}^s_{jk} = -d^s_{ij}$ for all $x^s \in X^0_{ik} \cap X^-_{ij} \cap X^+_{jk}$, $\tilde{d}^s_{jk} > -d^s_{ji}$ for all*

[8]That such a $w \in \mathcal{O}\langle V \rangle$ exists follows with Stiemke's Theorem, see Lemma 3.11, Step 2.

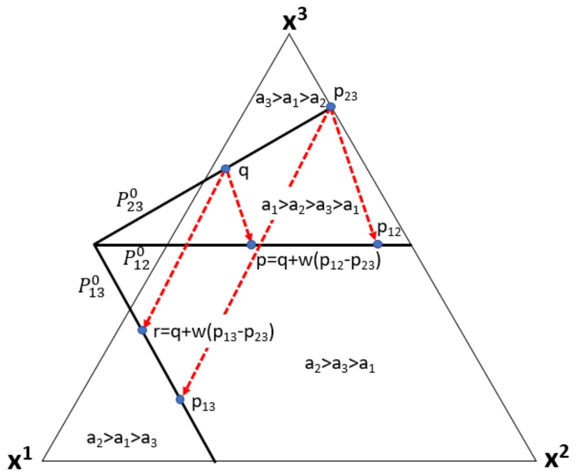
$x^s \in X_{ik}^+ \cap X_{ij}^- \cap X_{jk}^+$, *and* $\tilde{d}_{ik} = d_{ij} + \tilde{d}_{jk}$ *represents* $\succsim$ *on* $(\{a_i, a_j, a_k\}, X)$.

3) *Let* $\succsim$ *exhibit no reversals on* $(\{a_i, a_j, a_k\}, X)$. *Then any matrix of pairwise representations* $\mathcal{D}' = [\tilde{d}_{ij}, \tilde{d}_{jk}, \tilde{d}_{ij}]$ *with* $\tilde{d}_{ij} + \tilde{d}_{jk} = \tilde{d}_{ij}$ *represents* $\succsim$ *on* $(\{a_i, a_j, a_k\}, X)$.

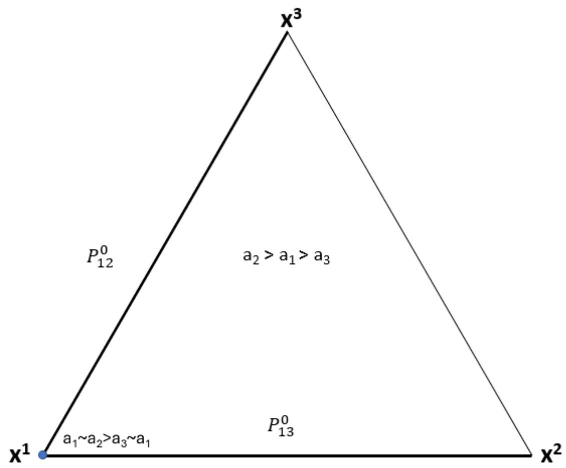
A special case of Theorem 3.10 arises for two-state decision problems. Since indifference sets are singletons here, one trivially has consistency with scaling transitivity.

Corollary 3.13. *(Representation for 3 Acts and 2 States)*
For every pairwise distinct $a_i, a_j, a_k \in A$ *and distinct* $x^\ell, x^m \in X$, $\succsim$ *admits a difference representation on* $(\{a_i, a_j, a_k\}, \{x^\ell, x^m\})$ *iff it is pairwise linear and transitive.*

Before moving on to $n > 3$ acts, examine the logical relationship between transitivity (Definition 3.1) and scaling transitivity (Definition 3.9). Lemma 3.11 and Figure 2b suggest a strong connection between transitivity of the indifference relation $\sim$ and scaling transitivity. In fact, one may check that scaling transitivity implies transitivity of $\sim$ whenever we observe reversals between all pairs of acts. However, as seen in Figures 3a, 3b below, $\succ$ and/or $\sim$ can be intransitive even if $\succsim$ is scaling transitive.



(a) Scaling transitivity, intransitive $\succ$.



(b) Scaling transitivity, intransitive $\sim$.

Figure 3: Scaling transitive, intransitive preferences.

3.iii Transitive Preference Sensitivity

For $n \geq 4$ acts, additional assumptions are needed for a difference representation. To see this, consider the preference for a decision problem $(\{a_1, a_2, a_3, a_4\}, \{x^1, x^2\})$ shown in Figure 4. This preference is pairwise linear, transitive, and (trivially) scaling transitive. With Corollary 3.13, $\succsim$ admits a representation on $(\{a_i, a_j, a_k\}, \{x^1, x^2\})$ for every $a_i, a_j, a_k \in \{a_1, a_2, a_3, a_4\}$. However, there is no representation on $(\{a_1, a_2, a_3, a_4\}, \{x^1, x^2\})$.

