More Reasoning, Less Outcomes: A Monotonicity Result for Reasoning in Dynamic Games

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Abstract

A focus function in a dynamic game describes, for every player and each of his information sets, the collection of opponents' information sets he reasons about. Every focus function induces a rationalizability procedure in which a player believes, whenever possible, that his opponents choose rationally at those information sets he reasons about. Under certain conditions, we show that if the players start reasoning about more information sets, then the set of outcomes induced by the associated rationalizability procedure becomes smaller or stays the same. This result does not hold on the level of strategies, unless the players only reason about present and future information sets. The monotonicity result enables us to derive existing theorems, such as the relation in terms of outcomes between forward and backward induction reasoning, but also paves the way for new results.

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Keywords: Dynamic games, reasoning, rationalizability, forward induction, backward induction, epistemic priority, focus function

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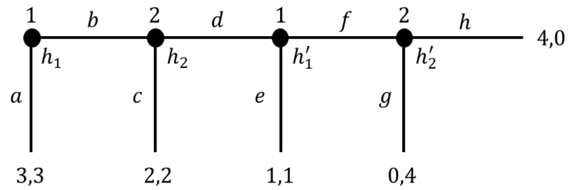

Figure 1: Reasoning about more information sets may not be more restrictive in terms of strategies

1 Introduction

To reach a meaningful decision in a dynamic game it is important to reason, at each of your information sets, about the opponents' rationality at some, or all, of their information sets. This is reflected in many equilibrium and rationalizability concepts for dynamic games. Consider, for instance, the forward induction concept of *strong rationalizability* (Pearce (1984), Battigalli (1997)), also known as *extensive-form rationalizability,* in which a player must believe, whenever possible, that his opponents choose rationally at all of their information sets. It thus requires the player to always reason about *all* the opponents' information sets. Compare this to the backward induction concepts of *backward dominance* (Perea (2014)) and *backwards rationalizability*[1] (Penta (2015), Catonini and Penta (2022), Perea (2014)) in which a player must always believe that his opponents will choose rationally at all present and future information sets. It only requires a player to actively reason about the *present* and *future* information sets of his opponents.

As strong rationalizability requires reasoning about more information sets than backward dominance, does this mean that the former concept will also be more restrictive than the latter? A partial answer is: *not* in terms of strategies.

To see this, consider the game in Figure 1, which is an adaptation of Figure 3 in Reny (1992). Acccording to strong rationalizability, player 2 must believe at h_2 that player 1 has chosen rationally at h_1, which is only possible if player 1 chooses the strategy (b, f). Indeed, amongst the player 1 strategies that reach h_2, only (b, f) can yield player 1 more than 3 – a utility he could have guaranteed by choosing a at h_1. As such, player 2 will choose the strategy (d, g). Player 1, anticipating on this, will choose a at the beginning.

In the backward dominance procedure, player 2 must believe at h_2 (i) that player 1 will choose rationally at h'_1, and (ii) that player 1 believes at h'_1 that player 2 will choose rationally at h'_2. Therefore, player 2 believes at h_2 that player 1 chooses (b, e), which leaves c as the optimal strategy for player 2. Player 1, anticipating on this, will again choose a at the beginning. We thus see that strong rationalizability and backward dominance lead to a unique, yet distinct, strategy for player 2, despite the fact that strong rationalizability requires the players to reason about more information sets. However, both concepts lead to the same *outcome* in the game, which is the unique backward induction outcome.

[1] The difference between the two concepts is that the latter requires the beliefs to satisfy forward consistency (also referred to as Bayesian updating), whereas the former does not.

Battigalli (1997) has shown that the latter is true for *all* games with perfect information where there are no relevant ties.[2] Later, the result has been generalized in Perea (2017) and Catonini (2020) by showing that in every dynamic game with observed past choices[3] the set of outcomes induced by strong rationalizability is always included in the set of outcomes induced by the backward dominance procedure[4]. For this specific scenario we can thus say that reasoning about more information sets leads to less (or possibly the same) outcomes.

In this paper we generalize this finding by showing that, under certain conditions, reasoning about more information sets always leads to a smaller (or equal) set of outcomes. To this purpose we introduce a family of rationalizability procedures parametrized by the collections of opponents' information sets that the players reason about at their various information sets. Formally, the parameter is a *focus function* f (Perea and Tsakas (2019)) which specifies for every player, and each of his information sets, the collection of opponents' information sets he reasons about. The associated belief restriction is that every player believes, whenever possible, that his opponents choose rationally at those information sets he reasons about. By iterating this condition we obtain a recursive reduction procedure that we call *f-rationalizability.*

It turns out that many of the existing rationalizability concepts in the literature, including strong rationalizability, backward dominance, and the Ben-Porath procedure (Ben-Porath (1997)), can be phrased as f-rationalizability concepts with an appropriately chosen focus function f. In our definition of the focus function f, the collection of opponents' information sets that a player reasons about is allowed to depend on the strategies that have already been eliminated at the various information sets. This enables us to model rationalizability procedures like *forward and backward rationalizability* (Meier and Perea (2023)) where different collections of opponents' information sets enter at different stages of the procedure, according to some given epistemic priority ordering.

In Theorem 4.1 we prove the existence of f-rationalizable strategies under appropriate conditions on the focus function f. The conditions are the following: (a) f must be *individually forward decreasing,* which means that as the game moves on, a player should not start to reason about information sets he did not reason about before; (b) f must *individually preserve focus on past information sets,* which means that if a player reasons, at a particular information set, about a past information set h, then he will still reason about h at all future information sets; and (c) f must be *monotone,* which means that a player can only start reasoning about more, or the same, information sets if more strategies get eliminated at the various information sets in the game. The conditions above rule out the possibility of dynamically inconsistent beliefs, which would endanger the existence of f-rationalizable strategies. Many rationalizability concepts for dynamic games in the literature satisfy these conditions, and their existence is thus guaranteed by Theorem 4.1.

Under the same conditions, we show in Theorem 4.2 that additionally imposing *forward consistency*[5] (often referred to as Bayesian updating) on beliefs does not alter the sets of f-rationalizable strategies. More precisely, we define f-rationalizability as an iterated strict dominance procedure where beliefs do not

[2] No relevant ties means that for every player, different actions always lead to different utilities for that player.
[3] That is, there may be simultaneous moves, but a player always knows which actions have been chosen in the past.
[4] Catonini (2020) shows this result for the stronger concept of backwards rationalizability.
[5] We adopt this terminology from Battigalli, Catonini and Manili (2023).

explicitly enter the picture. But in the light of Lemma 3 in Pearce (1984) it can equivalently be phrased as an iterated never-a-best-reply procedure in which at every information set we remove strategies that are not optimal for any belief that is allowed at that information set. Theorem 4.2 then shows that under the conditions (a), (b) and (c) above, the output of f-rationalizability would not change if we additionally require every player's beliefs to be forward consistent, and to require that every player believes that his opponents' beliefs are forward consistent at those information sets he reasons about. However, imposing the stronger requirement that a player must believe that his opponents satisfy forward consistency also at those information sets he does not reason about, may alter the sets of f-rationalizable strategies. At the same time, this stronger condition would force a player to reason about information sets that are not prescribed by f, and would thus go against the nature of the focus function f.

Our main result, Theorem 5.1, compares two focus functions f and g such that f requires reasoning about more information sets than g, and shows that under certain conditions every outcome induced by a combination of f-rationalizable strategies is also induced by a combination of g-rationalizable strategies. The proof is constructive, as it explicitly shows how to transform a combination of f-rationalizable strategies into a combination of g-rationalizable strategies that yields the same outcome. The theorem also shows that if in f the players only reason about present and future information sets, then every f-rationalizable strategy is g-rationalizable. That is, in this case the set inclusion even holds on the level of strategies.

In fact, Theorem 5.1 states something stronger than the above monotonicity result, by showing that the outcomes induced by combinations of f-rationalizable strategies are the *same* as the outcomes that result if we first apply the g-rationalizability procedure and then the f-rationalizability procedure. In the latter combined procedure we thus give epistemic priority to reasoning according to g above reasoning according to f. This result therefore highlights that in terms of outcomes it does not matter whether we stick to reasoning in line with f all the way, or whether we first give epistemic priority to reasoning in line with g before we start reasoning in line with f. Again, the proof is constructive as it shows how to transform a combination of strategies surviving the first procedure into a combination of strategies surviving the second combined procedure yielding the same outcome, and *vice versa.*

The conditions under which the theorem holds are the following: (a) the focus functions f and g must be individually forward decreasing, individually preserving focus on past information sets, and monotone – the conditions needed to guarantee the existence of f- and g-rationalizable strategies; (b) f must be *collectively forward decreasing*, which means that as the game proceeds a player will never start reasoning about information sets that players before him (including himself) did not reason about; (c) f must *collectively preserve focus on past information sets,* which means that if a player reasons, at a particular information set, about a past information set h, then the players at later information sets will still reason about h; (d) f must be *transitively closed,* which means that if a player reasons at information set h about an information set h' where the player reasons about a third information set h'', then the player at h must reason about h'' as well; and (e) the focus function f is *monotone with respect to g*, which means that if we start from applying g-rationalizability for some rounds followed by applying f-rationalizability for some rounds, then either applying g-rationalizability for one more round, or applying g-rationalizability by one round less followed by applying f-rationalizability for one more round, should lead the players to reason about more, or the same, information sets as before.

The last property (e) is automatically satisfied if in f the collection of opponents' information sets that a player reasons about does not depend upon the strategies that have been eliminated at the various information sets in the game. Note that conditions (b) and (c) are interactive versions of their individual counterparts in (a), and state that the reasoning patterns of the different players should be "sufficiently aligned".

On the basis of Theorem 5.1 we can derive many existing results in the literature, such as the theorems by Battigalli (1997), Perea (2017) and Catonini (2020) mentioned above. But we can also use Theorem 5.1 to prove new relevant results. For instance, it can be shown that for every focus function f satisfying the properties (a) – (d) above, the computationally convenient *backwards order of elimination,* in which we first perform the required eliminations at the last information sets, then perform the required eliminations at the last and second-to-last information sets, and so on, will lead to the same set of outcomes as the original f-rationalizability procedure.[6]

Or compare two collections of "focal" information sets $H^1 \subseteq H^2$ and consider the focus mappings f_1, f_2 where all players, at every information set, only reason about opponents' information sets in H^1 and H^2, respectively. Then, the f_1- and f_2-rationalizability procedure may be considered restricted forms of forward induction reasoning in which a player only evaluates the opponents' optimality at past focal information sets in H^i, and uses this to form a belief about the opponents' present, future and unobserved past actions at the focal information sets in H^i. As $H^1 \subseteq H^2$, the f_2-rationalizability procedure may be considered the more fine-grained forward induction procedure of the two. It then immediately follows from Theorem 5.1 that f_2-rationalizability is more, or equally, restrictive in terms of outcomes than f_1-rationalizability.

We believe that Theorem 5.1 may be important for implementation theory and mechanism design, where a planner wishes to design a game form in order to achieve a certain goal with respect to a specific rationalizability concept. In the light of our theorem, the set of induced outcomes will become smaller, or stay the same, if the planner assumes that the agents start reasoning about more information sets. Within this context, Battigalli and Catonini (2024) prove an outcome-monotonicity result that is similar, in spirit, to ours: In the realm of incomplete information it is shown that when the planner evaluates the agents' behavior by strong rationalizability, then imposing more restrictions on the players' initial beliefs about types leads to less, or the same, outcomes.

The outline of the paper is as follows: In Section 2 we present our model of a dynamic game, together with some derived objects such as strategies, strict dominance, beliefs, and expected utility. In Section 3 we lay out the notion of a focus function, explain how it gives rise to a family of rationalizability procedures, and show how many of the existing rationalizability procedures can be modelled as members of this family. In Section 4 we present the conditions under which we can guarantee the existence of rationalizable strategies, and such that the output of the rationalizability procedure will remain unchanged if we additionally impose the abovementioned forward consistency requirement. In Section 5 we present our main result, Theorem 5.1, and provide a sketch of the proof. In Section 6 we show how this theorem can be used to prove existing and new results. In Section 7 we provide some concluding remarks. All the proofs can be found in Section 8.

[6] To be precise, we also need that f is monotone with respect to this backwards order of elimination.

2 Games, Strategies and Strict Dominance

In this section we lay out the model of a dynamic game, together with some derived objects such as strategies, strict dominance, beliefs, expected utility, and optimality of strategies.

2.1 Dynamic Games

In this paper we consider finite dynamic games that allow for simultaneous moves and imperfect information. Formally, a *dynamic game* is a tuple $D = (I, P, I^a, (A_i, H_i)_{i \in I}, Z, (u_i)_{i \in I})$, where

(a) I is the finite set of *players*;

(b) P is the finite set of *past action profiles*, or *histories;*

(c) the mapping I^a assigns to every non-terminal history $p \in P$ the (possibly empty) set of *active players* $I^a(p) \subseteq I$ who must choose after history p. If $I^a(p)$ contains more than one player, there are simultaneous moves after p. If $I^a(p)$ is empty, the game terminates after p. We say that p is a *terminal history,* or an *outcome,* if $I^a(p) = \emptyset$, and p is called a *non-terminal history* otherwise. By P_i we denote the set of histories $p \in P$ with $i \in I^a(p)$;

(d) for every player i, the mapping A_i assigns to every history $p \in P_i$ the finite set of *actions* $A_i(p)$ from which player i can choose after history p. By $\varnothing$ we denote the empty history, marking the beginning of the game. It is also the unique history of length 0. For every $k \geq 1$, the histories of length k can then inductively be defined as the pairs $p' = (p, (a_i)_{i \in I^a(p)})$ where p is a non-terminal history of length $k-1$, and such that for every $i \in I^a(p)$ we have that $a_i \in A_i(p)$. We assume that the objects P, I^a and $(A_i)_{i \in I}$ are such that the histories in P are precisely those that are histories of length k for some $k \geq 0$;

(e) for every player i there is a partition H_i of the set of histories P_i where i is active. Every partition element $h_i \in H_i$ is called an *information set* for player i. In case h_i contains more than one history, the interpretation is that player i does not know at h_i which history in h_i has been realized. The objects A_i and H_i must be such that for every information set $h_i \in H_i$ and every two histories p, p' in h_i, we have that $A_i(p) = A_i(p')$. We can thus write $A_i(h_i)$ for the unique set of available actions at h_i. Moreover, it must be that $A_i(h_i) \cap A_i(h_i') = \emptyset$ for every two distinct information sets $h_i, h_i' \in H_i$;

(f) $Z \subseteq P$ is the collection of *terminal histories* or *outcomes*;

(g) for every player i there is a utility function $u_i : Z \to \mathbf{R}$.

This definition follows Osborne and Rubinstein (1994). We say that a history p *precedes* a history p' (or p' *follows* p) if p' results by adding some action profiles after p. Let $H := \cup_{i \in I} H_i$ be the collection of all information sets for all players. For every two information sets $h, h' \in H$, we say that h *precedes* h' (or h' *follows* h) if there is a history $p \in h$ and a history $p' \in h'$ such that p precedes p'. Two information sets h, h' are *simultaneous* if there is some history p which belongs to both h and h'. We say that h *weakly precedes* h' (or h' *weakly follows* h) if either h precedes h', or h, h' are simultaneous. The game has a *cycle-free ordering of information sets* if there are no information sets $h^1, h^2, ..., h^K$ such that h^k weakly precedes h^{k+1} for all $k \in \{1, ..., K-1\}$ and h^K precedes h^1.

For a player i, an information set $h_i \in H_i$, and a player j (possibly equal to i) we denote by $H_j^+(h_i)$ the collection of information sets in H_j that weakly follow h_i, and by $H_j^-(h_i)$ the collection of information sets in H_j that precede h_i. For a collection $\hat{H}_i \subseteq H_i$ we denote by $\hat{H}_i^{first}$ the collection of information sets in $\hat{H}_i$ that are not preceded by any other information set in $\hat{H}_i$. We say that $\hat{H}_i$ is *closed under weak followers* (*closed under precedessors*) if for every $h_i \in \hat{H}_i$ we have that $H_i^+(h_i) \subseteq \hat{H}_i$ ($H_i^-(h_i) \subseteq \hat{H}_i$).

The dynamic game satisfies *perfect recall* (Kuhn (1953)) if every player always remembers which actions he chose in the past, and which information he had about the opponents' past actions. Formally, for every player i, information set $h_i \in H_i$, and histories $p, p' \in h_i$, the sequence of player i actions in p and p' must be the same (and consequently, the collection of player i information sets that precede p and p' must be the same). In the sequel we will always assume that the dynamic game under consideration satisfies perfect recall and has a cycle-free ordering of information sets.

2.2 Strategies

A strategy for player i assigns an available action to every information set at which player i is active, and that is not excluded by earlier actions in the strategy. Formally, let $\tilde{s}_i$ be a mapping that assigns to *every* information set $h_i \in H_i$ some action $\tilde{s}_i(h_i) \in A_i(h_i)$. We call $\tilde{s}_i$ a *complete strategy.* Then, a history $p \in P$ is *excluded* by $\tilde{s}_i$ if there is some information set $h_i \in H_i$, with some history $p' \in h_i$ preceding p, such that $\tilde{s}_i(h_i)$ is different from the unique player i action at p' leading to p. An information set $h \in H$ is excluded by $\tilde{s}_i$ if all histories in h are excluded by $\tilde{s}_i$. The *strategy* induced by $\tilde{s}_i$ is the restriction of $\tilde{s}_i$ to those information sets in H_i that are not excluded by $\tilde{s}_i$. A mapping $s_i : \tilde{H}_i \to \cup_{h \in \tilde{H}_i} A_i(h)$, where $\tilde{H}_i \subseteq H_i$, is a *strategy* for player i if it is the strategy induced by a complete strategy.[7] By S_i we denote the set of strategies for player i, and by $S_{-i} := \times_{j \neq i} S_j$ the set of strategy combinations for i's opponents.

Consider a strategy profile $s = (s_i)_{i \in I}$ in $\times_{i \in I} S_i$. Then, s induces a unique outcome $z(s)$. We say that the strategy profile s *reaches* a history p if p precedes $z(s)$. Similarly, the strategy profile s is said to reach an information set h if s reaches a history in h.

For a given information set $h \in H$ and player i we define the sets

$$\begin{aligned} S(h) &:= \{s \in \times_{i \in I} S_i \mid s \text{ reaches } h\}, \\ S_i(h) &:= \{s_i \in S_i \mid \text{there is some } s_{-i} \in S_{-i} \text{ such that } (s_i, s_{-i}) \in S(h)\}, \text{ and} \\ S_{-i}(h) &:= \{s_{-i} \in S_{-i} \mid \text{there is some } s_i \in S_i \text{ such that } (s_i, s_{-i}) \in S(h)\}. \end{aligned}$$

It is well-known that under perfect recall we have, for every player i and every information set $h_i \in H_i$, that $S(h_i) = S_i(h_i) \times S_{-i}(h_i)$. For a given strategy $s_i \in S_i$ we denote by $H_i(s_i) := \{h_i \in H_i \mid s_i \in S_i(h_i)\}$ the collection of information sets for player i that the strategy s_i allows to be reached.