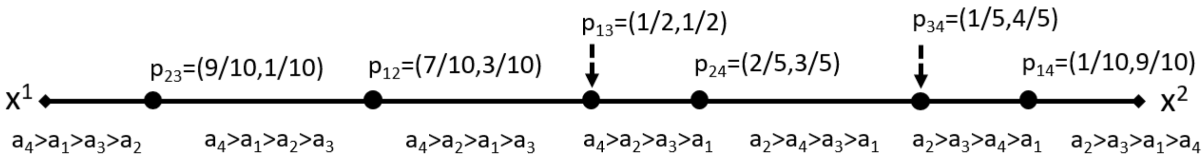


Figure 4: A pairwise linear and transitive preference that admits no difference representation

To see this, first use Theorem 2.6 to find representations for pairwise comparisons. This yields

$$\hat{d}_{12} = \left(\frac{3}{10}, -\frac{7}{10}\right),\ \hat{d}_{13} = \left(\frac{1}{2}, -\frac{1}{2}\right),\ \hat{d}_{14} = \left(-\frac{9}{10}, \frac{1}{10}\right),\ \hat{d}_{23} = \left(-\frac{1}{10}, \frac{9}{10}\right),\ \hat{d}_{24} = \left(-\frac{3}{5}, \frac{2}{5}\right),\ \hat{d}_{34} = \left(-\frac{4}{5}, \frac{1}{5}\right).$$

Note that each difference vector $\hat{d}_{ij}$ is such that $\|\hat{d}_{ij}\|_1 = 1$. This can be seen as a *canonical* pairwise comparison, in the sense that $p_{ij} \in \Delta(X)$ is only rotated around $\mathbf{0}$ and not rescaled to arrive at the representation $\hat{d}_{ij}$. I.e., using $R(\pm\frac{\pi}{2})$ to denote the standard rotation matrix in $\mathbb{R}^2$ and identifying x^1 with the horizontal axis, we have $\hat{d}_{ij} = R(\frac{\pi}{2})p_{ij}$ if $a_i \succ_{x^1} a_j$ and $\hat{d}_{ij} = R(-\frac{\pi}{2})p_{ij}$ otherwise.

Whenever reversals occur for all pairs of acts in a 3-act, 2-state problems, canonical representations make it easy to find triple representations as in Corollary 3.13. To see this, wlog assume $\hat{d}_{ij} = R(-\frac{\pi}{2})p_{ij}$ and take $\lambda_{ijk} \in \mathbb{R}$ such that $p_{ij} = \lambda_{ijk}p_{ik} + (1-\lambda_{ijk})p_{jk}$. Then the *elementary* representations satisfy $\hat{d}_{ij} = R(-\frac{\pi}{2})p_{ij} = R(-\frac{\pi}{2})\lambda_{ijk}p_{ik} + R(-\frac{\pi}{2})(1-\lambda_{ijk})p_{jk}$. Now since the representation is unique up to scaling, and since it must satisfy strong transitivity, one immediately gets that $\hat{d}_{ij} = |\lambda_{ijk}|\hat{d}_{ik} - |1-\lambda_{ijk}|\hat{d}_{jk}$.

Back to the example, it is now easy to find representations for triples $a_i, a_j, a_k \in \{a_1, a_2, a_3, a_4\}$.

$$\hat{d}_{12} = \frac{1}{2}\hat{d}_{13} - \frac{1}{2}\hat{d}_{23}, \quad \hat{d}_{12} = \hat{d}_{14} - 2\hat{d}_{24}, \quad \hat{d}_{13} = 3\hat{d}_{14} - 4\hat{d}_{34}, \quad \hat{d}_{23} = \frac{7}{2}\hat{d}_{24} - \frac{5}{2}\hat{d}_{34}.$$

The weights are unique up to scaling, so a representation on $(\{a_1, a_2, a_3, a_4\}, \{x^1, x^2\})$ would require

$$\hat{d}_{14} - 2\hat{d}_{24} = \hat{d}_{12} = \frac{1}{2}\hat{d}_{13} - \frac{1}{2}\hat{d}_{23} = \frac{3}{2}\hat{d}_{14} - \frac{3}{4}\hat{d}_{34} - \frac{7}{4}\hat{d}_{24}$$

– a contradiction since different weights on $\hat{d}_{14}$, $\hat{d}_{24}$, $\hat{d}_{34}$ are necessary to express $\hat{d}_{12}$ and $\frac{1}{2}\hat{d}_{13} - \frac{1}{2}\hat{d}_{23}$.

To intuitively see why the preference from Figure 4 is inconsistent with expected utility, examine the relationship between displacement vectors and difference-shifts. Toward a contradiction, assume the Figure-4 preference has a representation $\mathcal{D} = [d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}]$ and consider the displacement $p_{24} - p_{14}$. With strong transitivity, we must have $d_{14} \cdot (p_{24} - p_{14}) = (d_{12} + d_{24}) \cdot p_{24} = d_{12} \cdot (p_{24} - p_{12})$. I.e., the a_1, a_4-change associated with $p_{24} - p_{14}$ is of equal magnitude as the a_1, a_2-change associated with $p_{24} - p_{12}$. Now, using that all indifference points are collinear for the present two-state decision problem, let σ_{134} solve $p_{23} - p_{12} = \sigma_{134}(p_{24} - p_{12})$. Continuing the above derivation, we have

$$d_{14} \cdot (p_{24} - p_{14}) = d_{12} \cdot (p_{24} - p_{12}) = \frac{1}{\sigma_{134}} d_{12} \cdot (p_{23} - p_{12}) = \frac{1}{\sigma_{134}} d_{13} \cdot (p_{23} - p_{13}).$$

Proceeding in this way, while defining $p_{23} - p_{13} = \sigma_{124}(p_{34} - p_{13})$, $p_{24} - p_{14} = \sigma_{123}(p_{34} - p_{14})$, yields

$$\begin{aligned} d_{14} \cdot (p_{24} - p_{14}) = d_{12} \cdot (p_{24} - p_{12}) &= \frac{1}{\sigma_{134}} d_{12} \cdot (p_{23} - p_{12}) = \frac{1}{\sigma_{134}} d_{13} \cdot (p_{23} - p_{13}) \\ &= \frac{\sigma_{124}}{\sigma_{134}} d_{13} \cdot (p_{34} - p_{13}) = \frac{\sigma_{124}}{\sigma_{134}} d_{14} \cdot (p_{34} - p_{14}) = \frac{\sigma_{124}}{\sigma_{123}\sigma_{134}} d_{14} \cdot (p_{24} - p_{14}). \end{aligned}$$

So the existence of a difference representation requires $\sigma_{124} = \sigma_{123}\sigma_{134}$. For the preference from Figure 4 however, straightforward calculations show that $\sigma_{124} = -\frac{4}{3} \neq \sigma_{123}\sigma_{134} = 3 \cdot (-\frac{2}{3})$.

The numbers σ_{ijk}, $a_i, a_j, a_k \in A$ measure the sensitivity of $\succsim$ for a_i, a_j-comparisons relative to a_i, a_k-comparisons. In the setting with $n \geq 4$ acts, such sensitivities are derived from conditional preferences as follows. Given any fourth act $a_\ell \neq a_i, a_j, a_k$, the a_i, a_j-difference over $p_{j\ell} - p_{ij}$ and the a_i, a_k-difference over $p_{k\ell} - p_{ik}$ can be equated to the a_i, a_ℓ-differences over (respectively) $p_{j\ell} - p_{i\ell}$ and $p_{k\ell} - p_{i\ell}$. This way, comparisons with a_i, a_ℓ-differences reveal the magnitude of a_i, a_j-differences relative a_i, a_k-differences. Figure 5 illustrates how $\sigma_{134} = -\frac{2}{3}$ is revealed for the preference from Figure 4.

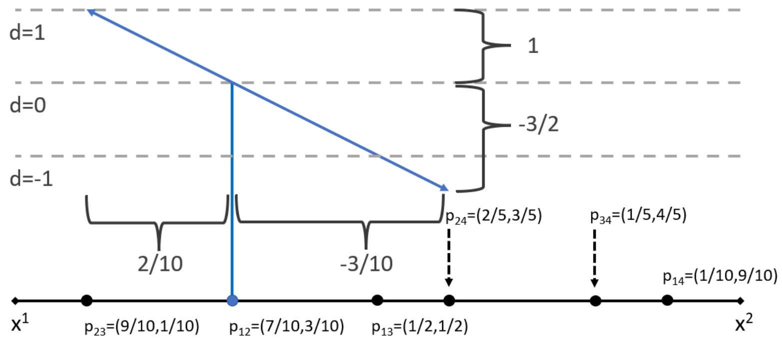


Figure 5: Retrieving $\sigma_{134} = -\frac{2}{3}$ for the preference from Figure 4.

In the figure, the $(p_{23} - p_{12})$- and $(p_{24} - p_{12})$-distances are associated with "d-levels" of utility differences. $(p_{23} - p_{12}) = -\frac{2}{3}(p_{24} - p_{12})$ means that the positive a_1, a_2-difference on $p_{23} - p_{12}$ corresponds to a $\frac{3}{2}$-larger and negative a_1, a_2-difference on $p_{24} - p_{12}$.

Figure 6 adds graphical derivations for σ_{123}, σ_{124}. As we see, fixing σ_{134} like in Figure 5, and fixing σ_{124} based on $(p_{23} - p_{13})$- and $(p_{34} - p_{13})$-distances makes it impossible to assign a unique value to σ_{123} or, equivalently, a unique slope to a_1, a_4-differences (yellow lines).

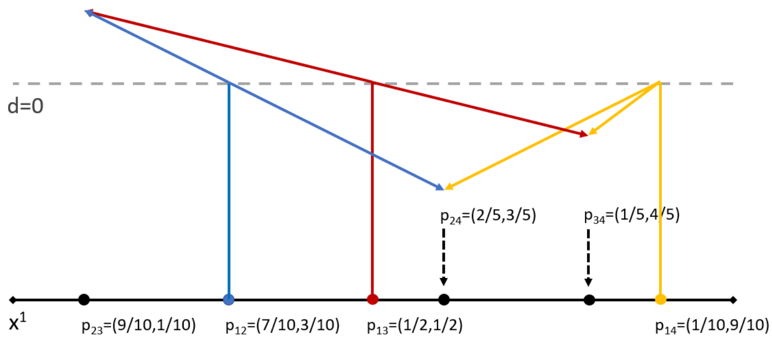


Figure 6: The Figure-4 preference is inconsistent with $\sigma_{123}\sigma_{134} = \sigma_{124}$.

To find a preference that is consistent with $\sigma_{123}\sigma_{134} = \sigma_{124}$, relocate any of the six indifference points in Figure 6, e.g. p_{14}. With $p_{14} = (0, 1)$, we get $\sigma_{123} = 2$, so that $\sigma_{123}\sigma_{134} = 2 \cdot (-\frac{2}{3}) = -\frac{4}{3} = \sigma_{124}$. The modified preference is illustrated in Figure 7 below. As we see, $\sigma_{123}\sigma_{134} = \sigma_{124}$ guarantees unique slopes for a_1, a_2-, a_1, a_3-, and a_1, a_4-differences.