A *decision problem* for player i at an information set $h_i \in H_i$ is a pair $(D_i(h_i), D_{-i}(h_i))$, where $D_i(h_i) \subseteq S_i(h_i)$ and $D_{-i}(h_i) \subseteq S_{-i}(h_i)$.

[7] What we call a "strategy" is sometimes called a "plan of action" in the literature (Rubinstein (1991)), and what we call a "complete strategy" is often called a "strategy".

2.3 Strict Dominance and Optimal Strategies

For a given player i and information set $h_i \in H_i$, let $(D_i(h_i), D_{-i}(h_i))$ be a decision problem for player i at h_i. A strategy $s_i \in D_i(h_i)$ is said to be *strictly dominated* in $(D_i(h_i), D_{-i}(h_i))$ if there is a randomized strategy $\mu_i \in \Delta(D_i(h_i))$ for player i such that

$$\sum_{s'_i \in D_i(h_i)} \mu_i(s'_i) \cdot u_i(z(s'_i, s_{-i})) > u_i(z(s_i, s_{-i})) \text{ for all } s_{-i} \in D_{-i}(h_i).$$

From Pearce (1984) it is well-known that a strategy is not strictly dominated in a decision problem precisely when it is optimal for some probabilistic belief there. Formally, a (probabilistic) *belief* for player i is a probability distribution $b_i \in \Delta(S_{-i})$ on the opponents' strategy combinations. Here, $\Delta(X)$ denotes the set of probability distributions on a finite set X. For a given strategy $s_i \in S_i$ and belief $b_i \in \Delta(S_{-i})$, we denote by

$$u_i(s_i, b_i) := \sum_{s_{-i} \in S_{-i}} b_i(s_{-i}) \cdot u_i(z(s_i, s_{-i}))$$

the *expected utility* induced by strategy s_i under the belief b_i. Now, consider a player i, an information set $h_i \in H_i$, a decision problem $(D_i(h_i), D_{-i}(h_i))$ at h_i, and a belief $b_i \in \Delta(D_{-i}(h_i))$. A strategy $s_i \in D_i(h_i)$ is said to be *optimal* in $D_i(h_i)$ for the belief b_i if $u_i(s_i, b_i) \geq u_i(s'_i, b_i)$ for all $s'_i \in D_i(h_i)$. The following result corresponds to Lemma 3 in Pearce (1984).

Lemma 2.1 (Pearce's lemma) *Consider a player i, an information set $h_i \in H_i$, and a decision problem $(D_i(h_i), D_{-i}(h_i))$ for player i at h_i. Then, a strategy $s_i \in D_i(h_i)$ is not strictly dominated in $(D_i(h_i), D_{-i}(h_i))$, if and only if, s_i is optimal in $D_i(h_i)$ for some belief $b_i \in \Delta(D_{-i}(h_i))$.*

The proof follows from a simple adaptation of the proof in Pearce (1984), and is therefore omitted.

3 Rationalizability Procedures

In this section we present a family of rationalizability procedures parametrized by a *focus function* – a mapping that specifies for every player, and each of his information sets, the collection of opponents' information sets he reasons about.

3.1 Collections of Decision Problems

A *collection of decision problems* is a profile $D = (D_i(h_i), D_{-i}(h_i))_{i \in I, h_i \in H_i}$ where for every player i and information set $h_i \in H_i$, the pair $(D_i(h_i), D_{-i}(h_i))$ is a decision problem for player i at h_i. If we write D, then for the remainder of this paper it will be understood that the induced decision problem for player i at $h_i \in H_i$ is $(D_i(h_i), D_{-i}(h_i))$. By D^{full} we denote the collection of full decision problems, specifying for every player i and information set $h_i \in H_i$ the full decision problem $(S_i(h_i), S_{-i}(h_i))$. For two collections of decision problems D, E we write $D \subseteq E$ if for every player i and every information set $h_i \in H_i$ we have that $D_i(h_i) \subseteq E_i(h_i)$ and $D_{-i}(h_i) \subseteq E_{-i}(h_i)$.

3.2 Reduction Operators

A *reduction operator* r assigns to every collection of decision problems D a collection of reduced decision problems $r(D) \subseteq D$. For a given D we set $r^0(D) := D$, and for every $k \geq 1$ we inductively define by $r^k(D) := r(r^{k-1}(D))$ the k-fold application of the reduction operator r to D. Suppose that K is such that $r^{K+1}(D) = r^K(D)$. Then, we denote by $r^\infty(D) := r^K(D)$ the *iterated* application of the reduction operator r to D. For two reduction operators r and t, and two numbers k and m, we denote by $(t^m \circ r^k)(D)$ the collection of decision problems obtained by first performing the k-fold application of r to D, followed by the m-fold application of t to $r^k(D)$. Similarly, $(t^\infty \circ r^\infty)(D)$ is the collection of decision problems obtained by first performing the iterated application of r to D, followed by the iterated application of t to $r^\infty(D)$.

3.3 Focus-Based Rationalizability Procedures

We will now consider some special reduction operators for dynamic games in which a player, at each of his information sets, reasons about the rationality of his opponents at *certain* – but not necessarily all – information sets. Based on Perea and Tsakas (2019) we formalize this type of reasoning by a *focus function.*

Definition 3.1 (Focus function) *A* ***focus function*** *f assigns to every player i, information set $h_i \in H_i$, opponent $j \neq i$ and collection of decision problems D a collection of information sets $f_{ij}(h_i, D) \subseteq H_j$.*

Intuitively, $f_{ij}(h_i, D)$ contains the player j information sets that player i reasons about when being at information set h_i and when the current collection of decision problems is D.[8] For two focus functions f, g we write $f \subseteq g$ if for every player i, information set $h_i \in H_i$, opponent $j \neq i$ and collection of decision problems D, we have that $f_{ij}(h_i, D) \subseteq g_{ij}(h_i, D)$.

Every focus function f induces a *focus based reduction operator* rf as follows. For every collection of decision problems D, the output will be the collection of reduced decision problems $E = rf(D)$ where

$$D^+_{-i}(h_i, f) := \{(s_j)_{j\neq i} \in D_{-i}(h_i) \mid \text{for all } j \neq i,\ s_j \in D_j(h_j) \text{ for all } h_j \in f_{ij}(h_i, D) \cap H_j(s_j)\},$$

$$E_{-i}(h_i) := \begin{cases} D^+_{-i}(h_i, f), & \text{if } D^+_{-i}(h_i, f) \neq \emptyset \\ D_{-i}(h_i), & \text{if } D^+_{-i}(h_i, f) = \emptyset \end{cases}, \text{ and}$$

$$E_i(h_i) := \{s_i \in D_i(h_i) \mid s_i \text{ not strictly dominated in } (D_i(h_i), E_{-i}(h_i))\}$$

for all players i and information sets $h_i \in H_i$. By definition of $E_{-i}(h_i)$ player i believes, whenever possible, at h_i that every opponent is "currently rational" at all opponents' information sets that i reasons about at h_i. Here, we say that opponent j is "currently rational" at information set h_j if he chooses a strategy $s_j \in D_j(h_j)$.

[8] Our notion of a focus function is more general than that used in Perea and Tsakas (2019). In the latter framework, the collection of opponents' information sets a player focuses on does not depend on the collection of decision problems at hand. Moreover, within our notion it is possible that for a given information set h where two opponents $j, k \neq i$ are active, player i reasons at a certain information set $h_i \in H_i$ about the rationality of opponent j, but not about the rationality of opponent k, at h. This is not possible in Perea and Tsakas (2019).

For a given collection of decision problems D we call $(rf^k(D))_{k=0}^{\infty}$ the *f-rationalizability procedure* starting at D. Let $(D^k)_{k=0}^{\infty}$ be the f-rationalizability procedure starting at D^{full}, and let K be such that $D^K = D^{K+1}$. Then, for every player i, a *strategy* $s_i \in S_i$ is called *f-rationalizable* if $s_i \in D_i^K(h_i)$ for all $h_i \in H_i(s_i)$. An *outcome* $z \in Z$ is called *f-rationalizable* if for every player i there is an f-rationalizable strategy s_i such that $z = z((s_i)_{i \in I})$.

For two focus functions g, f we define the *combined* focus function (g, f) by

$$(g, f)_{ij}(h_i, D) := \begin{cases} f_{ij}(h_i, D), & \text{if } D \subseteq rg^{\infty}(D^{full}) \\ g_{ij}(h_i, D), & \text{otherwise} \end{cases}$$

for all players i, opponents $j \neq i$, information sets $h_i \in H_i$ and collections of decision problems D. Hence, in (g, f) we give epistemic priority to g, and only resort to f once all eliminations under g have been exhausted. By construction, $(rf^{\infty} \circ rg^{\infty})(D^{full}) = r(g, f)^{\infty}(D^{full})$. For three or more focus functions $f_1, ..., f_k$ we inductively define the combined focus function $(f_1, ..., f_k)$ by $(f_1, ..., f_k) := ((f_1, ..., f_{k-1}), f_k)$. By construction we then have that $(rf_k^{\infty} \circ ... \circ rf_1^{\infty})(D^{full}) = r(f_1, ..., f_k)^{\infty}(D^{full})$. Hence, in $(f_1, ..., f_k)$ we give the highest epistemic priority to the focus function f_1 until all reductions of rf_1 have been exhausted, after which we give the second highest epistemic priority to f_2 until all reductions of rf_2 have been exhausted, and so on.

In view of Lemma 2.1, the sets $E_i(h_i)$ above in the definition of $rf(D)$ can equivalently be defined as

$$E_i(h_i) := \{s_i \in D_i(h_i) \mid s_i \text{ optimal in } D_i(h) \text{ for some belief } b_i \in \Delta(E_{-i}(h_i))\}.$$

With our definition above, the f-rationalizability procedure is phrased as an iterated strict dominance procedure, whereas the latter definition of $E_i(h_i)$ would describe it as an iterated never-a-best-reply procedure.[9]

3.4 Special Cases

We will show that many of the existing rationalizability concepts for dynamic games can be modelled as an f-rationalizability procedure for some focus function f, and that our framework can also be used to define new, natural rationalizability procedures.

3.4.1 Iterated Conditional Dominance Procedure

In the *iterated conditional dominance* procedure (Shimoji and Watson (1998)) a player, whenever possible, believes that his opponents choose optimally at *each* of their information sets. This procedure can be phrased as the f^{all}-rationalizability procedure where f^{all} is the focus function given by

$$f_{ij}^{all}(h_i, D) = H_j$$

[9] The elimination procedure in Perea and Tsakas (2019) is formulated as an iterated never-a-best-reply procedure. Another difference is that Perea and Tsakas (2019) allow players to entertain belief hierarchies about focus functions in which they, or their opponents, are (believed to be) wrong about the players' actual focus functions, whereas we implicitly assume that the players' actual focus functions are transparent to everyone.

for every two players $i \neq j$, information set $h_i \in H_i$, and collection of decision problems D. That is, a player always reasons about the opponents' optimality at each of their information sets.

In Shimoji and Watson (1998) it is shown that this procedure yields the same strategies as the *strong rationalizability* procedure (Pearce (1984), Battigalli (1997)). Battigalli and Siniscalchi (2002) show, in turn, that strong rationalizability can be characterized epistemically by *common strong belief in rationality,* which means that a player believes, whenever possible, that his opponents choose optimally at all of their information sets (i.e. strongly believes in the opponents' rationality), that a player believes, whenever possible, that his opponents choose optimally at all of their information sets while strongly believing in *their* opponents' rationality, and so on. This resembles the nature of the focus function f^{all}.

3.4.2 Backward Dominance Procedure

In the *backward dominance* procedure (Perea (2014)) a player always believes, at each of his information sets h, that his opponents choose optimally at all of their information sets that *weakly follow* h. Since believing so is always possible, this is equivalent to requiring that a player, at each of his information sets h, believes *whenever possible* that his opponents choose optimally at all information sets that weakly follow h. We can describe this procedure as the f^{future}-rationalizability procedure where

$$f_{ij}^{future}(h_i, D) = \{h_j \in H_j \mid h_j \text{ weakly follows } h_i\}$$

for all i, j, h_i and D.

Perea (2014) has epistemically characterized the strategies surviving the backward dominance procedure by *common belief in future rationality,* which states that a player always believes that his opponents choose optimally at all information sets that weakly follow (i.e. believes in the opponents' future rationality), always believes that his opponents believe in *their* opponents' future rationality, and so on. These conditions are reflected by the focus function f^{future}.

3.4.3 Ben-Porath Procedure

Ben-Porath (1997) proposes a procedure for generic[10] dynamic games with perfect information[11] in which we first eliminate all strategies that are weakly dominated at the beginning of the game, followed by the iterated elimination of strategies that are strictly dominated at the beginning of the game.[12] In that paper, the strategies surviving the procedure are epistemically characterized by *common certainty of rationality at the beginning,* which means that a player believes, at the beginning of the game, that all opponents choose optimally at all information sets (i.e. the player is certain of rationality at the beginning), that a player believes, at the beginning of the game, that all opponents are certain of rationality at the beginning, and so on. For general dynamic games, these epistemic conditions characterize the procedure where at the

[10] This means that for every player, all outcomes yield different utilities.

[11] That is, there is only one active player at every information set, and this player knows which actions have been chosen in the past.

[12] This procedure is also called the *Dekel-Fudenberg procedure* (Dekel and Fudenberg (1990)).

beginning of the game we first eliminate the strategies that are strictly dominated at some information set in the game, followed by the iterated elimination of strategies that are strictly dominated at the beginning of the game. We refer to this procedure as the *Ben-Porath procedure.*[13]

Consider a dynamic game in which all players are active at $\varnothing$ – the beginning of the game.[14] Then, the Ben-Porath procedure is the $f^{initial}$-rationalizability procedure, where

$$f_{ij}^{initial}(\varnothing, D) = H_j \text{ and } f_{ij}^{initial}(h_i, D) = \emptyset \text{ for all } h_i \in H_i \backslash \{\varnothing\},$$

for all i, j and D. The focus function $f^{initial}$ reflects precisely the reasoning in common certainty of rationality at the beginning.

3.4.4 Forward and Backward Rationalizability

The reasoning in *forward and backward rationalizability* (Meier and Perea (2023)) is as follows: To start, a player believes, whenever possible, that his opponents choose optimally at the last information sets (i.e. strongly believes in the opponents' rationality at the last information sets), believes, whenever possible, that his opponents choose optimally at the last information sets and that the opponents' strongly believe in *their* opponents' rationality at the last information sets, and so on. On top of that, a player believes, whenever possible, that his opponents choose optimally at the last and second-to-last information sets (i.e. strongly believes in the opponents' rationality at the last and second-to-last information sets), believes, whenever possible, that his opponents choose optimally at the last and second-to-last information sets and that the opponents' strongly believe in *their* opponents' rationality at the last and second-to-last information sets, and so on. By following this pattern we finally arrive at the last stages of the procedure where a player believes, whenever possible, that his opponents choose optimally at all information sets, and so on.

In a sense, this procedure applies the iterated conditional dominance procedure in a backward fashion, by first applying it to the last information sets, then applying it to the last and second-to-last information sets, and so on, until we reach the beginning of the game. To formulate this procedure as an f-rationalizability procedure we denote by H^1 the collection of information sets that are only followed by terminal histories, and for every $k \geq 2$ we recursively define H^k as the collection of information sets that are only followed by terminal histories or information sets in H^{k-1}. Then, $H^k \subseteq H^{k+1}$ for every $k \geq 1$. Since we assume that the game has a cycle-free ordering of information sets, it may be verified that there is some K with $H^K = H$.[15] We call $(H^1, H^2, ..., H^K)$ the *backwards ordering of the information sets.*

[13] This procedure corresponds to *weak* Δ*-rationalizability* in Battigalli (2003).

[14] This can be assumed without loss of generality, since for the players who are not really active at $\varnothing$ we can add a "dummy" set of actions at $\varnothing$ consisting of one action only.

[15] Here is the argument: First, it can be shown that $H^1 \neq \emptyset$. Indeed, if $H^1 = \emptyset$ then for every $h \in H$ there is some $h' \in H$ that follows h. Hence, starting from an arbitrary $h^1 \in H$ we can find an infinite sequence $h^1, h^2, ...$ where h^{k+1} follows h^k for every k. As there are only finitely many information sets, there must be some $k, m \geq 1$ such that $h^k = h^{k+m}$. This, however, would contradict the assumption that we have a cycle-free ordering of information sets. Next, if $H^1 \neq H$ it can be shown that $H^2 \backslash H^1 \neq \emptyset$. Indeed, if $H^2 \backslash H^1 = \emptyset$ then for every $h \in H \backslash H^1$ there is some $h' \in H \backslash H^1$ that follows h. In the same way as above it can be shown that this contradicts the assumption that the ordering of information sets is cycle-free. By repeating this argument it can be shown that $H^{k+1} \backslash H^k \neq \emptyset$ whenever $H \neq H^k$, for every $k \geq 1$. As H is finite, there must be some K with $H^K = H$.

For every $k \in \{1, ..., K\}$ let f_k^{all} be the focus function where

$$f_{k,ij}^{all}(h_i, D) = H_j \cap H^k$$

for all i, j, h_i and D. Then, forward and backward rationalizability matches the $(f_1^{all}, f_2^{all}, ..., f_K^{all})$-rationalizability procedure, provided we drop the forward consistency assumption in the former concept.

Hence, the highest epistemic priority is given to reasoning about the last information sets, the second-highest epistemic priority is given to reasoning about the last and second-to-last information sets, and so on. This reflects precisely the logic of forward and backward rationalizability outlined above.

3.4.5 Epistemic Priority

The forward and backward rationalizability procedure above prioritizes backward induction reasoning over forward induction reasoning. This can be seen from the structure of the procedure, the fact that it is governed by the focus function $(f_1^{all}, f_2^{all}, ..., f_K^{all})$ above, and the result in Meier and Perea (2023) that every forward and backward rationalizable strategy survives the backward dominance procedure – a pure backward induction procedure.

One could also opt for a more extreme epistemic priority between backward induction and forward induction by first performing the backward dominance procedure until no further eliminations are possible, followed by the iterated conditional dominance procedure which represents pure forward induction reasoning. In light of the above, such a procedure amounts to the (f^{future}, f^{all})-rationalizability procedure where epistemic priority is given to reasoning about the opponents' rationality at future information sets. Alternatively, it can be viewed as an instance of *strong* Δ*-rationalizability* (Battigalli (2003), Battigalli and Siniscalchi (2003)) where Δ represents the restriction on first-order beliefs embodied by the backward dominance procedure.