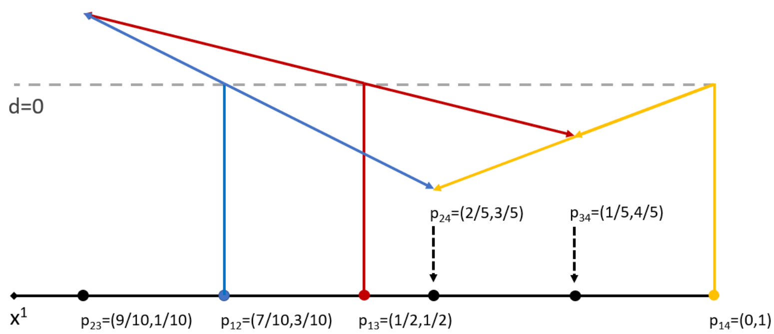


Figure 7: A preference with transitive preference sensitivity ($\sigma_{123}\sigma_{134} = \sigma_{124}$).

For general decision problems (A, X) and preferences $\succsim$ exhibiting reversals for all distinct pairs of acts $a_i, a_j \in A$, global consistency among preference sensitivities is necessary and sufficient for an n-act representation based on triple representations. This motivates the following property.

Definition 3.14. *(Transitive Preference Sensitivity)*
For distinct acts a_i, a_j, a_k, a_ℓ, let $p_{ij} \in P^0_{ij}$, $p_{ik} \in P^0_{ik}$, $p_{i\ell} \in P^0_{i\ell}$, $p_{jk} \in P^0_{jk}$, $p_{j\ell} \in P^0_{j\ell}$, $p_{k\ell} \in P^0_{k\ell}$, and scalars $\sigma_{ijk}, \sigma_{ik\ell}, \sigma_{ij\ell}$ satisfy

$$\sigma_{ijk} p_{k\ell} + (1 - \sigma_{ijk}) p_{i\ell} = p_{j\ell}, \;\; \sigma_{ik\ell} p_{j\ell} + (1 - \sigma_{ik\ell}) p_{ij} = p_{jk}, \;\; \text{and } \sigma_{ij\ell} p_{k\ell} + (1 - \sigma_{ij\ell}) p_{ik} = p_{jk}.$$

Then $\succsim$ has transitive preference sensitivity if $\sigma_{ij\ell} = \sigma_{ijk}\sigma_{ik\ell}$.

Transitive preference sensitivity extends intuitions from the two-state problems above to $m > 2$ states. The difference with m states is that not all indifference points need be collinear. Since sensitivities σ_{ijk} must be derived from collinear points $p_{i\ell}, p_{j\ell}, p_{k\ell}$, it is easy to see that the six indifference points must be colocated on an extended triangle as in Figure 8 below.

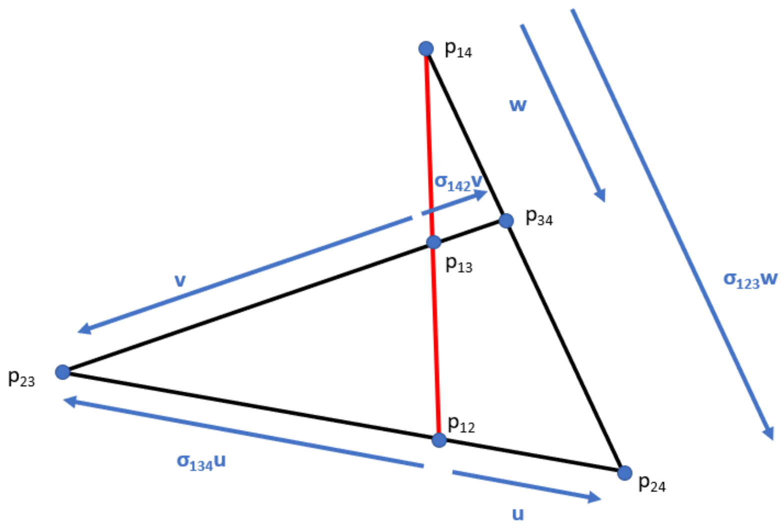


Figure 8: Transitive preference sensitivity $\sigma_{123}\sigma_{134} = \sigma_{124}$ on an extended triangle for $m > 2$ states.

The red line in Figure 8 shows an interesting consequence of Property 3.14: For six points on an extended triangle satisfying Property 3.14, the equation $\sigma_{123}\sigma_{134} = \sigma_{124}$ is the one from Menelaus's Theorem. I.e., p_{12}, p_{13}, p_{14} must be collinear whenever six indifference points for a quadruple $\{a_1, a_2, a_3, a_4\} \in A$ satisfy $\sigma_{123}\sigma_{134} = \sigma_{124}$.

The next Theorem proves that transitive preference sensitivity and the properties from Sections 2, 3.i, and 3.ii are necessary and sufficient for a difference representation over n acts forming a *fully connected component*.

In what follows, for any difference representation $\mathcal{D}_A$ on a problem (A, X) and any $B \subseteq A$, use $\mathcal{D}_{A|B}$ to denote the restriction of $\mathcal{D}_A$ to acts in B.

Theorem 3.15. Difference Representation for n Undominated Acts
Let (A, X) be a decision problem, and let $B \subseteq A$ be a fully connected component wrt $\succsim$. Then $\succsim$ admits a difference representation on (B, X) iff it satisfies pairwise linearity, transitivity, scaling transitivity, and transitive preference sensitivity.

For the proof, we will need the following Lemma.

Lemma 3.16. *(Line through Indifference Sets)*
For a decision problem (A, X), let $\succsim$ exhibit reversals for every pair of acts on $B \subseteq A$, and let $\mathcal{H} = \{\langle P^0_{ij}\rangle |\ a_i, a_j \in B,\ a_i \neq a_j\}$ define a hyperplane arrangement of rank 2. Then there exists a Line $L = \{v + \lambda w |\ \lambda \in \mathbb{R},\ v, w \in \mathbb{R}^m\}$ such that $L \cap P^0_{ij}$ is non-empty for every pair of acts $a_i, a_j \in B$.

Proof. Throughout, write $\langle P^*\rangle := \bigcap_{a_i,a_j \in B} \langle P^0_{ij}\rangle$ for the center of $\mathcal{H}$ or, equivalently, the subspace of total indifference between all acts in B. Since $\mathcal{H}$ has rank 2, it immediately follows that $\langle P^*\rangle$ is $m-2$-dimensional. Also, let $\mathcal{H}_\Delta := \{P^0_{ij} |\ a_i, a_j \in B,\ a_i \neq a_j\}$ collect the indifference sets induced by $\mathcal{H}$. Now distinguish two cases:

1) $\langle P^*\rangle \cap \Delta^+(X)$ is non-empty: Take any line $\tilde{L} \subset H^1(\mathbf{1})$ intersecting all distinct hyperplanes in $\mathcal{H}$ at different points, with some or all intersections possibly outside the simplex.[9] Now, taking any total indifference point $p^* \in \langle P^*\rangle \cap \Delta^+(X)$, for any distinct pair of acts $a_i, a_j \in B$ and $q_{ij} \in \tilde{L} \cap \langle P^0_{ij}\rangle$, we have that $\langle p^*, q_{ij}\rangle \in \langle P^0_{ij}\rangle$. In particular, for any $\alpha \in (0,1)$, $\tilde{L}_\alpha = \{\alpha p^* + (1-\alpha)(v + \lambda w)\}$ still intersects all indifference hyperplanes at distinct points $q_{ij}(\alpha) = \alpha p^* + (1-\alpha) q_{ij}$. Since $q_{ij}(1) = p^* \in \Delta^+(X)$ for all distinct pairs of acts $a_i, a_j \in B$, there must then exist $\overline{\alpha} \in [0,1)$ such that $q_{ij}(\alpha) \in P^0_{ij} = \langle P^0_{ij}\rangle \cap \Delta(X)$ for all distinct pairs of acts $a_i, a_j \in B$ and all $\alpha \in (\overline{\alpha}, 1]$.

2) $\langle P^*\rangle \cap \Delta^+(X)$ is empty: To start, recalling that $\langle P^*\rangle$ is $m-2$-dimensional, note that no two distinct indifference hyperplanes in $\mathcal{H}$ may intersect outside of $\langle P^*\rangle$ (or else they would immediately be coplanar). Now, since all distinct pairs $a_i, a_j \in B$ exhibit reversals and since $\langle P^*\rangle$ intersects at most the boundary of $\Delta(X)$, it follows that every indifference set $P^0_{ij} \in \mathcal{H}_\Delta$ splits the simplex into two convex polytopes $\Delta^1_{ij}, \Delta^2_{ij}$. Since this is true for all distinct pairs of acts $a_i, a_j \in B$ and since no two indifference hyperplanes in $\mathcal{H}$ may intersect outside $\langle P^*\rangle$, for any distinct pair $a_1, a_2 \in B$ and any other distinct $a_i, a_j \in B$ with $P^0_{12} \neq P^0_{ij}$, we must have either $P^0_{ij} \subset \Delta^1_{12}$ or $P^0_{ij} \subset \Delta^2_{12}$.

Use $\mathbf{P}^1_{12}$ to denote the set of indifference sets contained in Δ^1_{12} and similarly define $\mathbf{P}^2_{12}$. Now take any $P^0_{ij} \in \mathbf{P}^1_{12}$ and define $\Delta^1_{ij}, \Delta^2_{ij}$ such that $P^0_{12} \in \Delta^2_{ij}$. Letting $\mathbf{P}^2_{ij}$ denote the set of indifference sets contained in Δ^2_{ij}, by construction we must have $\mathbf{P}^2_{12} \cup \{P^0_{12}\} \subset \mathbf{P}^2_{ij}$. An immediate implications is that $\mathbf{P}^1_{ij} \subset \mathbf{P}^1_{12}$. Iterating this construction and recalling that $\mathcal{H}$ is finite, it is now easy to see that there exists an indifference set $\overline{P} \in \mathcal{H}_\Delta$ with associated convex polytopes $\overline{\Delta}^1$, $\overline{\Delta}^2$, such that all other indifference sets $P^0_{ij} \neq \overline{P}$ in $\mathcal{H}_\Delta$ are contained in $\overline{\Delta}^2$.

Moreover, redoing the same construction but starting from $P^0_{ij} \in \mathbf{P}^2_{12}$ yields a second $\underline{P} \in \mathcal{H}_\Delta$ with associated convex polytopes $\underline{\Delta}^1$, $\underline{\Delta}^2$, such that all other indifference sets $P^0_{ij} \neq \underline{P}$ in $\mathcal{H}_\Delta$ (including $\overline{P}$) are contained in $\underline{\Delta}^2$.