One can also turn the epistemic priority between backward and forward induction around, by first performing the iterated conditional dominance procedure followed by the backward dominance procedure. This would result in the (f^{all}, f^{future}) rationalizability procedure. It is comparable to the concept of *selective rationalizability* (Catonini (2019)) in that it refines the strategies delivered by strong rationalizability by imposing additional restrictions on first-order beliefs afterwards.[16] Catonini (2019) explains how selective rationalizability represents a mode of reasoning in which epistemic priority is given to the eliminations in strong rationalizability above those imposed by the restrictions on first-order beliefs.

3.4.6 Restricted Forward Induction Reasoning

The iterated conditional dominance procedure represents a pure form of forward induction reasoning in which a player, whenever possible, believes that his opponents choose optimally at all of their information sets. In particular, a player will base his belief about the opponents' present, future, and unobserved past actions on the past actions he observed – a typical forward induction argument. One could also imagine a

[16] The difference is that in selective rationalizability the additional restrictions come in the form of exogenous restrictions on first-order beliefs.

situation where there are certain "focal" information sets in the game, and where players base their forward induction reasoning on the opponents' observed past actions at these focal information sets only. Also this can be modeled as an f-rationalizability procedure. Let $H^* \subseteq H$ be the collection of "focal" information sets in the game, and define the focus function f^{H^*} by

$$f_{ij}^{H^*}(h_i, D) = H_j \cap H^*$$

for all i, j, h_i and D. In the f^{H^*}-rationalizability procedure a player thus believes, whenever possible, that his opponents have chosen optimally at the past information sets in H^* and will choose optimally at all present and future information sets in H^*. As such, it represents a restricted form of forward induction in which the beliefs about an opponent's present, future and unobserved past actions at the focal information sets in H^* are solely based on his observed past actions at the focal information sets in H^*.

The idea of "focal" information sets is also present in the concept of *jointly rational belief systems* (Reny (1993)) in which, for a given collection of "focal" information sets H^*, every player believes at each of his information sets in H^* that all opponents choose rationally at all of their information sets, every player believes at each of his information sets in H^* that all other players believe at each of their information sets in H^* that all opponents choose rationally at all of their information sets, and so on. Battigalli and Siniscalchi (1999) provide a procedural characterization of the largest jointly rational belief systems. However, there are dynamic games in which jointly rational belief systems for a given collection H^* of focal information sets do not exist, for instance because it will be impossible to believe at certain information sets that the opponents are choosing rationally at all of their information sets.

Suppose we modify the conditions above to requiring that a player, at each of his information sets in H^*, believes *whenever possible* that his opponents choose rationally at all of their information sets, and so on. Then this procedure would correspond to $\hat{f}^{H^*}$-rationalizability, where

$$\hat{f}_{ij}^{H^*}(h_i, D) = \begin{cases} H_j, & \text{if } h_i \in H^* \\ \emptyset, & \text{if } h_i \notin H^* \end{cases}$$

for all i, j, h_i and D. Note that $\hat{f}^{H^*}$ can be viewed as the "dual" to f^{H^*}: in f^{H^*} a player believes at every information set, whenever possible, that his opponents choose rationally at all information sets in H^*, whereas in $\hat{f}^{H^*}$ a player believes at every information set in H^*, whenever possible, that his opponemts choose rationally at all information sets.

4 Existence and Forward Consistency

4.1 Existence

We will show that, under some conditions on the focus function f, there will always be at least one f-rationalizable strategy for every player.

Definition 4.1 (Conditions on focus functions) *(a) A focus function f is* ***monotone*** *if for every two collections of decision problems D, E with $D \subseteq E$ we have that $f_{ij}(h_i, E) \subseteq f_{ij}(h_i, D)$ for every two players*

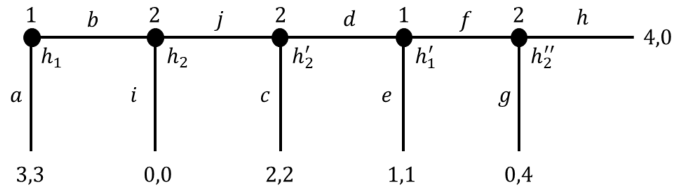

Figure 2: When f-rationalizable strategies fail to exist

$i \neq j$ and every information set $h_i \in H_i$.

*(b) A focus function f is **individually forward decreasing** if for every collection of decision problems D, every two players $i \neq j$, and every two information sets $h_i, h_i' \in H_i$ where h_i' follows h_i, it holds that $f_{ij}(h_i', D) \subseteq f_{ij}(h_i, D)$.*

*(c) A focus function f **individually preserves focus on past information sets** if for every collection of decision problems D, every two players $i \neq j$, and every two information sets $h_i, h_i' \in H_i$ where h_i' follows h_i, it holds that $f_{ij}(h_i, D) \cap H_j^-(h_i) \subseteq f_{ij}(h_i', D)$.*

Note that monotonicity is automatically satisfied if $f_{ij}(h_i, D)$ is always independent of D. The individual forward decreasing property states that a player must not start reasoning about new information sets as the game proceeds, whereas individual preservation of focus on past information sets requires a player to always reason about all past information sets he focused on before.

The following result states that under the conditions above, the existence of f-rationalizable strategies is guaranteed.

Theorem 4.1 (Existence) *Let f be a focus function that is monotone, individually forward decreasing and individually preserving focus on past information sets. Then, for every player there is at least one f-rationalizable strategy.*

To show that f-rationalizable strategies may fail to exist if some of the conditions are dropped, consider the game in Figure 2. This game is obtained by adding the "dummy moves" i and j for player 2 to the game in Figure 1. Let f be the focus function where

$$f_{12}(h_1, D) = f_{12}(h_1', D) = H_2, \; f_{21}(h_2, D) = H_1, \; f_{21}(h_2', D) = \{h_1'\} \text{ and } f_{21}(h_2'', D) = \emptyset$$

for all D. Note that f does not individually preserve focus on past information sets since $h_1 \in f_{21}(h_2, D) \cap H_1^-(h_2)$ but $h_1 \notin f_{21}(h_2', D)$. It is, however, individually forward decreasing and monotone.

To run the f-rationalizability procedure we start from the full decision problems in Table 1, where we put the active player's strategies in the rows, the opponent's strategies in the columns, and the active player's

h_1	i	(j,c)	(j,d,g)	(j,d,h)
a	3	3	3	3
(b,e)	0	2	1	1
(b,f)	0	2	0	4

h_2	(b,e)	(b,f)
i	0	0
(j,c)	2	2
(j,d,g)	1	4
(j,d,h)	1	0

h_2'	(b,e)	(b,f)
(j,c)	2	2
(j,d,g)	1	4
(j,d,h)	1	0

h_1'	(j,d,g)	(j,d,h)
(b,e)	1	1
(b,f)	0	4

h_2''	(b,f)
(j,d,g)	4
(j,d,h)	0

Table 1: Full decision problems for the game in Figure 2

utilities in the associated cells. In round 1 we eliminate the strictly dominated strategy (b,e) at h_1, the strictly dominated strategy i at h_2, and the strictly dominated strategy (j,d,h) at h_2, h_2' and h_2''. In round 2, at h_1, we first eliminate 2's strategies i and (j,d,h), after which we eliminate 1's strictly dominated strategy (b,f). At h_2 we first eliminate 1's strategy (b,e), because $h_1 \in f_{21}(h_2, D)$, after which we eliminate 2's strictly dominated strategy (j,c). At h_2' we cannot eliminate anything as $h_1 \notin f_{21}(h_2', D)$. At h_1' we first eliminate 2's strategy (j,d,h), after which we eliminate 1's strictly dominated strategy (b,f). At h_2'' nothing can be eliminated in round 2. In round 3, at h_1, we eliminate 2's strategy (j,c). At h_2' we start by eliminating 1's strategy (b,f), as $h_1' \in f_{21}(h_2', D)$, after which we eliminate 2's strictly dominated strategy (j,d,g). Then, the procedure terminates.

We thus see that only strategy (j,d,g) is left for player 2 at h_2, whereas only (j,c) is left at h_2'. As such, there is no f-rationalizable strategy for player 2. The reason is that the focus function f leads to dynamically inconsistent beliefs for player 2: at h_2 player 2 must believe that player 1 will choose f in the future, whereas at h_2' he must believe that player 1 chooses e in the future.

In a similar way it can be shown that also the individual forward decreasing property is indispensable for existence. To see this, consider the alternative focus function g where

$$g_{12}(h_1, D) = g_{12}(h_1', D) = H_2,\ g_{21}(h_2, D) = \{h_1'\},\ g_{21}(h_2', D) = H_1 \text{ and } g_{21}(h_2'', D) = H_1$$

for all D. Then, g is not individually forward decreasing as $h_1 \notin g_{21}(h_2, D)$ but $h_1 \in g_{21}(h_2', D)$. However, g is monotone and individually preserves focus on past information sets. In a similar way as above it can be shown that in the g-rationalizability procedure, only the strategy (j,c) is left for player 2 at h_2 whereas only the strategy (j,d,g) is left at h_2'. Therefore, there is no g-rationalizable strategy for player 2. The problem, again, is caused by dynamically inconsistent beliefs: at h_2 player 2 must believe that player 1 will choose e in the future, whereas at h_2' he must believe that player 1 will choose f in the future.

In general, the conditions in Theorem 4.1 guarantee that a player never ends up with dynamically inconsistent beliefs (see Lemma 8.6 in the proofs section) which in turn allows the player to have an f-rationalizable strategy. At this stage we do not know whether monotonicity can be dropped without losing existence.

It may be verified that all of the focus functions discussed in Sections 3.4.1–3.4.6, except (f^{all}, f^{future}) and $\hat{f}^{H^*}$, satisfy the conditions in Theorem 4.1. In particular, it may be verified that the focus function $(f_1^{all}, f_2^{all}, ..., f_K^{all})$, associated with forward and backward rationalizability, is monotone since $f_1^{all} \subseteq f_2^{all} \subseteq ... \subseteq f_K^{all}$.[17] As such, Theorem 4.1 implies that all of the associated rationalizability procedures always provide non-empty strategy sets.

4.2 Forward Consistency

In Section 3.3 we have seen that f-rationalizability can alternatively be phrased as an iterated never-a-best-reply procedure by requiring, at every information set h_i, that player i must choose a strategy that is optimal for some belief that only assigns positive probability to opponents' strategy combinations that have remained so far at h_i. But we did not require these beliefs to be forward consistent. At the same time, most rationalizability concepts in the literature do require forward consistency. This raises the question: What happens to f-rationalizability if we add the forward consistency condition?

We will see that under the conditions of Theorem 4.1 the set of f-rationalizable strategies will not change if we require a player to have forward consistent beliefs, and to believe, at each of his information sets, that his opponents will have forward consistent beliefs *at those information sets he reasons about.*

Consider for player i a belief vector $\tilde{b}_i = (b_i(h_i))_{h_i \in H_i}$, where $b_i(h_i) \in \Delta(S_{-i}(h_i))$ for all $h_i \in H_i$, and a collection $\hat{H}_i \subseteq H_i$ of information sets. We say that $\tilde{b}_i$ *is forward consistent on* $\hat{H}_i$ if for every two information sets $h_i, h_i' \in \hat{H}_i$ where h_i' follows h_i and $b_i(h_i)(S_{-i}(h_i')) > 0$, it holds that

$$b_i(h_i')(s_{-i}) = \frac{b_i(h_i)(s_{-i})}{b_i(h_i)(S_{-i}(h_i'))}$$

for all $s_{-i} \in S_{-i}(h_i')$.

For a given focus function f we will now introduce a reduction procedure $(D^{bu,k})_{k=0}^{\infty}$ that incorporates forward consistency. Set $D^{bu,0} := D^{full}$, $D^{bu,1} := rf(D^{full})$, and for every $k \geq 2$ we inductively define $D^{bu,k}$ by

$$\begin{aligned} D_{-i}^{bu,k-1+}(h_i, f) := \{(s_j)_{j \neq i} \in D_{-i}^{bu,k-1}(h_i) \mid & \text{ for all } j \neq i \text{ there is a belief vector } (b_j(h_j))_{h_j \in H_j} \\ \text{that is forward consistent on } f_{ij}(h_i, D^{bu,k-1}) & \text{ such that for all } h_j \in f_{ij}(h_i, D^{bu,k-1}) \cap H_j(s_j), \\ b_j(h_j) \in \Delta(D_{-j}^{bu,k-1}(h_j)),\ s_j \in D_j^{bu,k-2}(h_j) & \text{ and } s_j \text{ is optimal in } D_j^{bu,k-2}(h_j) \text{ for } b_j(h_j)\}, \end{aligned}$$

$$D_{-i}^{bu,k}(h_i) := \begin{cases} D_{-i}^{bu,k-1+}(h_i, f), & \text{if } D_{-i}^{bu,k-1+}(h_i, f) \neq \emptyset \\ D_{-i}^{bu,k-1}(h_i), & \text{if } D_{-i}^{bu,k-1+}(h_i, f) = \emptyset \end{cases}, \text{ and}$$

[17] Formally, it follows from a repeated application of Lemma 8.8 in the proofs section.

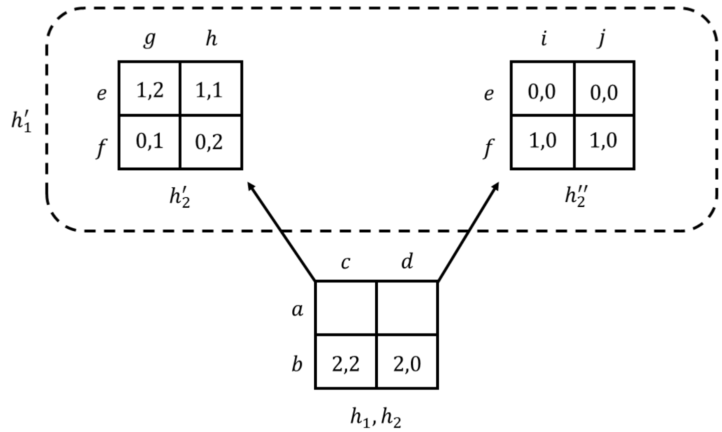

Figure 3: When forward consistency matters

$$D_i^{bu,k}(h_i) := \{s_i \in D_i^{bu,k-1}(h_i) \mid s_i \text{ is optimal in } D_i^{bu,k-1}(h_i) \text{ for some } b_i \in \Delta(D_{-i}^{bu,k}(h_i))\}$$

for all players i and information sets $h_i \in H_i$. We call $(D^{bu,k})_{k=0}^{\infty}$ the *f-rationalizability procedure under forward consistency* that starts at D^{full}.

Let K be such that $D^{bu,K} = D^{bu,K+1}$. Then, a strategy $s_i \in S_i$ is called *f-rationalizable under forward consistency* if there is a belief vector $(b_i(h_i))_{h_i \in H_i}$ that is forward consistent on H_i such that for every $h_i \in H_i(s_i)$ we have that $b_i(h_i) \in \Delta(D_{-i}^{bu,K}(h_i))$, $s_i \in D_i^{bu,K}(h_i)$ and s_i is optimal in $D_i^{bu,K}(h_i)$ for $b_i(h_i)$.

The following theorem states that the new procedure, which incorporates forward consistency, will induce the same output as the original f-rationalizability procedure.

Theorem 4.2 (Forward consistency) *Let f be a focus function that is monotone, individually forward decreasing and individually preserving focus on past information sets, and let $(D^{bu,k})_{k=0}^{\infty}$ be the f-rationalizability procedure under forward consistency that starts at D^{full}. Then,*

(a) $D^{bu,k} = rf^k(D^{full})$ for all $k \geq 0$, and

(b) for every player, the f-rationalizable strategies are precisely the strategies that are f-rationalizable under forward consistency.

Important in this theorem is that we only require a player to believe that his opponent satisfies forward consistency at those information sets *he reasons about.* If we impose the stronger requirement that a player must believe that his opponents satisfy forward consistency at *all* information sets, the equivalence result may break down. To see this, consider the game in Figure 3 which is taken from Perea (2012, 2014). Here, player 1's actions are in the rows and player 2's actions in the columns. Player 1's information set h_1' indicates that at this stage player 1 does not know whether player 2 has chosen c or d at the beginning.

Consider the focus function f^{future} from Section 3.4.2, inducing the backward dominance procedure. It can be shown that strategy (c, h) for player 2 is f^{future}-rationalizable. However, it is no longer f^{future}-rationalizable if we additionally require player 2 to believe at h_2' that player 1 satisfies forward consistency at h_1 and h_1'. To see this, note that player 1 will believe at h_1 that player 2 chooses either (c, g) or (c, h), as choosing d will always yield him a lower utility. Hence, if player 1 is forward consistent when the game moves from h_1 to h_1', he must still believe at h_1' that player 2 chooses (c, g) or (c, h), which implies that player 1 must choose e when h_1' is reached. Thus, if player 2 believes at h_2' that player 1 is forward consistent at h_1 and h_1', he must believe that player 1 chooses e at h_1', and he must therefore choose (c, g) himself.

The concept that combines the backward dominance procedure with common belief in forward consistency at *all* information sets is the *backwards rationalizability procedure* (Penta (2015), Catonini and Penta (2022), Perea (2014)). The example above thus shows that strategy (c, h) survives the backward dominance procedure but is not backwards rationalizable. The reason was that player 2 is now required to believe at h_2' that player 1 is forward consistent at h_1 and h_1'. However, this forces player 2 at h_2' to reason about h_1 – contrary to what the associated focus function f^{future} prescribes. This argument holds in general: Requiring a player to believe that his opponent is forward consistent at information sets he does not reason about goes against the spirit of the focus functions.

If instead we take the focus function f^{all} from Section 3.4.1, inducing the iterated conditional dominance procedure, then the f^{all}-rationalizability procedure under forward consistency already requires a player to believe that his opponents are forward consistent at *all* information sets. That is, the latter procedure imposes common belief in forward consistency at all information sets. Theorem 4.2 above then guarantees that this does not matter for the output of the iterated conditional dominance procedure (and hence of the strong rationalizability procedure) – a result that has already been shown in Shimoji and Watson (1998).

5 Monotonicity Result

Consider two focus functions f and g where $g \subseteq f$. We will show that, under certain conditions, every f-rationalizable outcome is also g-rationalizable.

5.1 Theorem

To formulate these conditions we need some new definitions. For a given focus function f, collection of decision problems D, player i and information set $h_i \in H_i$ we define the set

$$f_{ii}(h_i, D) := \{h_i\} \cup \{h_i' \in H_i \mid \text{there is some } j \neq i \text{ and } h_j \in f_{ij}(h_i, D) \text{ such that } h_i' \in f_{ji}(h_j, D)\}.$$

It thus represents the collection of own information sets that player i indirectly reasons about while being at h_i.