In combination, the convex polytopes $\overline{\Delta}^1$, $\underline{\Delta}^1$, $\overline{\Delta}^2 \cap \underline{\Delta}^2$ are such that all $P^0_{ij} \neq \overline{P}, \underline{P}$, for distinct $a_i, a_j \in B$ are contained in $\overline{\Delta}^2 \cap \underline{\Delta}^2$. Now take any $\overline{p} \in \overline{\Delta}^1$, $\underline{p} \in \underline{\Delta}^1$ and define $L = \{\overline{p} + \lambda(\underline{p} - \overline{p}),\ \lambda \in \mathbb{R}\}$.

By construction, for any distinct pair of acts $a_i, a_j \in B$, $L \cap \Delta(X)$ intersects both halfspaces defined by the indifference hyperplane $\langle P^0_{ij}\rangle$. Consequently, $L \cap \Delta(X)$ must then also intersect each of the hyperplanes $\langle P^0_{ij}\rangle$ inside the simplex.

The proof of Lemma 3.16 is now complete. ∎

[9] Since each distinct indifference hyperplane defines a distinct affine hyperplane in $\mathbb{R}^{m-1}$ when intersected with $H^1(\mathbf{1})$, any line $\tilde{L}$ that is not parallel to any of these affine hyperplanes will feature the desired distinct intersections.

Proof. (*Theorem 3.15*)

$\Rightarrow$: Take p_{ij}, p_{ik}, p_{jk}, $p_{i\ell}$,$p_{j\ell}$, $p_{k\ell}$ as in Definition 3.14. If $\succsim$ has a difference representation on (B, X), then

$$\begin{aligned}\sigma_{ij\ell} d_{ik}(p_{k\ell} - p_{ik}) &= d_{ik}(p_{jk} - p_{ik}) = d_{ij}(p_{jk} - p_{ij}) \\ &= \sigma_{ik\ell} d_{ij}(p_{j\ell} - p_{ij}) = \sigma_{ik\ell} d_{i\ell}(p_{j\ell} - p_{i\ell}) \\ &= \sigma_{ijk}\sigma_{ik\ell} d_{i\ell}(p_{k\ell} - p_{i\ell}) = \sigma_{ijk}\sigma_{ik\ell} d_{ik}(p_{k\ell} - p_{ik})\end{aligned}$$

where strong transitivity was used for the second, fourth, and sixth equalities.

It follows that $\sigma_{ij\ell} = \sigma_{ijk}\sigma_{ik\ell}$.

$\Leftarrow$: First assume that $P^0_{ij} = P^0_{ik}$ for some distinct pair $a_i, a_j \in B$ and all $a_k \in B \backslash \{a_i, a_j\}$. By transitivity, it then follows that there is $P \subset \Delta(X)$ such that $P = P^0_{k\ell}$ for all distinct $a_k, a_\ell \in B$. Now take any $p \in \Delta(X) \backslash P$. By completeness and transitivity, there must be a strict local ranking over all acts in B at p. Wlog, let $a_1 \succ_p \cdots \succ_p a_{|B|}$. Since P separates $H^+(d_{ij})$ and $H^-(d_{ij})$ for all distinct pairs $a_i, a_j \in B$, it follows that $H^+(d_{ij}) = H^+(d_{k\ell})$ and $H^-(d_{ij}) = H^-(d_{k\ell})$ for all a_i, a_j, a_k, a_ℓ such that $i < j$ and $k < \ell$. Now take any difference representation $d_{1,|B|}$ for $\succsim$ on $(\{a_1, a_{|B|}\}, \Delta(X))$. By construction, $d_{1,|B|}$ also represents $\succsim$ on $(\{a_i, a_j\}, \Delta(X))$ for any $a_i, a_j \in B$ with $1 \le i < j \le |B|$. It then remains to find scalars α_{ij} for every $1 \le i < j \le |B|$ such that each $\alpha_{ij} d_{1,|B|}$ represents $\succsim$ on $(\{a_i, a_j\}, \Delta(X))$, and such that all triples of difference vectors satisfy strong transitivity. So let $\alpha_{1,|B|} = 1$ and, for every $a_i \ne a_1$, choose $\alpha_{1i} > 0$ such that $\alpha_{12} < \cdots < \alpha_{1,|B|-1} < \alpha_{1,|B|}$. Moreover, for any $a_i, a_j \ne a_1$, define $\alpha_{ij} = \alpha_{1j} - \alpha_{1i}$. By construction, we have $\alpha_{ij} > 0$ for every a_i, a_j with $i < j$, and it follows that $d_{ij} := \alpha_{ij} d_{1,|B|}$ represents $\succsim$ on $(\{a_i, a_j\}, \Delta(X))$ for every $a_i, a_j \in B$ with $1 \le i < j \le |B|$. To show that strong transitivity is satisfied, first note that $d_{1i} + d_{ij} = (\alpha_{1i} + \alpha_{ij}) d_{1|B|} = \alpha_{1j} d_{1|B|} = d_{1j}$ for all $a_i, a_j \ne a_1$. Next, using strong transitivity on triples involving a_1, we can observe that $d_{ij} + d_{jk} = d_{1j} - d_{1i} + d_{1k} - d_{1j} = d_{1k} - d_{1i} = d_{ik}$ for all $a_i, a_j, a_k \ne a_1$. It now follows that $\mathcal{D} = \left[d_{12}, \ldots, d_{|B|-1,|B|}\right]$ is a difference representation for $\succsim$ on $(B, \Delta(X))$.

Henceforth assume that there exist distinct $a_i, a_j \in B$ and $a_k \in B \backslash \{a_i, a_j\}$ such that $P^0_{ij} \ne P^0_{ik}$. Note that by transitivity it then follows that $P^0_{ij}, P^0_{jk}, P^0_{ik}$ is a triple of pairwise distinct indifference sets. By Theorem 3.10, there then exist a difference representation for $\succsim$ on $(\{a_i, a_j, a_k\}, X)$ such that d_{ij}, d_{jk}, d_{ik} are unique up to a common positive scalar.

The remainder of the proof proceeds by induction over $|B| \ge 3$. The induction start is the previous paragraph.

So let $|B| \ge 3$ and suppose there is a difference representation $\mathcal{D}_C$ on (C, X) for some $C \subset B$, $|C| = |B - 1|$. Let $\{a_B\} = C \backslash B$. Note that, since B is fully connected, there is a ratio-scale unique difference vector $\tilde{d}_{iB}$ representing $\succsim$ on any $(\{a_i, a_B\}, X)$, $a_i \in C$.

Now consider any distinct $a_i, a_j \in C$. Using Theorem 3.10, there exist positive scalars α^{ijB}_{iB}, α^{ijB}_{jB} such that

$$\mathbf{0} = d_{ij} + \alpha^{ijB}_{jB} \tilde{d}_{jB} - \alpha^{ijB}_{iB} \tilde{d}_{iB}.$$

Furthermore, take any $a_k \in C$ such that $P^0_{ij}, P^0_{jk}, P^0_{ik}$ are pairwise distinct. Then, again using Theorem 3.10, there exist positive scalars α^{ikB}_{iB}, α^{ikB}_{kB}, α^{jkB}_{jB}, α^{jkB}_{kB} such that

$$\mathbf{0} = d_{jk} + \alpha^{jkB}_{kB} \tilde{d}_{kB} - \alpha^{jkB}_{jB} \tilde{d}_{jB}$$

$$\mathbf{0} = d_{ik} + \alpha_{kB}^{ikB}\tilde{d}_{kB} - \alpha_{iB}^{ikB}\tilde{d}_{iB}$$

Adding the first and second equation and subtracting the third one yields:

$$\mathbf{0} = d_{ij} + d_{jk} - d_{ik} + \left(\alpha_{jB}^{ijB} - \alpha_{jB}^{jkB}\right)\tilde{d}_{jB} + \left(\alpha_{kB}^{jkB} - \alpha_{kB}^{ikB}\right)\tilde{d}_{kB} + \left(\alpha_{iB}^{ikB} - \alpha_{iB}^{ijB}\right)\tilde{d}_{iB} \tag{1}$$

And using $\mathbf{0} = d_{ij} + d_{jk} - d_{ik}$ this further reduces to

$$\mathbf{0} = \left(\alpha_{jB}^{ijB} - \alpha_{jB}^{jkB}\right)\tilde{d}_{jB} + \left(\alpha_{kB}^{jkB} - \alpha_{kB}^{ikB}\right)\tilde{d}_{kB} + \left(\alpha_{iB}^{ikB} - \alpha_{iB}^{ijB}\right)\tilde{d}_{iB}.$$

Now distinguish the following cases:

1) $P_{mB}^0 = P_{m\ell}^0$ for some $a_m, a_\ell \in \{a_i, a_j, a_k\}$:
First, note that $P_{mB}^0 = P_{m\ell}^0$ only for one pair a_m, a_ℓ out of $\{a_i, a_j, a_k\}$. Otherwise, using transitivity, $P_{ij}^0, P_{jk}^0, P_{ik}^0$ could not be pairwise distinct. Wlog let $P_{iB}^0 = P_{ik}^0 = P_{kB}^0$.

Since $\tilde{d}_{iB}$ and $\tilde{d}_{kB}$ are the linearly dependent, there exists a scalar β such that

$$\mathbf{0} = \left(\alpha_{jB}^{ijB} - \alpha_{jB}^{jkB}\right)\tilde{d}_{jB} + \left[\left(\alpha_{kB}^{jkB} - \alpha_{kB}^{ikB}\right)\beta + \left(\alpha_{iB}^{ikB} - \alpha_{iB}^{ijB}\right)\right]\tilde{d}_{iB}.$$

Note that $\tilde{d}_{jB}$ and $\tilde{d}_{iB}$ must be linearly independent. (Otherwise, with transitivity, one would have $P_{ik}^0 = P_{ij}^0$.) It follows that $\alpha_{jB}^{ijB} = \alpha_{jB}^{jkB} =: \alpha_{jB}$.

Then, adding the second strong transitivity equation above to the first, one has

$$\begin{aligned}\mathbf{0} &= d_{ij} + d_{jk} + (\alpha_{jB} - \alpha_{jB})\tilde{d}_{jB} - \alpha_{iB}^{ijB}\tilde{d}_{iB} + \alpha_{kB}^{jkB}\tilde{d}_{kB} \\ &= d_{ik} + \alpha_{kB}^{jkB}\tilde{d}_{kB} - \alpha_{iB}^{ijB}\tilde{d}_{iB}.\end{aligned}$$

Hence, $\alpha_{iB}^{ikB} = \alpha_{iB}^{ijB} =: \alpha_{iB}$ and $\alpha_{kB}^{jkB} = \alpha_{kB}^{ikB} =: \alpha_{kB}$ yields a ratio-scale unique difference representation on $(\{a_i, a_j, a_k, a_B\}, X)$.