Consider two focus functions f, g, and two collections of decision problems D, E where $E = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$. We say that D is a (g, f)-*semi reduction* of E if either $D = (rf^k \circ rg^{m+1})(D^{full})$, or $D = (rf^{k+1} \circ rg^{m-1})(D^{full})$ (provided $m \geq 1$).

Definition 5.1 (Conditions on focus functions) *(a) A focus function f is* ***collectively forward decreasing*** *if for every collection of decision problems D, every three players i, j, k (where some of these players may be equal), and every two information sets $h_i \in H_i, h_j \in H_j$ where h_j weakly follows h_i, it holds that $f_{jk}(h_j, D) \subseteq f_{ik}(h_i, D)$.*

(b) A focus function f ***collectively preserves focus on past information sets*** *if for every collection of decision problems D, every three players i, j, k (where some of these players may be equal), and every two information sets $h_i \in H_i, h_j \in H_j$ where h_j weakly follows h_i, it holds that $f_{ik}(h_i, D) \cap H_k^-(h_i) \subseteq f_{jk}(h_j, D)$.*

(c) A focus function f is ***transitively closed*** *if for every collection of decision problems D, every three players i, j, k (where some of these players may be equal), and every three information sets $h_i \in H_i$, $h_j \in H_j$ and $h_k \in H_k$ with $h_j \in f_{ij}(h_i, D)$ and $h_k \in f_{jk}(h_j, D)$, it holds that $h_k \in f_{ik}(h_i, D)$.*

(d) A focus function f is ***monotone with respect to another focus function*** *g if for every two collections of decision problems D, E where $E = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$ and D is a (g, f)-semi reduction of E, we have that $f_{ij}(h_i, E) \subseteq f_{ij}(h_i, D)$ for every two players $i \neq j$ and every information set $h_i \in H_i$.*

Conditions (a) and (b) require that the focus functions of the different players should not only be individually forward decreasing and individually preserving focus on past information sets (as defined in Section 4.1), but should satisfy these requirements collectively. As such, the various focus functions of the different players should be "appropriately aligned". It may be verified that (a) and (b) imply that f is individually forward decreasing, and individually preserving focus on past information sets, respectively. Condition (c) states that whenever a player *indirectly* reasons about an information set h, by reasoning about an information set where the respectively player reasons about h, he should also *directly* reason about h.[18] Condition (d) states that if we first apply g-rationalizability for some rounds followed by f-rationalizability for some rounds, then applying one more round of g-rationalizability, or applying one round less of g-rationalizability followed by one more round of f-rationalizability, should induce the players to reason about more, or the same, information sets. Note that this condition is automatically satisfied if $f_{ij}(h_i, D)$ is always independent of D.

To state part (c) of the theorem below we need the following definition: We say that a focus function f is *future oriented* if $f_{ij}(h_i, D) \subseteq H_j^+(h_i)$ for every two players $i \neq j$, every information set $h_i \in H_i$ and every collection of decision problems D. That is, players only reason about information sets that weakly follow (but not necessarily all of these).

Theorem 5.1 (Monotonicity theorem) *Let f, g be two focus functions with $g \subseteq f$ that are monotone, individually forward decreasing and individually preserving focus on past information sets. Assume moreover that f is collectively forward decreasing, collectively preserving focus on past information sets, transitively closed, and monotone with respect to g. Then,*

(a) the set of f-rationalizable outcomes is the same as the set of (g, f)-rationalizable outcomes,

[18] Perea and Tsakas (2019) also make this assumption.

(b) every f-rationalizable outcome is g-rationalizable, and

(c) if f is future oriented, then every f-rationalizable strategy is g-rationalizable.

Part (a) thus reveals that, under the conditions of the theorem, it does not matter for the induced outcomes whether we only use the focus function f, or first iteratedly use g and then f, thereby giving epistemic priority to g over f. Part (b) immediately follows from (a) as, by construction, every (g, f)-rationalizable outcome is in particular g-rationalizable. Part (c) states that the monotonicity property in (b) even holds in terms of strategies if f only involves backward induction reasoning, and no forward induction reasoning.

An immediate consequence of this theorem is that amongst the focus functions that satisfy the conditions above, f^{all} yields the most restrictive rationalizability procedure in terms of outcomes.

5.2 Proof Sketch

The proof of part (a) is constructive: We explicitly show how to transform a combination $(s_i)_{i\in I}$ of f-rationalizable strategies into a combination $(\sigma_i(s_i))_{i\in I}$ of (g, f)-rationalizable strategies that induces the same outcome, and *vice versa.* Here, σ_i represents a transformation mapping for every player i, transforming an f-rationalizable strategy s_i into a (g, f)-rationalizable strategy $\sigma_i(s_i)$ that preserves the behavior of s_i at those information sets that can be reached under $(s_j)_{j\in I}$.

To formally describe $\sigma_i(s_i)$ we need some new terminology. By

$$D_i^+(h_i, f) := \{s_i \in S_i(h_i) \mid s_i \in D_i(h_i') \text{ for all } h_i' \in f_{ii}(h_i, D) \cap H_i(s_i)\}$$

we denote the set of strategies for player i that are "currently rational" at all of i's information sets that player i indirectly reasons about while being at h_i. Recall the definition of $D_{-i}^+(h_i, f)$ in Section 3.3. Then,

$$H_i^e(D, f) := \{h_i \in H_i \mid D_i^+(h_i, f) \neq \emptyset \text{ and } D_{-i}^+(h_i, f) \neq \emptyset\}$$

denotes the collection of *explicable* information sets for player i at the collection of decision problems D and focus function f. In words, these are the information sets at which it is possible for player i to believe that (i) the opponents believe that player i is "currently rational" at all information sets they reason about, and (ii) the opponents are "currently rational" at all information sets player i reasons about.

Let $D^{f,\infty} := rf^\infty(D^{full})$ and $D^{(g,f),\infty} := r(g, f)^\infty(D^{full})$ be the collection of decision problems that results from the f-rationalizability procedure, and the (g, f)-rationalizability procedure, respectively. Then, the transformation mapping σ_i transforms every strategy s_i into a strategy $\sigma_i(s_i)$ that

$$\text{coincides with } s_i \text{ at the information sets in } H_i^e(D^{f,\infty}, f) \text{ and} \tag{5.1}$$

$$\text{coincides at every information set } h_i \text{ outside } H_i^e(D^{f,\infty}, f) \text{ with a strategy in } D_i^{(g,f),\infty}(h_i). \tag{5.2}$$

See the left-hand panel of Figure 4 for an illustration.

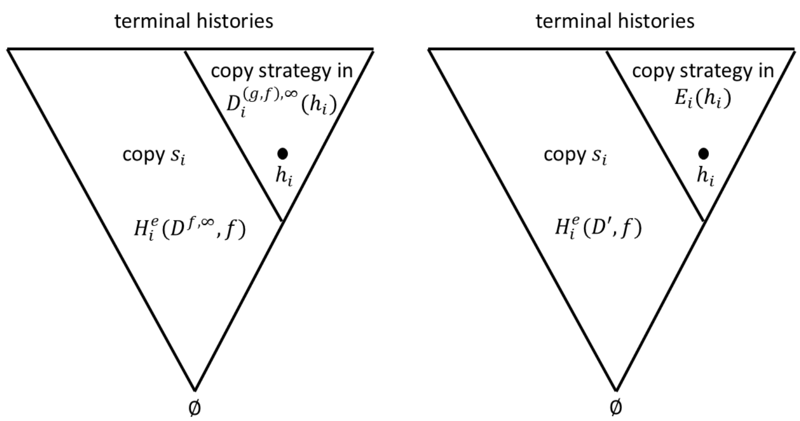

Figure 4: Strategy transformations in proof of Theorem 5.1

It is shown that σ_i transforms every f-rationalizable strategy s_i into a (g, f)-rationalizable strategy $\sigma_i(s_i)$. Moreover, by construction, the combination $(\sigma_i(s_i))_{i\in I}$ of (g, f)-rationalizable strategies yields the same outcome as the combination $(s_i)_{i\in I}$ of f-rationalizable strategies. To see this, it can be verified that every information set h reached by $(s_i)_{i\in I}$ is in $H_j^e(D^{f,\infty}, f)$ for every player j that is active at h, and therefore, by (5.1) above, $(\sigma_i(s_i))_{i\in I}$ leads to the same outcome as $(s_i)_{i\in I}$.

To show that σ_i transforms every f-rationalizable strategy s_i into a (g, f)-rationalizable strategy $\sigma_i(s_i)$ it suffices, in view of (5.2) above, to show the following property:

$$\text{if } h_i \in H_i^e(D^{f,\infty}, f) \text{ and } s_i \in D_i^{f,\infty+}(h_i, f) \text{ then } \sigma_i(s_i) \in D_i^{(g,f),\infty+}(h_i, f). \tag{5.3}$$

This property is shown in steps: Let M be such that $rg^M(D^{full}) = rg^{M+1}(D^{full})$, and for every $m \in \{0, ..., M\}$ let $D^{(g^{\leq m},f),\infty}$ be the collection of decision problems obtained if we first apply rg during m rounds, after which we iteratedly apply rf. For every $m \in \{0, ..., M-1\}$ and strategy s_i, let $\sigma_i^m(s_i)$ be the strategy given by the right-hand panel of Figure 4, where $D' := D^{(g^{\leq m},f),\infty}$ and $E := D^{(g^{\leq m+1},f),\infty}$. Then, we show the following property:

$$\text{if } h_i \in H_i^e(D^{(g^{\leq m},f),\infty}, f) \text{ and } s_i \in D_i^{(g^{\leq m},f),\infty+}(h_i, f) \text{ then } \sigma_i^m(s_i) \in D_i^{(g^{\leq m+1},f),\infty+}(h_i, f). \tag{5.4}$$

Since $\sigma_i = (\sigma_i^{M-1} \circ ... \circ \sigma_i^0)$ is the consecutive application of the transformations $\sigma_i^0, \sigma_i^1, ..., \sigma_i^{M-1}$, (5.3) follows from a repeated application of (5.4).

Also property (5.4) is shown in steps: Let K be such that $r(g^{\leq m}, f)^K(D^{full}) = r(g^{\leq m}, f)^{K+1}(D^{full})$ for every m. For every $m \in \{0, ..., M\}$ and $k \in \{0, ..., K\}$, let $D^{(g^{\leq m},f),k}$ be the collection of decision problems obtained if we start at $rg^m(D^{full})$ and subsquently apply k rounds of rf. Now, let either

$$(D = D^{(g^{\leq m+1},f),k} \text{ and } E = D^{(g^{\leq m},f),k}) \text{ or } (D = D^{(g^{\leq m},f),k+1} \text{ and } E = D^{(g^{\leq m+1},f),k}). \tag{5.5}$$

Let $D' = D^{full}$ if $D = D^{(g^{\leq m+1},f),0}$, let $D' = D^{(g^{\leq m+1},f),k-1}$ if $k \geq 1$ and $D = D^{(g^{\leq m+1},f),k}$, and let $D' = D^{(g^{\leq m},f),k}$ if $k \geq 0$ and $D = D^{(g^{\leq m},f),k+1}$. For every strategy s_i let $\sigma_i^{k,m}(s_i)$ be the strategy given by the right-hand panel of Figure 4. Then, by induction on k we show the following property:

$$\text{if } h_i \in H_i^e(D', f) \text{ and } s_i \in D_i^+(h_i, f) \text{ then } \sigma_i^{k,m}(s_i) \in E_i^+(h_i, f). \tag{5.6}$$

This property is shown in Lemma 8.12 and Lemma 8.13, which constitute the heart of our proof. By induction on k we then conclude that (5.6) holds for $D' = D = D^{(g^{\leq m},f),\infty}$ and $E = D^{(g^{\leq m+1},f),\infty}$, which is equivalent to (5.4) since $\sigma_i^m = \sigma_i^{K,m}$.[19]

Hence, we can construct transformation mappings σ_i such that for every combination $(s_i)_{i\in I}$ of f-rationalizable strategies, $(\sigma_i(s_i))_{i\in I}$ is a combination of (g, f)-rationalizable strategies that yields the same outcome. By (5.5) and (5.6) above we can also show the converse: For every combination $(s_i)_{i\in I}$ of (g, f)-rationalizable strategies there is a combination $(\tau_i(s_i))_{i\in I}$ of f-rationalizable strategies that yields the same outcome. This completes the proof of (a).

As we have seen, (a) implies (b). To prove (c) we show that in the construction above, $H_i^e(D^{f,\infty}, f) = H_i$ whenever f is future oriented. Then, by (5.1) above, σ_i is the identity mapping, which implies part (c).

6 Implications

We now show how Theorem 5.1 can be used to prove some existing results in the literature, but also how it generates new results.

6.1 Forward versus Backward Induction Reasoning

Perea (2017) and Catonini (2020) have shown, for dynamic games with observed past choices, that every outcome induced by strong rationalizability is also induced by the backward dominance procedure. It also follows from the arguments in Chen and Micali (2013). This property is an immediate consequence of Theorem 5.1: It may be verified that $f := f^{all}$ and $g := f^{future}$ satisfy all the conditions in Theorem 5.1. In particular, f^{all} is monotone with respect to f^{future} since it is constant across collections of decision problems. As such, by Theorem 5.1 (b), every f^{all}-rationalizable outcome is f^{future}-rationalizable. Since f^{all}-rationalizability and f^{future}-rationalizability correspond to strong rationalizability and the backward dominance procedure, respectively, we obtain the following corollary.

Corollary 6.1 (Forward versus backward induction reasoning) *Every outcome induced by strong rationalizability is also induced by the backward dominance procedure.*

[19] Battigalli and Catonini (2024) use a similar argument in the proof of their main theorem. Also Perea's (2018) proof of Battigalli's theorem proceeds along these lines.

As this property holds also for games with non-observed past choices, it generalizes the results in Perea (2017) and Catonini (2020). On the basis of Theorem 5.1 (a) we can say even more:

Corollary 6.2 (Epistemic priority) *The outcomes induced by strong rationalizability are the same as those induced by first applying the backward dominance procedure, followed by applying strong rationalizability.*

That is, for the induced outcomes it does not matter whether we give epistemic priority to the backward induction concept of backward dominance over the forward induction concept of strong rationalizability, or whether we restrict ourselves to strong rationalizability all the way. This result is similar, in spirit, to Catonini (2020) where it is shown that under certain conditions, first imposing exogenous restrictions on the first-order beliefs and then applying the strong rationalizability procedure yields the same outcomes as reversing this order. Hence, under these conditions the epistemic priority between the exogenous restrictions on first-order beliefs and the rationality restrictions of strong rationalizability does not matter for the induced outcomes.

Finally, let us apply Corollary 6.1 to a game with perfect information without relevant ties. As the backward dominance procedure leads to the unique backward induction strategies (see Perea (2014)), and hence to the unique backward induction outcome, and strongly rationalizable strategies always exist, we obtain the following result which has first been shown by Battigalli (1997).

Corollary 6.3 (Battigalli's theorem) *Consider a dynamic game with perfect information and without relevant ties. Then, strong rationalizability uniquely induces the backward induction outcome.*

Alternative proofs can be found in Heifetz and Perea (2015) and Perea (2018). Reny (1992), in Proposition 3, has shown a similar result for the alternative forward induction concept of *explicable equilibrium.*

6.2 Forward and Backward Rationalizability

Recall from Section 3.4.4 that the concept of forward and backward rationalizability (Meier and Perea (2023)), if we drop the forward consistency assumption,[20] can be phrased as the $(f_1^{all}, f_2^{all}, ..., f_K^{all})$-rationalizability procedure. Here, the focus function f_k requires a player to reason about the information sets in H^k, and $H^1, H^2, ..., H^K$ is the backwards ordering of the information sets. As such, it applies forward induction reasoning in a backward inductive fashion, thereby giving epistemic priority to backward induction reasoning. This is also reflected by a result in Meier and Perea (2023) which shows that the concept refines backward dominance in terms of strategies.

It may be verified that all conditions in Theorem 5.1 are satisfied by $f := f^{all}$ and $g := (f_1^{all}, f_2^{all}, ..., f_K^{all})$. As $f_K^{all} = f^{all}$ it follows that (g, f)-rationalizability is the same as g-rationalizability in this case. Hence, we reach the following conclusion on the basis of part (a) of Theorem 5.1.

[20] To be more precise, forward and backward rationalizability imposes common belief in forward consistency at *all* information sets, also at those that players are not required to reason about according to the focus function at hand.

Corollary 6.4 (Forward and backward rationalizability) *The outcomes induced by the strong rationalizability procedure are the same as those induced by the forward and backward rationalizability procedure (if we drop the forward consistency assumption).*

Meier and Perea (2023) show the same result for the version with forward consistency.

6.3 Backwards Order of Elimination

From a computational point of view it is often convenient to use a *backwards order of elimination,* in which we first perform the eliminations at the last information sets in the game, after which we turn to performing the eliminations at the last and second-to-last information sets, and so on, until we reach the beginning of the game. We can use Theorem 5.1 to show that this will not alter the outcomes of the rationalizability procedure.

To see this, take a focus function f that satisfies the conditions in Theorem 5.1, except the monotonicity with respect to g. Let $H^1, H^2, ..., H^K$ be the backwards ordering of the information sets as defined in Section 3.4.4. For every $k \in \{1, ..., K\}$ let f_k be the focus function given by

$$f_{k,ij}(h_i, D) = f_{ij}(h_i, D) \cap H^k$$

for all i, j, h_i and D. Then, $(f_1, f_2, ..., f_K)$ is the focus mapping in which f is applied according to the backwards order of elimimination. By means of Theorem 5.1 (a) we can show the following result.

Corollary 6.5 (Backwards order of elimination) *Let f be a focus mapping that is monotone, collectively forward decreasing, collectively preserving focus on past information sets, transitively closed, and monotone with respect to $(f_1, f_2, ..., f_K)$. Then, the f-rationalizable outcomes are the same as the $(f_1, f_2, ..., f_K)$-rationalizable outcomes.*

Indeed, it may be verified that f and $g := (f_1, f_2, ..., f_K)$ satisfy all the conditions in Theorem 5.1. In particular, since $f_1 \subseteq f_2 \subseteq ... \subseteq f_K$, it follows from Lemma 8.8 that $(f_1, f_2, ..., f_K)$ is monotone. As $f_K = f$ we see that (g, f)-rationalizability is the same as g-rationalizability in this case. By part (a) of Theorem 5.1 we thus obtain the result above.

Corollary 6.4 thus represents a special case where $f = f^{all}$. By part (c) of Theorem 5.1 we can conclude that if f is future oriented, then the f-rationalizable *strategies* are the same as the $(f_1, f_2, ..., f_K)$-rationalizable strategies. That is, the backwards order of elimination yields the same *strategies* as the original procedure.

6.4 Restricted Forward Induction Reasoning

Recall from Section 3.4.6 the focus function f^{H^*}, for some collection $H^* \subseteq H$ of "focal" information sets. It induces a restricted form of forward induction reasoning in which a player, at every information set, only evaluates the opponents' past optimality at focal information sets in H^*, and uses it to form a belief about the opponents' present, future, and unobserved past actions at the focal information sets in H^*. Now, compare two collections of focal information sets $H^1, H^2 \subseteq H$ where $H^1 \subseteq H^2$. Then, it may be verified that $f := f^{H^2}$ and $g := f^{H^1}$ satisfy all the conditions in Theorem 5.1. In light of part (b) of that theorem we thus reach the following conclusion.

Corollary 6.6 (Restricted forward induction reasoning) *Consider two collections of focal information sets $H^1, H^2 \subseteq H$ with $H^1 \subseteq H^2$. Then, every f^{H^2}-rationalizable outcome is f^{H^1}-rationalizable.*

That is, if we expand the collection of information sets on which the players perform forward induction reasoning, then the set of induced outcomes can only become smaller. Also this set inclusion only holds in terms of outcomes, not in terms of strategies. To see this, consider the game in Figure 1 while setting $H^1 := \{h_2, h_1', h_2'\}$ and $H^2 := H$. Then, the only f^{H^2}-rationalizable strategy for player 2 is (d, g), whereas the only f^{H^1}-rationalizable strategy for player 2 is c. However, both concepts lead to the same outcome since a is the unique f^{H^1}- and f^{H^2}-rationalizable strategy for player 1.

7 Concluding Remarks

Epistemic characterization. In this paper we have presented a family of rationalizability procedures parametrized by a focus function f, describing for every player, and at each of his information sets, the collection of opponents' information sets he reasons about. The key requirement is that a player, at each of his information sets, believes whenever possible that his opponents choose optimally at the information sets he reasons about. By iterating this condition we arrive at the f-rationalizability procedure.

Perea and Tsakas (2019) provide an epistemic characterization for the concept of f-rationalizability by means of *common strong belief in rationality with respect to* f, similarly to how Battigalli and Siniscalchi (2002) have epistemically characterized strong rationalizability by means of common strong belief in rationality.[21] Unlike Perea and Tsakas (2019), we allow the collection of opponents' information sets that a player reasons about to depend on the particular collection of decision problems at hand. As we have seen in Sections 3.4.4 and 3.4.5, this enables us to model rationalizability procedures in which the player gives epistemic priority to one mode of reasoning over another. A natural question that remains to be investigated is whether, and how, we can provide an epistemic chacterization of such rationalizability procedures that involve epistemic priority.

Order independence. Recall Corollary 6.1, in which it is shown that every strongly rationalizable outcome is also induced by the backward dominance procedure. Perea (2017) and Catonini (2020) prove this result by showing that the strong rationalizability procedure is *order independent with respect to outcomes,* which means that every "slow" elimination order of strong rationalizability leads to the same outcomes as the original procedure. Chen and Micali (2013) also prove this property. As the backward dominance procedure

[21] As mentioned earlier, the framework in Perea and Tsakas (2019) allows players to hold belief hierarchies about the focus functions of the players, where players may well be wrong about the opponents' actual focus functions. As such, their epistemic characterization is really based on the players' belief hierarchies about the focus functions.

can be phrased as a "slow", unfinished elimination order of the strong rationalizability procedure, the result obtains.

This raises the question whether Theorem 5.1, for the special case of $f = f^{all}$, can be shown on the basis of the order independence with respect to outcomes of strong rationalizability, and hence of the f^{all}-rationalizability procedure. The answer is "no". The reason is that for many focus functions $g \subseteq f^{all}$, the g-rationalizability procedure does not correspond to a "slow" (finished or unfinished) elimination order of f^{all}.

To see this, consider the game from Figure 1, let $H^* = \{h_2, h_1', h_2'\}$, and let f^{H^*} be the associated focus function as defined in Section 3.4.6. In the first round of f^{H^*}-rationalizability we eliminate at h_1 player 1's strictly dominated strategy (b, e), and at h_2 and h_2' player 2's strictly dominated strategy (d, h). In round 2 we eliminate at h_1 and h_1' player 2's strategy (d, h), after which we can eliminate at h_1 and h_1' player 1's strictly dominated strategy (b, f). In round 3 we eliminate at h_2 player 1's strategy (b, f) (since player 2 reasons at h_2 about h_1'), after which we can eliminate player 2's strictly dominated strategy (d, g) there.

However, if at round 3 we would use the reduction operator rf^{all} instead of rf^{H^*} then, since in f^{all} player 2 reasons at h_2 about h_1, we cannot eliminate any strategy for player 1 at h_2 as both of player 1's strategies (b, e) and (b, f) that reach h_2 have already been eliminated at h_1. As a result, we cannot eliminate any more strategies for player 2 at h_2 under rf^{all}. But recall that under rf^{H^*} we *can* eliminate player 2's strategy (d, g) at h_2 at this stage. As such, f^{H^*}-rationalizability eliminates in round 3 more at h_2 than f^{all}-rationalizability would for that particular collection of decision problems. This means that f^{H^*}-rationalizability does not correspond to a "slow" elimination order of f^{all}-rationalizability.

Incomplete information. A route that remains to be explored is how to extend our analysis to games with incomplete information. Many of the rationalizability concepts, such as strong Δ-rationalizability (Battigalli (2003)), which includes strong rationalizability as a special case, and backwards rationalizability (Penta (2015), Catonini and Penta (2022)), have already been defined for games with incomplete information. Extending the monotonicity-result to the context of incomplete information may be very valuable for implementation theory and mechanism design, for instance.

Cautious reasoning. The analysis could also be extended to capture *cautious reasoning,* in which a player never discards any opponent's strategy completely from consideration. Rationalizability concepts for dynamic games that incorporate cautious reasoning are, for instance, the iterated elimination of weakly dominated strategies in the normal form, the Dekel-Fudenberg procedure (Dekel and Fudenberg (1990)), quasi-perfect rationalizability (Asheim and Perea (2005)) and perfect backwards rationalizability (Meier and Perea (2024)).

An important step in this direction has been taken in Catonini (2024) by showing that there are dynamic games where not every outcome induced by the iterated elimination of weakly dominated strategies in the normal form is induced by strong rationalizability. This is remarkable, as the first step in the iterated elimination of weakly dominated strategies (eliminating all weakly dominated strategies from the full game) is more restrictive than the first step in strong rationalizability (eliminating all strategies that are not optimal, at some information set, for any belief in the full game). However, since both concepts are non-monotonic, this property does not necessarily carry over at further elimination steps.

Non-monotonicity. It is exactly this non-monotonicity that makes the analysis of forward induction concepts so difficult, and yet so interesting at the same time. This also applies to our paper, where the non-monotonic rationalizability concepts are exactly those that correspond to a focus function in which the players are required to reason about the opponents' rationality at some of the *past* information sets. One could categorize those as the forward induction concepts within the family of f-rationalizability procedures. The non-monotonicity of these forward induction concepts heavily complicated the analysis in this paper, and is largely responsible for the long proof we needed to show the monotonicity result.

8 Proofs

8.1 Proofs of Section 4

8.1.1 Preparatory Results

For a focus function f, collection of decision problems D, player i and information set $h_i \in H_i$ we define the set

$$D_i^*(h_i) := \{s_i \in S_i(h_i) \mid s_i \text{ not strictly dominated in } (S_i(h_i), D_{-i}(h_i))\}.$$

Lemma 8.1 (Optimal strategies in rationalizability procedure) *Let f be a focus function, and let $D = rf^k(D^{full})$ for some $k \geq 1$. Then, $D_i(h_i) = D_i^*(h_i)$.*

Proof. Let $D = rf^k(D^{full})$. We prove, by induction on $k \geq 1$, that $D_i(h_i) = D_i^*(h_i)$.

Induction start. Let $k = 1$, which means that $D = rf(D^{full})$. By definition,

$$D_i(h_i) = \{s_i \in S_i(h_i) \mid s_i \text{ not strictly dominated in } (S_i(h_i), D_{-i}(h_i))\} = D_i^*(h_i).$$

Induction step. Suppose next that $k \geq 2$, and that the statement holds for $k-1$. Then, $D = rf(D^{k-1})$, where $D^{k-1} := rf^{k-1}(D^{full})$. We start by showing that $D_i(h_i) \subseteq D_i^*(h_i)$. Take some $s_i \in D_i(h_i)$. Then, by definition, $s_i \in D_i^{k-1}(h_i)$ and s_i is not strictly dominated in $(D_i^{k-1}(h_i), D_{-i}(h_i))$. By Lemma 2.1 we then know that s_i is optimal in $D_i^{k-1}(h_i)$ for a belief $b_i \in \Delta(D_{-i}(h_i))$.

We will show that s_i is optimal in $S_i(h_i)$ for b_i. Suppose not. Then, there is some $s_i^* \in S_i(h_i)$ such that

$$u_i(s_i, b_i) < u_i(s_i^*, b_i) \tag{8.1}$$

and s_i^* is optimal in $S_i(h_i)$ for b_i. As $D_{-i}(h_i) \subseteq D_{-i}^{k-1}(h_i)$ it follows that $b_i \in \Delta(D_{-i}^{k-1}(h_i))$. Hence, we conclude by Lemma 2.1 that s_i^* is not strictly dominated in $(S_i(h_i), D_{-i}^{k-1}(h_i))$, and thus $s_i^* \in D_i^{k-1*}(h_i)$. Since, by the induction assumption, $D_i^{k-1*}(h_i) = D_i^{k-1}(h_i)$ it follows that $s_i^* \in D_i^{k-1}(h_i)$. However, together with (8.1) this would contradict the fact that s_i is optimal in $D_i^{k-1}(h_i)$ for b_i. We thus conclude that s_i is optimal in $S_i(h_i)$ for b_i. Since $b_i \in \Delta(D_{-i}(h_i))$ we know by Lemma 2.1 that s_i is not strictly dominated in $(S_i(h_i), D_{-i}(h_i))$, and hence $s_i \in D_i^*(h_i)$. As this holds for every $s_i \in D_i(h_i)$ we see that $D_i(h_i) \subseteq D_i^*(h_i)$.

We next show that $D_i^*(h_i) \subseteq D_i(h_i)$. Take some $s_i \in D_i^*(h_i)$. Then, by definition, s_i is not strictly dominated in $(S_i(h_i), D_{-i}(h_i))$. As $D_{-i}(h_i) \subseteq D_{-i}^{k-1}(h_i)$ it follows that s_i is not strictly dominated in $(S_i(h_i), D_{-i}^{k-1}(h_i))$ and hence $s_i \in D_i^{k-1*}(h_i)$. By the induction assumption we know that $D_i^{k-1*}(h_i) = D_i^{k-1}(h_i)$ and hence $s_i \in D_i^{k-1}(h_i)$. We thus see that $s_i \in D_i^{k-1}(h_i)$ and s_i is not strictly dominated in $(S_i(h_i), D_{-i}(h_i))$. In particular, s_i is not strictly dominated in $(D_i^{k-1}(h_i), D_{-i}(h_i))$, which implies that $s_i \in D_i(h_i)$. As this holds for every $s_i \in D_i^*(h_i)$ we conclude that $D_i^*(h_i) \subseteq D_i(h_i)$.

Together with the insight above that $D_i(h_i) \subseteq D_i^*(h_i)$ we conclude that $D_i(h_i) = D_i^*(h_i)$. By induction on k, the proof is complete. ■

For a collection of decision problems D, focus function f, player i and information set $h_i \in H_i$, recall that

$$D_{-i}^+(h_i, f) := \{(s_j)_{j \neq i} \in D_{-i}(h_i) \mid \text{for all } j \neq i, \ s_j \in D_j(h_j) \text{ for all } h_j \in f_{ij}(h_i, D) \cap H_j(s_j)\}.$$

We define

$$D_{-i}^{+*}(h_i, f) := \{(s_j)_{j \neq i} \in S_{-i}(h_i) \mid \text{for all } j \neq i, \ s_j \in D_j(h_j) \text{ for all } h_j \in f_{ij}(h_i, D) \cap H_j(s_j)\}.$$

Hence, $D_{-i}^+(h_i, f) = D_{-i}^{+*}(h_i, f) \cap D_{-i}(h_i)$.

Lemma 8.2 (Opponents' strategies in rationalizability procedure) *Consider a monotone focus function f and let $D = rf^k(D^{full})$ for some $k \geq 0$. Then, $D_{-i}^{+*}(h_i, f) = D_{-i}^+(h_i, f)$ for every player i and information set $h_i \in H_i$.*

Proof. Let $D = rf^k(D^{full})$. We show, by induction on $k \geq 0$, that $D_{-i}^{+*}(h_i, f) = D_{-i}^+(h_i, f)$ for every player i and information set $h_i \in H_i$.

Induction start. For $k = 0$ we have that $D_{-i}(h_i) = S_{-i}(h_i)$, and therefore $D_{-i}^+(h_i, f) = D_{-i}^{+*}(h_i, f) \cap D_{-i}(h_i) = D_{-i}^{+*}(h_i, f)$.

Induction step. Suppose next that $k \geq 1$ and that the property holds for $k-1$. Note that $D = rf(D^{k-1})$, where $D^{k-1} := rf^{k-1}(D^{full})$. Since, by construction, $D_{-i}^+(h_i, f) \subseteq D_{-i}^{+*}(h_i, f)$ it only remains to show that $D_{-i}^{+*}(h_i, f) \subseteq D_{-i}^+(h_i, f)$. To show this, take some $s_{-i} = (s_j)_{j \neq i} \in D_{-i}^{+*}(h_i, f)$. Then, $s_{-i} \in S_{-i}(h_i)$ and for all $j \neq i$ we have that

$$s_j \in D_j(h_j) \text{ for all } h_j \in f_{ij}(h_i, D) \cap H_j(s_j). \tag{8.2}$$

Note that $D \subseteq D^{k-1}$. By (8.2) we thus conclude that $s_j \in D_j^{k-1}(h_j)$ for all $h_j \in f_{ij}(h_i, D) \cap H_j(s_j)$. As f is monotone and $D \subseteq D^{k-1}$ we know that $f_{ij}(h_i, D^{k-1}) \subseteq f_{ij}(h_i, D)$. Therefore, $s_j \in D_j^{k-1}(h_j)$ for all $h_j \in f_{ij}(h_i, D^{k-1}) \cap H_j(s_j)$, which means that $s_{-i} \in D_{-i}^{k-1+*}(h_i, f)$. By the induction assumption we then know that $s_{-i} \in D_{-i}^{k-1+}(h_i, f)$. Hence, $D_{-i}^{k-1+}(h_i, f) \neq \emptyset$. Since $D = rf(D^{k-1})$ we know that $D_{-i}(h_i) = D_{-i}^{k-1+}(h_i, f)$, and thus $s_{-i} \in D_{-i}(h_i)$. Recall the assumption above that $s_{-i} \in D_{-i}^{+*}(h_i, f)$. Hence, $s_{-i} \in D_{-i}^{+*}(h_i, f) \cap D_{-i}(h_i) = D_{-i}^+(h_i, f)$. As this holds for every $s_{-i} \in D_{-i}^{+*}(h_i, f)$ we conclude that

$D_{-i}^{+*}(h_i, f) \subseteq D_{-i}^{+}(h_i, f)$. Together with the fact that $D_{-i}^{+}(h_i, f) \subseteq D_{-i}^{+*}(h_i, f)$ we see that $D_{-i}^{+}(h_i, f) = D_{-i}^{+*}(h_i, f)$. By induction on k the proof is complete. ■

Consider a player i, an information set $h_i \in H_i$, a belief $b_i \in \Delta(S_{-i}(h_i))$, and an information set $h_i' \in H_i$ following h_i with $b_i(S_{-i}(h_i')) > 0$. Then, the belief $b_i' \in \Delta(S_{-i}(h_i'))$ given by

$$b_i'(s_{-i}) := \frac{b_i(h_i)(s_{-i})}{b_i(h_i)(S_{-i}(h_i'))}$$

for all $s_{-i} \in S_{-i}(h_i')$ is called the *forward consistent update* of b_i at h_i'.

Lemma 8.3 (Preservation of optimality) *Consider a player i, an information set $h_i \in H_i$, a belief $b_i \in \Delta(S_{-i}(h_i))$, a strategy s_i that is optimal in $S_i(h_i)$ for b_i, and an information set $h_i' \in H_i(s_i)$ following h_i with $b_i(S_{-i}(h_i')) > 0$. Let b_i' be the forward consistent update of b_i at h_i'. Then, s_i is optimal in $S_i(h_i')$ for b_i'.*

The proof can be found in Perea (2012), proof of Lemma 8.14.9.

We say that a collection of decision problems D is *dynamically consistent* if for every player i and every two information sets $h_i, h_i' \in H_i$ where h_i precedes h_i', it holds that $D_{-i}(h_i) \cap S_{-i}(h_i') \subseteq D_{-i}(h_i')$. For a given history p and player i, let

$$H_i^-(p) := \{h_i \in H_i \mid \text{there is some history } p' \in h_i \text{ that precedes } p\}.$$

For a given strategy $s_i \in S_i$ and a collection of information sets $\hat{H} \subseteq H$, let

$$s_i|_{\hat{H}} := (s_i(h_i))_{h_i \in H_i(s_i) \cap \hat{H}}$$

be its restriction to information sets in $\hat{H}$.

Lemma 8.4 (Perfectionating a strategy) *Consider a dynamically consistent collection of decision problems D, a player i, a strategy $s_i \in S_i$ and a history p such that s_i selects all player i actions in p. Let $H_i^* \subseteq H_i^-(p)$ be such that $s_i \in D_i^*(h_i)$ for all $h_i \in H_i^*$. Then, there is some strategy s_i^* such that $s_i^*|_{H_i^-(p)} = s_i|_{H_i^-(p)}$ and $s_i^* \in D_i^*(h_i)$ for all $h_i \in H_i^* \cup (H_i(s_i^*) \backslash H_i^-(p))$. Moreover, if $p \in h_i$ for some $h_i \in H_i$ and $b_i \in \Delta(D_{-i}(h_i))$, then s_i^* can be constructed such that s_i^* is optimal in $S_i(h_i)$ for b_i.*

Proof. Let $H_i^+(p)$ be the collection of information sets $h_i \in H_i$ such that either $p \in h_i$ or there is some history $p' \in h_i$ that follows p. We first define collections of information sets $H_i^1, H_i^2, ...$, and associated beliefs $b_i(h_i)$ for $h_i \in H_i^1 \cup H_i^2 \cup ...$, as follows. Let h_i^1 be the first information set, if any, in H_i^*. By assumption, $s_i \in D_i^*(h_i^1)$. Hence, by definition, s_i is not strictly dominated in $(S_i(h_i^1), D_{-i}(h_i^1))$. By Lemma 2.1 there is a belief $b_i(h_i^1) \in \Delta(D_{-i}(h_i^1))$ such that s_i is optimal in $S_i(h_i^1)$ for $b_i(h_i^1)$. Let

$$H_i^1 := \{h_i \in H_i^+(h_i^1) \backslash H_i^+(p) \mid b_i(h_i^1)(S_{-i}(h_i)) > 0\}.$$

For every $h_i \in H_i^1 \backslash \{h_i^1\}$ let $b_i(h_i)$ be the forward consistent update of $b_i(h_i^1)$. Then, we know from Lemma 8.3 that for every $h_i \in H_i^1 \cap H_i(s_i)$ the strategy s_i is optimal in $S_i(h_i)$ for $b_i(h_i)$.