2) $P_{mB}^0 \neq P_{m\ell}^0$ for all $a_m, a_\ell \in \{a_i, a_j, a_k\}$:
To begin, note that the matrix $\tilde{\mathcal{D}} = [d_{ij}, d_{ik}, d_{jk}, \tilde{d}_{iB}, \tilde{d}_{jB}, \tilde{d}_{kB}]$. Must be of rank 2 or 3. If not, $\text{Rank}(\tilde{\mathcal{D}}) = 1$ would imply $P_{ij}^0 = P_{ik}^0$ for all distinct $a_i, a_j, a_k \in B$, whereas $\text{Rank}(\tilde{\mathcal{D}}) = 4$ would require that the matrix of difference vectors for one triple out of $\{a_i, a_j, a_k, a_B\}$ be of rank larger than 2 – a contradiction with Theorem 3.10.

Now, if $\text{Rank}(\tilde{\mathcal{D}}) = 3$, then $\tilde{\mathcal{D}}$ has a 3-dimensional null space using the rank-nullity theorem. Furthermore, since $\text{Rank}([d_{ij}, d_{ik}, d_{jk}]]) = 2$ by Theorem 3.10, $\tilde{d}_{iB}, d_{ij}, d_{ik}$ must be linearly independent. Hence, the three strong transitivity equations involving a_B are are linearly independent in $\mathcal{N}(\tilde{\mathcal{D}})$. Since the fourth strong transitivity equation $d_{ij} + d_{jk} - d_{ik} = \mathbf{0}$ must then linearly depend on these other three in $\mathcal{N}(\tilde{\mathcal{D}})$, and since equation 1 is the unique way one may express $\mathbf{0} = d_{ij} + d_{jk} - d_{ik}$ in terms of the three other equations, it follows that $\alpha_{iB}^{ijB} = \alpha_{iB}^{ikB} =: \alpha_{iB}$, $\alpha_{jB}^{ijB} = \alpha_{jB}^{jkB} =: \alpha_{jB}$, $\alpha_{kB}^{ikB} = \alpha_{kB}^{jkB} =: \alpha_{kB}$, yielding a ratio-scale unique difference representation on $(\{a_i, a_j, a_k, a_B\}, X)$.

Otherwise, if $\text{Rank}(\tilde{\mathcal{D}}) = 2$, the hyperplane arrangement $\{\langle P_{m\ell}^0 \rangle |\ a_m, a_\ell \in \{a_i, a_j, a_k, a_B\},\ a_m \neq a_\ell\}$ is of rank 2 as well. So, using Lemma 3.16, fix a line L passing at distinct points through each distinct indifference set. For each distinct $a_m, a_n \in \{a_i, a_j, a_k, a_B\}$ let $p_{mn} \in P_{mn}^0$ be the corresponding intersection. Using that all p_{mn} lie on the same line, and that all pairwise indifference sets among $\{a_i, a_j, a_k, a_B\}$ are pairwise distinct, there exist unique and non-zero $\sigma_{ijk}, \sigma_{ikB}, \sigma_{ijB}$ such that

$$p_{jB} = \sigma_{ijk}(p_{kB} - p_{iB}) + p_{iB}, \quad p_{jk} = \sigma_{ikB}(p_{jB} - p_{ij}) + p_{ij}, \quad p_{jk} = \sigma_{ijB}(p_{kB} - p_{ik}) + p_{ik}.$$

Using strong transitivity among the triples, it now follows that

$$\begin{aligned}
\alpha_{iB}^{ijB} \tilde{d}_{iB} \cdot (p_{jB} - p_{iB}) &= d_{ij} \cdot (p_{jB} - p_{ij}) \\
&= \frac{1}{\sigma_{ikB}} d_{ij} \cdot (p_{jk} - p_{ij}) = \frac{1}{\sigma_{ikB}} d_{ik} \cdot (p_{jk} - p_{ik}) \\
&= \frac{\sigma_{ijB}}{\sigma_{ikB}} d_{ik} \cdot (p_{kB} - p_{ik}) = \frac{\sigma_{ijB}}{\sigma_{ikB}} \alpha_{iB}^{ikB} \tilde{d}_{iB} \cdot (p_{kB} - p_{iB}) \\
&= \frac{\sigma_{ijB}}{\sigma_{ijk}\sigma_{ikB}} \alpha_{iB}^{ikB} \tilde{d}_{iB} \cdot (p_{jB} - p_{iB}).
\end{aligned}$$

And using transitive preference sensitivity $\sigma_{ijk}\sigma_{ikB} = \sigma_{ijB}$ yields $\alpha_{iB}^{ijB} = \alpha_{iB}^{ikB} =: \alpha_{iB}$.

Further, like in case (1), it follows that $\alpha_{jB} =: \alpha_{jB}^{ijB}$ and $\alpha_{kB} =: \alpha_{kB}^{ikB}$ also solve the strong transitivity equation for a_j, a_k, a_B, again yielding a ratio-scale unique difference representation on $(\{a_i, a_j, a_k, a_B\}, X)$.

Since a_i, a_j, a_k where arbitrary and since each case yielded a unique difference representation up to a common positive scalar, it follows that the weights α_{iB} for any $a_i \in B$ are independent of which other acts $a_j, a_k \in B$ are used in the construction. It follows that there exists a ratio-scale unique representation for $\succsim$ on (B, X). The induction step and hence the proof is now complete. □

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