Moreover, we can show that $b_i(h_i) \in \Delta(D_{-i}(h_i))$ for every $h_i \in H_i^1 \backslash \{h_i^1\}$. To see this, take some $h_i \in H_i^1 \backslash \{h_i^1\}$ and recall that $b_i(h_i)$ is the forward consistent update of $b_i(h_i^1)$ at h_i. Then, by construction, $b_i(h_i) \in \Delta(D_{-i}(h_i^1) \cap S_{-i}(h_i))$. As h_i follows h_i^1 and D is dynamically consistent, we know that $D_{-i}(h_i^1) \cap S_{-i}(h_i) \subseteq D_{-i}(h_i)$, and thus $b_i(h_i) \in \Delta(D_{-i}(h_i))$.

Let h_i^2 be the first information set, if any, in $H_i^* \backslash H_i^1$. By assumption, $s_i \in D_i^*(h_i^2)$. Hence, by a similar argument as above, there is a belief $b_i(h_i^2) \in \Delta(D_{-i}(h_i^2))$ such that s_i is optimal in $S_i(h_i^2)$ for $b_i(h_i^2)$. Set

$$H_i^2 := \{h_i \in H_i^+(h_i^2) \backslash H_i^+(p) \mid b_i(h_i^2)(S_{-i}(h_i)) > 0\}.$$

For every $h_i \in H_i^2 \backslash \{h_i^2\}$ let $b_i(h_i)$ be the forward consistent update of $b_i(h_i^2)$ at h_i. Then, we know from Lemma 8.3 that for every $h_i \in H_i^2 \cap H_i(s_i)$ the strategy s_i is optimal in $S_i(h_i)$ for $b_i(h_i)$. In the same way as above it can be shown that $b_i(h_i) \in \Delta(D_{-i}(h_i))$.

We would then select h_i^3 as the first information set, if any, in $H_i^* \backslash (H_i^1 \cup H_i^2)$, and so on. We continue this construction, by defining collections of information sets H_i^k and associated beliefs $b_i(h_i) \in \Delta(D_{-i}(h_i))$ for every $h_i \in H_i^k$, until we arrive at a collection H_i^K where $H_i^* \backslash (H_i^1 \cup ... \cup H_i^K)$ is empty. We then set

$$s_i^*(h_i) := s_i(h_i) \text{ for all } h_i \in (H_i^1 \cup ... \cup H_i^K) \cap H_i(s_i). \tag{8.3}$$

Moreover, we define

$$s_i^*(h_i) := s_i(h_i) \text{ for all } h_i \in H_i^-(p) \backslash H_i^*. \tag{8.4}$$

We then set

$$\hat{H}_i^1 := (H_i \backslash (H_i^-(p) \cup H_i^1 \cup ... \cup H_i^K))^{first}.$$

For every $h_i^1 \in \hat{H}_i^1$ we select a belief $b_i(h_i^1) \in \Delta(D_{-i}(h_i^1))$, and let $s_i[h_i^1] \in S_i(h_i^1)$ be a strategy that is optimal in $S_i(h_i^1)$ for $b_i(h_i^1)$. We define

$$\hat{H}_i^{1+} := \{h_i \in H_i \mid h_i \text{ weakly follows some } h_i^1 \in \hat{H}_i^1 \text{ with } b_i(h_i^1)(S_{-i}(h_i)) > 0\}.$$

Take some $h_i \in \hat{H}_i^{1+} \backslash \hat{H}_i^1$, and let $h_i^1 \in \hat{H}_i^1$ be the unique information set in $\hat{H}_i^1$ that precedes h_i. Define $b_i(h_i)$ to be the forward consistent update of $b_i(h_i^1)$ at h_i. Then, we know from Lemma 8.3 that strategy $s_i[h_i^1]$ is optimal in $S_i(h_i)$ for $b_i(h_i)$ whenever $h_i \in H_i(s_i[h_i^1])$. Moreover, it can be shown similarly as above that $b_i(h_i) \in \Delta(D_{-i}(h_i))$.

We then set

$$\hat{H}_i^2 := (H_i \backslash (\hat{H}_i^1 \cup H_i^-(p) \cup H_i^1 \cup ... \cup H_i^K))^{first}.$$

For every $h_i^2 \in \hat{H}_i^2$ we select a belief $b_i(h_i^2) \in \Delta(D_{-i}(h_i^2))$, and let $s_i[h_i^2] \in S_i(h_i^2)$ be a strategy that is optimal in $S_i(h_i^2)$ for $b_i(h_i^2)$. We define

$$\hat{H}_i^{2+} := \{h_i \in H_i \mid h_i \text{ weakly follows some } h_i^2 \in \hat{H}_i^2 \text{ with } b_i(h_i^2)(S_{-i}(h_i)) > 0\}.$$

Take some $h_i \in \hat{H}_i^{2+} \backslash \hat{H}_i^2$, and let $h_i^2 \in \hat{H}_i^2$ be the unique information set in $\hat{H}_i^2$ that precedes h_i. Define $b_i(h_i)$ to be the forward consistent update of $b_i(h_i^2)$ at h_i. Then, we know from Lemma 8.3 that strategy $s_i[h_i^2]$ is optimal in $S_i(h_i)$ for $b_i(h_i)$ whenever $h_i \in H_i(s_i[h_i^2])$. Moreover, it can be shown similarly as above that $b_i(h_i) \in \Delta(D_{-i}(h_i))$.

We continue this construction until we reach a set $\hat{H}_i^M$ such that

$$H_i = \hat{H}_i^{1+} \cup ... \cup \hat{H}_i^{M+} \cup H_i^-(p) \cup H_i^1 \cup ... \cup H_i^K.$$

On $\hat{H}_i^{1+} \cup ... \cup \hat{H}_i^{M+}$ define the strategy s_i^* such that for every $m \in \{1, ..., M\}$, every $h_i^m \in \hat{H}_i^m$, and every $h_i \in \hat{H}_i^{m+}$ weakly following h_i^m,

$$s_i^*(h_i) := (s_i[h_i^m])(h_i) \text{ whenever } h_i \in H_i(s_i^*). \tag{8.5}$$

This completes the construction of the strategy s_i^*.

By (8.3) and (8.4) it follows that $s_i^*|_{H_i^-(p)} = s_i|_{H_i^-(p)}$. We will now show that $s_i^* \in D_i^*(h_i)$ for all $h_i \in H_i^* \cup (H_i(s_i^*) \backslash H_i^-(p))$. We separate two cases: (1) $h_i \in H_i^k \cap H_i(s_i^*)$ for some $k \in \{1, ..., K\}$, and (2) $h_i \in \hat{H}_i^{m+} \cap H_i(s_i^*)$ for some $m \in \{1, ..., M\}$.

Case 1. Suppose that $h_i \in H_i^k \cap H_i(s_i^*)$ for some $k \in \{1, ..., K\}$. Then, by (8.3), s_i^* coincides with s_i on $H_i^k \cap H_i(s_i^*)$. Recall from above that strategy s_i is optimal in $S_i(h_i)$ for $b_i(h_i)$. As, by construction of H_i^k, the expected utility $u_i(s_i^*, b_i(h_i))$ only depends on the behavior of s_i^* at information sets in H_i^k, it follows that s_i^* is optimal in $S_i(h_i)$ for $b_i(h_i)$ as well. Since we have seen above that $b_i(h_i) \in \Delta(D_{-i}(h_i))$, we conclude by Lemma 2.1 that s_i^* is not strictly dominated in $(S_i(h_i), D_{-i}(h_i))$. That is, $s_i^* \in D_i^*(h_i)$, which was to show.

Case 2. Suppose that $h_i \in \hat{H}_i^{m+} \cap H_i(s_i^*)$ for some $m \in \{1, ..., M\}$. Let h_i^m be the unique information set in $\hat{H}_i^m$ that weakly precedes h_i. Then, we know from (8.5) that s_i^* coincides with $s_i[h_i^m]$ on $\hat{H}_i^{m+}$. Recall from above that $s_i[h_i^m]$ is optimal in $S_i(h_i)$ for $b_i(h_i)$. As, by construction of $\hat{H}_i^{m+}$, the expected utility $u_i(s_i^*, b_i(h_i))$ only depends on the behavior of s_i^* at information sets in $\hat{H}_i^{m+}$, it follows that s_i^* is optimal in $S_i(h_i)$ for $b_i(h_i)$ as well. Since we have seen above that $b_i(h_i) \in \Delta(D_{-i}(h_i))$, we conclude by Lemma 2.1 that s_i^* is not strictly dominated in $(S_i(h_i), D_{-i}(h_i))$. That is, $s_i^* \in D_i^*(h_i)$, which was to show.

Suppose now that $p \in h_i$ for some $h_i \in H_i$, and that the belief $b_i \in \Delta(D_{-i}(h_i))$ is given. Then, h_i will be included in $\hat{H}_i^1$, and we can define $b_i(h_i) := b_i$. As $s_i^*|_{H_i^-(p)} = s_i|_{H_i^-(p)}$ and s_i selects all the player i actions in p, it follows that s_i^* selects all player i actions in p as well, and hence $s_i^* \in S_i(h_i)$. Since, by construction, s_i^* is optimal for $b_i(h_i)$ on $S_i(h_i)$, it follows that s_i^* is optimal for b_i on $S_i(h_i)$. Hereby, the proof is complete. ■

Lemma 8.5 (Optimal planning) *Consider a dynamically consistent collection of decision problems D. Then, for every player i,*

(a) there is a strategy s_i such that $s_i \in D_i^*(h_i)$ for every $h_i \in H_i(s_i)$, and

(b) for every information set $h_i \in H_i$ there is a strategy $s_i \in S_i(h_i)$ such that $s_i \in D_i^*(h_i')$ for every $h_i' \in H_i(s_i) \backslash H_i^-(h_i)$.

Proof. **(a)** By Lemma 8.4 applied to the history $p = \varnothing$ marking the beginning of the game, there is a strategy s_i^* such that $s_i^* \in D_i^*(h_i)$ for all $h_i \in H_i(s_i^*)$.

(b) Consider an information set $h_i \in H_i$, a history $p \in h_i$ and a strategy $s_i \in S_i$ such that s_i selects all player i actions in p. Then, by Lemma 8.4 there is some strategy s_i^* such that $s_i^*|_{H_i^-(p)} = s_i|_{H_i^-(p)}$ and $s_i^* \in D_i^*(h_i')$ for all $h_i' \in H_i(s_i^*) \backslash H_i^-(p)$. By perfect recall we have that $H_i^-(p) = H_i^-(h_i)$, and hence $s_i^* \in D_i^*(h_i')$ for all $h_i' \in H_i(s_i^*) \backslash H_i^-(h_i)$. Moreover, since $s_i^*|_{H_i^-(p)} = s_i|_{H_i^-(p)}$ and s_i selects all player i actions in p, it follows that s_i^* selects all player i actions in p as well. Since $p \in h_i$ we conclude that $s_i^* \in S_i(h_i)$. This completes the proof. ■

Lemma 8.6 (Sufficient conditions for dynamic consistency) *Consider a focus function f that is monotone, individually forward decreasing and individually preserving focus on past information sets, and let $D = rf^k(D^{full})$ for some $k \geq 0$. Then, D is dynamically consistent.*

Proof. Let $D = rf^k(D^{full})$ for some $k \geq 0$. We show the statement by induction on k.

Induction start. For $k = 0$ we have that $D = D^{full}$. It can easily be verified that D^{full} is dynamically consistent.

Induction step. Now take some $k \geq 1$, and suppose that $D^{k-1} := rf^{k-1}(D^{full})$ is dynamically consistent. Note that $D = rf(D^{k-1})$. Take some player i and some information sets h_i, h_i' where h_i precedes h_i'. We will show that $D_{-i}(h_i) \cap S_{-i}(h_i') \subseteq D_{-i}(h_i')$. We distinguish two cases: (1) $D_{-i}^{k-1+}(h_i, f) \neq \emptyset$, and (2) $D_{-i}^{k-1+}(h_i, f) = \emptyset$.

Case 1. Suppose that $D_{-i}^{k-1+}(h_i, f) \neq \emptyset$. Take some $(s_j)_{j \neq i} \in D_{-i}(h_i) \cap S_{-i}(h_i')$. Then, in particular, $(s_j)_{j \neq i} \in D_{-i}^{k-1}(h_i) \cap S_{-i}(h_i')$. Since, by the induction assumption, D^{k-1} is dynamically consistent, we conclude that $(s_j)_{j \neq i} \in D_{-i}^{k-1}(h_i')$.

As $D_{-i}^{k-1+}(h_i, f) \neq \emptyset$ we have that $D_{-i}(h_i) = D_{-i}^{k-1+}(h_i, f)$, and hence $(s_j)_{j \neq i} \in D_{-i}^{k-1+}(h_i, f) \cap S_{-i}(h_i')$. Thus, for every $j \neq i$ we have that $s_j \in D_j^{k-1}(h_j)$ for all $h_j \in f_{ij}(h_i, D^{k-1}) \cap H_j(s_j)$. As f is individually forward decreasing we know that $f_{ij}(h_i', D^{k-1}) \subseteq f_{ij}(h_i, D^{k-1})$. Hence, for every $j \neq i$ we have that $s_j \in D_j^{k-1}(h_j)$ for all $h_j \in f_{ij}(h_i', D^{k-1}) \cap H_j(s_j)$. Since we have seen above that $(s_j)_{j \neq i} \in D_{-i}^{k-1}(h_i')$, and $D = rf(D^{k-1})$, it follows that $(s_j)_{j \neq i} \in D_{-i}(h_i')$. Thus, $D_{-i}(h_i) \cap S_{-i}(h_i') \subseteq D_{-i}(h_i')$.

Case 2. Suppose that $D_{-i}^{k-1+}(h_i, f) = \emptyset$. We show that $D_{-i}^{k-1+}(h_i', f) = \emptyset$ as well. Suppose not. Then, there is some $(s_j)_{j \neq i} \in D_{-i}^{k-1+}(h_i', f)$. Hence, for every $j \neq i$ we have that $s_j \in D_j^{k-1}(h_j)$ for all $h_j \in f_{ij}(h_i', D^{k-1}) \cap H_j(s_j)$. Fix a player $j \neq i$. As f individually preserves focus on past information sets, $f_{ij}(h_i, D^{k-1}) \cap H_j^-(h_i) \subseteq f_{ij}(h_i', D^{k-1})$. Therefore,

$$s_j \in D_j^{k-1}(h_j) \text{ for all } h_j \in f_{ij}(h_i, D^{k-1}) \cap H_j^-(h_i) \cap H_j(s_j). \tag{8.6}$$

Take some $s_i \in S_i(h_i')$. As $(s_m)_{m\neq i} \in D_{-i}^{k-1+}(h_i', f)$, we know in particular that $(s_m)_{m\neq i} \in S_{-i}(h_i')$, and hence $(s_i, (s_m)_{m\neq i})$ reaches h_i'. As h_i precedes h_i' and the game satisfies perfect recall, we know that $(s_i, (s_m)_{m\neq i})$ reaches h_i as well. Let p be the history in h_i reached by $(s_i, (s_m)_{m\neq i})$. Then, $H_j^-(p) \subseteq H_j^-(h_i)$, and hence by (8.6) we know that $s_j \in D_j^{k-1}(h_j)$ for all $h_j \in f_{ij}(h_i, D^{k-1}) \cap H_j^-(p) \cap H_j(s_j)$. By Lemma 8.1 we know that $D_j^{k-1}(h_j) = D_j^{k-1*}(h_j)$, and hence $s_j \in D_j^{k-1*}(h_j)$ for all $h_j \in f_{ij}(h_i, D^{k-1}) \cap H_j^-(p) \cap H_j(s_j)$. Moreover, by the induction assumption, we know that D^{k-1} is dynamically consistent. Using Lemma 8.4, choosing $H_j^* := f_{ij}(h_i, D^{k-1}) \cap H_j^-(p)$, we then know that there is some s_j^* with $s_j^*|_{H_j^-(p)} = s_j|_{H_j^-(p)}$ such that $s_j^* \in D_j^{k-1*}(h_j)$ for all $h_j \in f_{ij}(h_i, D^{k-1}) \cap H_j(s_j^*)$. Since, by Lemma 8.1, we know that $D_j^{k-1}(h_j) = D_j^{k-1*}(h_j)$, it follows that

$$s_j^* \in D_j^{k-1}(h_j) \text{ for all } h_j \in f_{ij}(h_i, D^{k-1}) \cap H_j(s_j^*). \tag{8.7}$$

As $(s_i, (s_j)_{j\neq i})$ reaches p and $s_j^*|_{H_j^-(p)} = s_j|_{H_j^-(p)}$ for every $j \neq i$, it follows that $(s_i, (s_j^*)_{j\neq i})$ reaches p as well. Since $p \in h_i$ we conclude that $(s_j^*)_{j\neq i} \in S_{-i}(h_i)$. Together with (8.7) we conclude that $(s_j^*)_{j\neq i} \in D_{-i}^{k-1+*}(h_i, f)$. Recall that f is monotone. By Lemma 8.2 it then follows that $D_{-i}^{k-1+*}(h_i, f) = D_{-i}^{k-1+}(h_i, f)$, and thus $(s_j^*)_{j\neq i} \in D_{-i}^{k-1+}(h_i, f)$. This, however, is a contradiction, since $D_{-i}^{k-1+}(h_i, f) = \emptyset$.

We must therefore conclude that $D_{-i}^{k-1+}(h_i', f) = \emptyset$ as well. As $D = rf(D^{k-1})$ we must have, by definition, that $D_{-i}(h_i) = D_{-i}^{k-1}(h_i)$ and $D_{-i}(h_i') = D_{-i}^{k-1}(h_i')$. Hence,

$$D_{-i}(h_i) \cap S_{-i}(h_i') = D_{-i}^{k-1}(h_i) \cap S_{-i}(h_i') \subseteq D_{-i}^{k-1}(h_i') = D_{-i}(h_i'),$$

where the set inclusion follows from the induction assumption that D^{k-1} is dynamically consistent.

By combining Case 1 and Case 2 we conclude that D^k is dynamically consistent. By induction on k, the proof is complete. ■

8.1.2 Proof of Theorem 4.1

Proof of Theorem 4.1. Let $(D^k)_{k=0}^\infty$ be the f-rationalizability procedure starting at D^{full}, and let K be such that $D^{K+1} = D^K$. By Lemma 8.6 we know that D^K is dynamically consistent. As such, Lemma 8.5 (a) guarantees that for every player i there is a strategy s_i such that $s_i \in D_i^{K*}(h_i)$ for all $h_i \in H_i(s_i)$. Since, by Lemma 8.1, $D_i^{K*}(h_i) = D_i^K(h_i)$ for all players i and all $h_i \in H_i$, we conclude that $s_i \in D_i^K(h_i)$ for all $h_i \in H_i(s_i)$. That is, s_i is f-rationalizable. This completes the proof. ■

8.1.3 Proof of Theorem 4.2

To prove Theorem 4.2 we need the following preparatory result.

Lemma 8.7 (Forward consistency) *Consider a collection of decision problems D that is dynamically consistent, a player i, a strategy s_i, and a collection of information sets $\hat{H}_i \subseteq H_i$. Suppose that for every $h_i \in \hat{H}_i \cap H_i(s_i)$ there is a belief $b_i(h_i) \in \Delta(D_{-i}(h_i))$ such that s_i is optimal in $S_i(h_i)$ for $b_i(h_i)$. Then, there is a belief vector $(\tilde{b}_i(h_i))_{h_i\in H_i}$ that is forward consistent on $\hat{H}_i$ such that for all $h_i \in \hat{H}_i \cap H_i(s_i)$ we have that $\tilde{b}_i(h_i) \in \Delta(D_{-i}(h_i))$ and s_i is optimal in $S_i(h_i)$ for $\tilde{b}_i(h_i)$.*

Proof. We construct the conditional belief vector $(\tilde{b}_i(h_i))_{h_i \in H_i}$ as follows. Let $H_i^1 := \hat{H}_i^{first}$, and define

$$\tilde{b}_i(h_i^1) := b_i(h_i^1) \text{ for all } h_i \in H_i^1.$$

Let

$$H_i^{1+} := \{h_i \in \hat{H}_i \mid h_i \text{ weakly follows some } h_i^1 \in H_i^1 \text{ with } b_i(h_i^1)(S_{-i}(h_i)) > 0\}.$$

For every $h_i \in H_i^{1+} \backslash H_i^1$, let $h_i^1[h_i]$ be the unique information set in H_i^1 that precedes h_i. We then define $\tilde{b}_i(h_i)$ as the forward consistent update of $b_i(h_i^1[h_i])$ at h_i.

Next, let $H_i^2 := (\hat{H}_i \backslash H_i^{1+})^{first}$ and define

$$\tilde{b}_i(h_i^2) := b_i(h_i^2) \text{ for all } h_i \in H_i^2.$$

Let

$$H_i^{2+} := \{h_i \in \hat{H}_i \mid h_i \text{ weakly follows some } h_i^2 \in H_i^2 \text{ with } b_i(h_i^2)(S_{-i}(h_i)) > 0\}.$$

For every $h_i \in H_i^{2+} \backslash H_i^2$, let $h_i^2[h_i]$ be the unique information set in H_i^2 that precedes h_i. We then define $\tilde{b}_i(h_i)$ as the forward consistent update of $b_i(h_i^2)$ at h_i.

We next define $H_i^3 := (\hat{H}_i \backslash (H_i^{1+} \cup H_i^{2+}))^{first}$, and so on. We continue this construction until we reach some K with $H_i^{1+} \cup ... \cup H_i^{K+} = \hat{H}_i$. We finally define $\tilde{b}_i(h_i) \in \Delta(S_{-i}(h_i))$ arbitrarily for all $h_i \in H_i \backslash \hat{H}_i$. Then, by construction, the belief vector $\tilde{b}_i = (\tilde{b}_i(h_i))_{h_i \in H_i}$ is forward consistent on $\hat{H}_i$.

We will now show, for all $h_i \in \hat{H}_i$, that $\tilde{b}_i(h_i) \in \Delta(D_{-i}(h_i))$ and that s_i is optimal in $S_i(h_i)$ for $\tilde{b}_i(h_i)$. Take some $h_i \in \hat{H}_i$, and let $h_i \in H_i^{k+}$ for some $k \in \{1, ..., K\}$. Let $h_i^k[h_i]$ be the unique information set in H_i^k that weakly precedes h_i.

If $h_i = h_i^k[h_i]$ then, by construction, $\tilde{b}_i(h_i) = b_i(h_i) \in \Delta(D_{-i}(h_i))$. Moreover, we know by assumption that s_i is optimal in $S_i(h_i)$ for $b_i(h_i)$, and hence s_i is optimal in $S_i(h_i)$ for $\tilde{b}_i(h_i)$.

If $h_i \neq h_i^k[h_i]$ then, by construction, $\tilde{b}_i(h_i)$ is the forward consistent update of $b_i(h_i^k[h_i])$ at h_i. As $b_i(h_i^k[h_i]) \in \Delta(D_{-i}(h_i^k[h_i]))$ it follows, by the definition of the forward consistent update, that $\tilde{b}_i(h_i) \in \Delta(D_{-i}(h_i^k[h_i])) \cap S_{-i}(h_i))$. Since, by assumption, D is dynamically consistent, we know that $D_{-i}(h_i^k[h_i]) \cap S_{-i}(h_i) \subseteq D_{-i}(h_i)$, and thus $\tilde{b}_i(h_i) \in \Delta(D_{-i}(h_i))$.

To show that s_i is optimal in $S_i(h_i)$ for $\tilde{b}_i(h_i)$ note that, by assumption, s_i is optimal in $S_i(h_i^k[h_i])$ for $b_i(h_i^k[h_i])$. As $\tilde{b}_i(h_i)$ is the forward consistent update of $b_i(h_i^k[h_i])$ at h_i, it follows from Lemma 8.3 that s_i is optimal in $S_i(h_i)$ for $\tilde{b}_i(h_i)$. We thus see, for all $h_i \in \hat{H}_i$, that $\tilde{b}_i(h_i) \in \Delta(D_{-i}(h_i))$ and that s_i is optimal in $S_i(h_i)$ for $\tilde{b}_i(h_i)$. This completes the proof. ■

Proof of Theorem 4.2. **(a)** Let $(D^k)_{k=0}^{\infty}$ be the f-rationalizability procedure that starts at D^{full}. We show, by induction on $k \geq 0$, that (i) $D^{bu,k} = D^k$ and (ii) $D_{-i}^{bu,k+}(h_i) = D_{-i}^{k+}(h_i)$ whenever $k \geq 1$, for all players i, and all $h_i \in H_i$.

Induction start. For $k = 0$ we have by definition that $D^{bu,0} = D^0$, and hence (i) is satisfied. Also, (ii) is vacuously satisfied since $k < 1$.

Induction step. Let $k \geq 1$, and suppose that (i) and (ii) hold for $k-1$. We start by showing (i), that is, we must show that $D^{bu,k} = D^k$. If $k = 1$ then we have, by definition, that $D^{bu,1} = D^1$. Assume next that $k \geq 2$. Take some player i and information set $h_i \in H_i$. We distinguish two cases: (1) $D_{-i}^{bu,k-1+}(h_i) \neq \emptyset$, and (2) $D_{-i}^{bu,k-1+}(h_i) = \emptyset$.

Case 1. Suppose that $D_{-i}^{bu,k-1+}(h_i) \neq \emptyset$. Then, by definition, $D_{-i}^{bu,k}(h_i) = D_{-i}^{bu,k-1+}(h_i)$. By the induction assumption we know that $D_{-i}^{bu,k-1+}(h_i) = D_{-i}^{k-1+}(h_i)$, and thus $D_{-i}^{k-1+}(h_i) \neq \emptyset$ as well. We then have that $D_{-i}^{bu,k}(h_i) = D_{-i}^{bu,k-1+}(h_i) = D_{-i}^{k-1+}(h_i) = D_{-i}^{k}(h_i)$.

Case 2. Suppose that $D_{-i}^{bu,k-1+}(h_i) = \emptyset$. Then, by definition, $D_{-i}^{bu,k}(h_i) = D_{-i}^{bu,k-1}(h_i)$. By the induction assumption we know that $D_{-i}^{bu,k-1+}(h_i) = D_{-i}^{k-1+}(h_i)$, and thus $D_{-i}^{k-1+}(h_i) = \emptyset$ as well. By definition, we then have that $D_{-i}^{k}(h_i) = D_{-i}^{k-1}(h_i)$. We also know, by the induction assumption, that $D_{-i}^{bu,k-1}(h_i) = D_{-i}^{k-1}(h_i)$. Hence, we see that $D_{-i}^{bu,k}(h_i) = D_{-i}^{bu,k-1}(h_i) = D_{-i}^{k-1}(h_i) = D_{-i}^{k}(h_i)$.

We next show (ii). Hence, we must show that $D_{-i}^{bu,k+}(h_i) = D_{-i}^{k+}(h_i)$ for all players i and all $h_i \in H_i$. Fix a player i and an information set $h_i \in H_i$. We first show that $D_{-i}^{bu,k+}(h_i) \subseteq D_{-i}^{k+}(h_i)$. Take some $(s_j)_{j \neq i} \in D_{-i}^{bu,k+}(h_i)$. Then, $(s_j)_{j \neq i} \in D_{-i}^{bu,k}(h_i)$ and for every $j \neq i$ there is a belief vector $(b_j(h_j))_{h_j \in H_j}$ that is forward consistent on $f_{ij}(h_i, D^{bu,k})$, and such that for all $h_j \in f_{ij}(h_i, D^{bu,k}) \cap H_j(s_j)$ we have that $b_j(h_j) \in \Delta(D_{-j}^{bu,k}(h_j))$, $s_j \in D_j^{bu,k-1}(h_j)$ and s_j is optimal in $D_j^{bu,k-1}(h_j)$ for $b_j(h_j)$.

We have shown in (i) that $D^{bu,k} = D^k$ and we know by the induction assumption that $D^{bu,k-1} = D^{k-1}$. Hence, $(s_j)_{j \neq i} \in D_{-i}^{k}(h_i)$ and for every $j \neq i$ there is a belief vector $(b_j(h_j))_{h_j \in H_j}$ that is forward consistent on $f_{ij}(h_i, D^k)$, and such that for all $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$ we have that $b_j(h_j) \in \Delta(D_{-j}^{k}(h_j))$, $s_j \in D_j^{k-1}(h_j)$ and s_j is optimal in $D_j^{k-1}(h_j)$ for $b_j(h_j)$. By Lemma 2.1 it then follows that for every $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$, strategy s_j is in $D_j^{k-1}(h_j)$ and is not strictly dominated in $(D_j^{k-1}(h_j), D_{-j}^{k}(h_j))$, and hence $s_j \in D_j^{k}(h_j)$. Put together, we conclude that $(s_j)_{j \neq i} \in D_{-i}^{k}(h_i)$ and for every $j \neq i$ and every $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$ we know that $s_j \in D_j^{k}(h_j)$. Hence, by definition, $(s_j)_{j \neq i} \in D_{-i}^{k+}(h_i)$. As this holds for every $(s_j)_{j \neq i} \in D_{-i}^{bu,k+}(h_i)$ we conclude that $D_{-i}^{bu,k+}(h_i) \subseteq D_{-i}^{k+}(h_i)$.

We next show that $D_{-i}^{k+}(h_i) \subseteq D_{-i}^{bu,k+}(h_i)$. Take some $(s_j)_{j \neq i} \in D_{-i}^{k+}(h_i)$. Then, by definition, $(s_j)_{j \neq i} \in D_{-i}^{k}(h_i)$ and for every $j \neq i$ we have that $s_j \in D_j^{k}(h_j)$ for all $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$. Fix a player $j \neq i$. By Lemma 8.1 we know that $D_j^{k}(h_j) = D_j^{k*}(h_j)$. Hence, for all $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$ we have that $s_j \in D_j^{k*}(h_j)$, and thus s_j is not strictly dominated in $(S_j(h_j), D_{-j}^{k}(h_j))$. By Lemma 2.1 we then conclude that for every $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$, strategy s_j is optimal in $S_j(h_j)$ for some $b_j(h_j) \in \Delta(D_{-j}^{k}(h_j))$.

Recall that f is monotone, individually forward decreasing and individually preserves focus on past information sets. By Lemma 8.6 it then follows that $D^k = rf^k(D^{full})$ is dynamically consistent. Hence, by Lemma 8.7 we can find a belief vector $(\tilde{b}_j(h_j))_{h_j \in H_j}$ that is forward consistent on $f_{ij}(h_i, D^k)$ and such that for every $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$ we have that $\tilde{b}_j(h_j) \in \Delta(D_{-j}^{k}(h_j))$ and strategy s_j is optimal in $S_j(h_j)$ for $\tilde{b}_j(h_j)$. This holds for every $j \neq i$. Recall that $(s_j)_{j \neq i} \in D_{-i}^{k}(h_i)$. Since we have seen in (i) that

$D^k = D^{bu,k}$, we conclude that $(s_j)_{j \neq i} \in D_{-i}^{bu,k}(h_i)$ and for every $j \neq i$ there is a belief vector $(\tilde{b}_j(h_j))_{h_j \in H_j}$ that is forward consistent on $f_{ij}(h_i, D^{bu,k})$ and such that for every $h_j \in f_{ij}(h_i, D^{bu,k}) \cap H_j(s_j)$ we have that $\tilde{b}_j(h_j) \in \Delta(D_{-j}^{bu,k}(h_j))$ and strategy s_j is optimal in $S_j(h_j)$ for $\tilde{b}_j(h_j)$. Recall that $s_j \in D_j^k(h_j)$ for all $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$. As $D^k = D^{bu,k}$ we conclude that $s_j \in D_j^{bu,k}(h_j)$ for all $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$ and hence, in particular, $s_j \in D_j^{bu,k-1}(h_j)$ for all $h_j \in f_{ij}(h_i, D^k) \cap H_j(s_j)$. Together with our insight above, we see that for every $h_j \in f_{ij}(h_i, D^{bu,k}) \cap H_j(s_j)$ we have that $\tilde{b}_j(h_j) \in \Delta(D_{-j}^{bu,k}(h_j))$, $s_j \in D_j^{bu,k-1}(h_j)$ and strategy s_j is optimal in $S_j(h_j)$ for $\tilde{b}_j(h_j)$. Then, by definition, $(s_j)_{j \neq i} \in D_{-i}^{bu,k+}(h_i)$.

As this holds for every $(s_j)_{j \neq i} \in D_{-i}^{k+}(h_i)$ we conclude that $D_{-i}^{k+}(h_i) \subseteq D_{-i}^{bu,k+}(h_i)$. We have already seen that $D_{-i}^{bu,k+}(h_i) \subseteq D_{-i}^{k+}(h_i)$, and hence $D_{-i}^{bu,k+}(h_i) = D_{-i}^{k+}(h_i)$. We have thus shown property (ii).

By induction on k, the proof of (a) is complete.

(b) Let K be such that $D^{bu,K} = D^{bu,K+1}$. Then, we know from (a) that $D^K = D^{K+1}$. We first show that every f-rationalizable strategy is f-rationalizable under forward consistency. Consider a player i and a strategy $s_i \in S_i$ that is f-rationalizable. Then, $s_i \in D_i^K(h_i)$ for all $h_i \in H_i(s_i)$. By Lemma 8.1 we know that $D_i^K(h_i) = D_i^{K*}(h_i)$, and hence $s_i \in D_i^{K*}(h_i)$ for all $h_i \in H_i(s_i)$. As $D^K = D^{K+1}$, this means that for all $h_i \in H_i(s_i)$ the strategy s_i is not strictly dominated in $(S_i(h_i), D_{-i}^K(h_i))$. Hence, by Lemma 2.1, for all $h_i \in H_i(s_i)$ there is a belief $b_i(h_i) \in \Delta(D_{-i}^K(h_i))$ such that s_i is optimal in $S_i(h_i)$ for $b_i(h_i)$.

Recall that f is monotone, individually forward decreasing and individually preserves focus on past information sets. By Lemma 8.6 it then follows that $D^K = rf^K(D^{full})$ is dynamically consistent. Hence, by Lemma 8.7 we can find a belief vector $(\tilde{b}_i(h_i))_{h_i \in H_i}$ that is forward consistent on H_i such that for all $h_i \in H_i(s_i)$ we have that $\tilde{b}_i(h_i) \in \Delta(D_{-i}^K(h_i))$ and s_i is optimal in $S_i(h_i)$ for $\tilde{b}_i(h_i)$. Recall that $D^K = D^{bu,K}$. Hence, the belief vector $(\tilde{b}_i(h_i))_{h_i \in H_i}$ is forward consistent on H_i and for all $h_i \in H_i(s_i)$ we have that $\tilde{b}_i(h_i) \in \Delta(D_{-i}^{bu,K}(h_i))$ and s_i is optimal in $S_i(h_i)$ for $\tilde{b}_i(h_i)$. As $s_i \in D_i^K(h_i) = D_i^{bu,K}(h_i)$ for all $h_i \in H_i(s_i)$, we conclude for all $h_i \in H_i(s_i)$ that $\tilde{b}_i(h_i) \in \Delta(D_{-i}^{bu,K}(h_i))$, $s_i \in D_i^{bu,K}(h_i)$ and s_i is optimal in $D_i^{bu,K}(h_i)$ for $\tilde{b}_i(h_i)$. Thus, by definition, s_i is f-rationalizable under forward consistency.

We now show that every strategy which is f-rationalizable under forward consistency is also f-rationalizable. Consider a player i and a strategy $s_i \in S_i$ that is f-rationalizable under forward consistency. Then, there is a belief vector $(b_i(h_i))_{h_i \in H_i}$ that is forward consistent on H_i and such that for all $h_i \in H_i(s_i)$ we have that $b_i(h_i) \in \Delta(D_{-i}^{bu,K}(h_i))$, $s_i \in D_i^{bu,K}(h_i)$ and s_i is optimal in $D_i^{bu,K}(h_i)$ for $b_i(h_i)$. In particular, for every $h_i \in H_i(s_i)$ the strategy s_i is in $D_i^{bu,K}(h_i)$. As $D^{bu,K} = D^K$ we conclude that for every $h_i \in H_i(s_i)$ the strategy s_i is in $D_i^K(h_i)$. That is, s_i is f-rationalizable.

We thus conclude that a strategy is f-rationalizable precisely when it is f-rationalizable under forward consistency. This completes the proof. ■

8.2 Proof of Section 5

8.2.1 Preparatory Results

Consider two focus functions f, g and some $m \geq 0$. Then, we denote by $(g^{\leq m}, f)$ the focus function given by

$$(g^{\leq m}, f)_{ij}(h_i, D) := \begin{cases} f_{ij}(h_i, D), & \text{if } D \subseteq rg^m(D^{full}) \\ g_{ij}(h_i, D), & \text{otherwise} \end{cases}$$

for every player i, opponent $j \neq i$, and information set $h_i \in H_i$. By construction we then have that $r(g^{\leq m}, f)^{\infty}(D^{full}) = (rf^{\infty} \circ rg^m)(D^{full})$. We can say a bit more: Let $m^* \leq m$ be the smallest number such that $rg^{m^*}(D^{full}) = rg^m(D^{full})$. Then, $r(g^{\leq m}, f)^k(D^{full}) = rg^k(D^{full})$ for all $k \leq m^*$, and $r(g^{\leq m}, f)^{m^*+k}(D^{full}) = (rf^k \circ rg^m)(D^{full})$ for all $k \geq 1$.

Lemma 8.8 (Combination of monotone focus mappings) *Consider two monotone focus functions f, g with $g \subseteq f$. Then, for every $m \geq 0$, the focus function $(g^{\leq m}, f)$ is monotone.*

Proof. Take two collections of decisions problems $D \subseteq E$. We distinguish three cases: (1) $D, E \nsubseteq rg^m(D^{full})$, (2) $D \subseteq rg^m(D^{full})$ and $E \nsubseteq rg^m(D^{full})$, and (3) $D, E \subseteq rg^m(D^{full})$.

Case 1. Suppose that $D, E \nsubseteq rg^m(D^{full})$. Then, for every player i, opponent $j \neq i$ and information set $h_i \in H_i$ we have

$$(g^{\leq m}, f)_{ij}(h_i, E) = g_{ij}(h_i, E) \subseteq g_{ij}(h_i, D) = (g^{\leq m}, f)_{ij}(h_i, D),$$

where the set inclusion follows from the monotonicity of g.

Case 2. Suppose that $D \subseteq rg^m(D^{full})$ and $E \nsubseteq rg^m(D^{full})$. Then, for every player i, opponent $j \neq i$ and information set $h_i \in H_i$ we have

$$(g^{\leq m}, f)_{ij}(h_i, E) = g_{ij}(h_i, E) \subseteq g_{ij}(h_i, D) \subseteq f_{ij}(h_i, D) = (g^{\leq m}, f)_{ij}(h_i, D),$$

where the first set inclusion follows from the monotonicity of g, and the second set inclusion follows from $g \subseteq f$.

Case 3. Suppose that $D, E \subseteq rg^m(D^{full})$. Then, for every player i, opponent $j \neq i$ and information set $h_i \in H_i$ we have

$$(g^{\leq m}, f)_{ij}(h_i, E) = f_{ij}(h_i, E) \subseteq f_{ij}(h_i, D) = (g^{\leq m}, f)_{ij}(h_i, D),$$

where the set inclusion follows from the monotonicity of f.

By Cases 1–3 we conclude that $(g^{\leq m}, f)$ is monotone. This completes the proof. ■

For a focus function f, collection of decision problems D, player i and information set $h_i \in H_i$ we define the sets

$$D_i^+(h_i, f) := \{s_i \in S_i(h_i) \mid s_i \in D_i(h_i') \text{ for all } h_i' \in f_{ii}(h_i, D) \cap H_i(s_i)\} \text{ and}$$
$$D_{-i}^{++}(h_i, f) := \{(s_j)_{j \neq i} \in D_{-i}(h_i) \mid \text{for all } j \neq i, \ s_j \in D_j^+(h_j, f) \text{ for all } h_j \in f_{ij}(h_i, D) \cap H_j(s_j)\}.$$

In the proof of the following lemma we use a new definition: Fix a focus function f, a collection of decision problems D, two players i and j (where $i = j$ is allowed) and two information set $h_i \in H_i$ and $h_j \in H_j$. Then, h_j is called *reachable* from h_i under f and D if either $h_j = h_i$, or there are players $i^1, i^2, ..., i^K$ with associated information sets $h^1 \in H_{i^1}, h^2 \in H_{i^2}, ..., h^K \in H_{i^K}$ such that (i) $i^1 = i$, $h^1 = h_i$, $i^K = j$, $h^K = h_j$, (ii) $i_k \neq i_{k+1}$ for all $k \in \{1, ..., K-1\}$ and (iii) $h^{k+1} \in f_{i^k i^{k+1}}(h^k, D)$ for all $k \in \{1, ..., K-1\}$. Hence, if h_j is reachable from h_i then player i either directly, or indirectly, reasons about h_j while being at h_i. If h_j is reachable from h_i under f and D, and f is *transitively closed*, then it follows that $h_j \in f_{ij}(h_i, D)$.

Lemma 8.9 (Decision problems for transitively closed focus function) *Consider a transitively closed focus function f, a collection of decision problems D, a player i and an information set $h_i \in H_i$. Then, $D^+_{-i}(h_i, f) = D^{++}_{-i}(h_i, f)$.*

Proof. If $D^+_{-i}(h_i, f) = \emptyset$ then we have that $D^{++}_{-i}(h_i, f) = \emptyset$ also, since $D^{++}_{-i}(h_i, f) \subseteq D^+_{-i}(h_i, f)$. Suppose now that $D^+_{-i}(h_i, f) \neq \emptyset$. Since $D^{++}_{-i}(h_i, f) \subseteq D^+_{-i}(h_i, f)$ it remains to show that $D^+_{-i}(h_i, f) \subseteq D^{++}_{-i}(h_i, f)$. Take some $(s_j)_{j \neq i} \in D^+_{-i}(h_i, f)$. Then, $(s_j)_{j \neq i} \in D_{-i}(h_i)$ and for all $j \neq i$,

$$s_j \in D_j(h_j) \text{ for all } h_j \in f_{ij}(h_i, D) \cap H_j(s_j). \tag{8.8}$$

Consider some $h_j \in f_{ij}(h_i, D) \cap H_j(s_j)$. We will show that $s_j \in D^+_j(h_j, f)$. To see this, choose some $h'_j \in f_{jj}(h_j, D) \cap H_j(s_j)$. Then, either (i) $h'_j = h_j$ or (ii) there is some $k \neq j$ and some $h_k \in f_{jk}(h_j, D)$ such that $h'_j \in f_{kj}(h_k, D)$. If $h_j = h'_j$ then $s_j \in D_j(h'_j)$ since we have, by (8.8), that $s_j \in D_j(h_j)$. Suppose now that $h'_j \neq h_j$. As $h_j \in f_{ij}(h_i, D)$ we conclude from (ii) that h_k is reachable from h_i under f and D. Since f is transitively closed, we conclude that $f_{kj}(h_k, D) \subseteq f_{ij}(h_i, D)$. As $h'_j \in f_{kj}(h_k, D)$ it follows that $h'_j \in f_{ij}(h_i, D)$. Since $h'_j \in H_j(s_j)$ also, we see that $h'_j \in f_{ij}(h_i, D) \cap H_j(s_j)$. Together with (8.8) this implies that $s_j \in D_j(h'_j)$. Since this holds for every $h'_j \in f_{jj}(h_j, D) \cap H_j(s_j)$ we conclude that $s_j \in D^+_j(h_j, f)$.

Thus, we see that $s_j \in D^+_j(h_j, f)$ for all $h_j \in f_{ij}(h_i, D) \cap H_j(s_j)$. As this holds for every $j \neq i$, and since $(s_j)_{j \neq i} \in D_{-i}(h_i)$, we conclude that $(s_j)_{j \neq i} \in D^{++}_{-i}(h_i, f)$. This is true for every $(s_j)_{j \neq i} \in D^+_{-i}(h_i, f)$, and hence $D^+_{-i}(h_i, f) \subseteq D^{++}_{-i}(h_i, f)$. Together with the fact that $D^{++}_{-i}(h_i, f) \subseteq D^+_{-i}(h_i, f)$ this completes the proof. ■

For a given collection of decision problems D, focus function f and player i we define the set

$$H^e_i(D, f) := \{h_i \in H_i \mid D^+_i(h_i, f) \neq \emptyset \text{ and } D^+_{-i}(h_i, f) \neq \emptyset\},$$

and we refer to $H^e_i(D, f)$ as the collection of *explicable* information sets for player i under D and f.

For a set of strategy combinations $D \subseteq \times_{i \in I} S_i$ we define

$$H(D) := \{h \in H \mid \text{there is some } s \in D \text{ that reaches } h\}.$$

Consider a focus function f, two collections of decision problems D, E and a profile of transformation mappings $\sigma = (\sigma_i)_{i \in I}$ where $\sigma_i : S_i \to S_i$ for every player i. For some collection of decision problems D' we write $D \subseteq_{\sigma,f,D'} E$ if either $D' = D^{full}$ or $D = rf(D')$, and if for every player i and every strategy $s_i \in S_i$ the following three properties hold:

(T1) $\sigma_i(s_i)|_{H_i^e(D',f)} = s_i|_{H_i^e(D',f)}$,

(T2) $\sigma_i(s_i) \in E_i^+(h_i, f)$ whenever $h_i \in H_i^e(D', f) \cap H_i(\sigma_i(s_i))$ and $s_i \in D_i^+(h_i)$, and

(T3) $\sigma_i(s_i) \in E_i(h_i)$ whenever $h_i \in H_i(\sigma_i(s_i)) \backslash H_i^e(D', f)$.

Lemma 8.10 (Explicable information sets) *Consider two focus functions f, g with $g \subseteq f$ that are monotone, individually forward decreasing and individually preserving focus on past information sets. Assume moreover that f is collectively forward decreasing, collectively preserving focus on past information sets, transitively closed and monotone with respect to g. Then, the following three properties hold:*

(a) For every collection of decision problems $D = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$, every player i, every information set $h_i \in H_i^e(D, f)$, and every player j, it holds that

$$H(D_i^+(h_i, f) \times D_{-i}^+(h_i, f)) \cap H_j \subseteq H_j^e(D, f).$$

(b) For every collection of decision problems $D = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$, and every player i, the set $H_i^e(D, f)$ is closed under predecessors.

(c) Suppose that either $D = (rf^{k+1} \circ rg^m)(D^{full})$ and $E = (rf^k \circ rg^{m+1})(D^{full})$, or $D = (rf^k \circ rg^{m+1})(D^{full})$ and $E = (rf^k \circ rg^m)(D^{full})$, for some $k, m \geq 0$, and let $D \subseteq_{\sigma,f,D'} E$ for some σ and D'. Then, $H_i^e(D, f) \subseteq H_i^e(E, f)$ for every player i.

Proof. **(a)** Let $D = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$. Let $m^* \leq m$ be the smallest number such that $rg^{m^*}(D^{full}) = rg^m(D^{full})$. Then, $D = r(g^{\leq m}, f)^{m^*+k}(D^{full})$. Moreover, as f, g are monotone and $g \subseteq f$ we know by Lemma 8.8 that $(g^{\leq m}, f)$ is monotone.

Take some player i, some information set $h_i \in H_i^e(D, f)$, some player j, and some $h_j' \in H(D_i^+(h_i, f) \times D_{-i}^+(h_i, f)) \cap H_j$. Hence, there is some $s_i \in D_i^+(h_i, f)$ and some $(s_n)_{n \neq i} \in D_{-i}^+(h_i, f)$ such that $(s_i, (s_n)_{n \neq i})$ reaches h_j'. Moreover, $(s_i, (s_n)_{n \neq i})$ reaches h_i as well since $D_i^+(h_i, f) \subseteq S_i(h_i)$, $D_{-i}^+(h_i, f) \subseteq S_{-i}(h_i)$ and, by perfect recall, $S(h_i) = S_i(h_i) \times S_{-i}(h_i)$. Let p and p' be the histories in h_i and h_j', respectively, reached by $(s_i, (s_n)_{n \neq i})$. Then, either p' weakly follows p (that is, either p' follows p or $p' = p$) or p' precedes p. We will show that p' is reached by some strategy profile in $D_j^+(h_j', f) \times D_{-j}^+(h_j', f)$, thereby showing that $h_j' \in H_j^e(D, f)$. To do so we distinguish four cases: (1) p' weakly follows p and $j = i$, (2) p' precedes p and $j = i$, (3) p' weakly follows p and $j \neq i$, and (4) p' precedes p and $j \neq i$.

Case 1. Suppose that p' weakly follows p and $j = i$. We will show that $s_i \in D_i^+(h_i', f)$ and $(s_n)_{n \neq i} \in D_{-i}^+(h_i', f)$. Since $s_i \in D_i^+(h_i, f)$ we know, by definition, that $s_i \in D_i(h_i'')$ for every $h_i'' \in f_{ii}(h_i, D) \cap H_i(s_i)$. As h_i' weakly follows h_i and f is collectively forward decreasing we know that $f_{ii}(h_i', D) \subseteq f_{ii}(h_i, D)$. As such, $s_i \in D_i(h_i'')$ for every $h_i'' \in f_{ii}(h_i', D) \cap H_i(s_i)$. Since $(s_i, (s_n)_{n \neq i})$ reaches p' we also know that $s_i \in S_i(h_i')$. Therefore, $s_i \in D_i^+(h_i', f)$.

Take some $n \neq i$. As $(s_n)_{n\neq i} \in D^+_{-i}(h_i, f)$ we know, by definition, that $s_n \in D_n(h_n)$ for every $h_n \in f_{in}(h_i, D) \cap H_n(s_n)$. As h'_i weakly follows h_i and f is collectively forward decreasing we know that $f_{in}(h'_i, D) \subseteq f_{in}(h_i, D)$. As such, $s_n \in D_n(h_n)$ for every $h_n \in f_{in}(h'_i, D) \cap H_n(s_n)$. Since $(s_i, (s_n)_{n\neq i})$ reaches p' we also know that $(s_n)_{n\neq i} \in S_{-i}(h'_i)$. We therefore conclude that $(s_n)_{n\neq i} \in D^{+*}_{-i}(h'_i, f)$. Recall that $D = r(g^{\leq m}, f)^{m^*+k}(D^{full})$ and that $(g^{\leq m}, f)$ is monotone. By Lemma 8.2 we then know that

$$D^{+*}_{-i}(h'_i, f) = D^{+*}_{-i}(h'_i, (g^{\leq m}, f)) = D^+_{-i}(h'_i, (g^{\leq m}, f)) = D^+_{-i}(h'_i, f)$$

and hence $(s_n)_{n\neq i} \in D^+_{-i}(h'_i, f)$. Thus, we see that $s_i \in D^+_i(h'_i, f)$ and $(s_n)_{n\neq i} \in D^+_{-i}(h'_i, f)$, and hence $h'_i \in H^e_i(D, f)$.

Case 2. Suppose that p' precedes p and $j = i$. We show that we can construct $s^*_i \in D^+_i(h'_i, f)$ and $(s^*_n)_{n\neq i} \in D^+_{-i}(h'_i, f)$. Since $s_i \in D^+_i(h_i, f)$ we know, by definition, that $s_i \in D_i(h''_i)$ for every $h''_i \in f_{ii}(h_i, D) \cap H_i(s_i)$. As we know, by Lemma 8.1, that $D_i(h''_i) = D^*_i(h''_i)$, we see that

$$s_i \in D^*_i(h''_i) \text{ for every } h''_i \in f_{ii}(h_i, D) \cap H_i(s_i). \tag{8.9}$$

Recall that $D = r(g^{\leq m}, f)^{m^*+k}(D^{full})$ and that $(g^{\leq m}, f)$ is monotone. Moreover, as f, g are individually forward decreasing and individually preserve focus on past information sets, the focus function $(g^{\leq m}, f)$ inherits these properties. It then follows by Lemma 8.6 that D is dynamically consistent. But then, it follows from (8.9) and Lemma 8.4, setting $H^*_i := f_{ii}(h_i, D) \cap H^-_i(p')$, that there is some s^*_i with $s^*_i|_{H^-_i(p')} = s_i|_{H^-_i(p')}$ such that $s^*_i \in D^*_i(h''_i)$ for every $h''_i \in (H_i(s^*_i)\backslash H^-_i(p')) \cup (f_{ii}(h_i, D) \cap H^-_i(p'))$. By Lemma 8.1 we know that $D^*_i(h''_i) = D_i(h''_i)$, and hence $s^*_i \in D_i(h''_i)$ for every $h''_i \in (H_i(s^*_i)\backslash H^-_i(p')) \cup (f_{ii}(h_i, D) \cap H^-_i(p'))$.

As h'_i precedes h_i and f collectively preserves focus on past information sets, we conclude that $f_{ii}(h'_i, D) \cap H^-_i(p') \subseteq f_{ii}(h_i, D)$, and hence $s^*_i \in D_i(h''_i)$ for every $h''_i \in (H_i(s^*_i)\backslash H^-_i(p')) \cup (f_{ii}(h'_i, D) \cap H^-_i(p'))$. In particular, $s^*_i \in D_i(h''_i)$ for every $h''_i \in f_{ii}(h'_i, D) \cap H_i(s^*_i)$. Moreover, as $s^*_i|_{H^-_i(p')} = s_i|_{H^-_i(p')}$ and s_i selects all player i actions in p', we see that s^*_i selects all player i actions in p' as well, which means that $s^*_i \in S_i(h'_i)$. Altogether, we conclude that $s^*_i \in D^+_i(h'_i, f)$.

Take some $n \neq i$. As $(s_n)_{n\neq i} \in D^+_{-i}(h_i, f)$ we know, by definition, that $s_n \in D_n(h_n)$ for every $h_n \in f_{in}(h_i, D) \cap H_n(s_n)$. As we know, by Lemma 8.1, that $D_n(h_n) = D^*_n(h_n)$, we see that

$$s_n \in D^*_n(h_n) \text{ for every } h_n \in f_{in}(h_i, D) \cap H_n(s_n). \tag{8.10}$$

Recall from above that D is dynamically consistent. But then, it follows from (8.10) and Lemma 8.4, setting $H^*_n := f_{in}(h_i, D) \cap H^-_n(p')$, that there is some s^*_n with $s^*_n|_{H^-_n(p')} = s_n|_{H^-_n(p')}$ such that $s^*_n \in D^*_n(h_n)$ for every $h_n \in (H_n(s^*_n)\backslash H^-_n(p')) \cup (f_{in}(h_i, D) \cap H^-_n(p'))$. By Lemma 8.1 we know that $D^*_n(h_n) = D_n(h_n)$, and hence $s^*_n \in D_n(h_n)$ for every $h_n \in (H_n(s^*_n)\backslash H^-_n(p')) \cup (f_{in}(h_i, D) \cap H^-_n(p'))$.

As h'_i precedes h_i and f collectively preserves focus on past information sets, we conclude that $f_{in}(h'_i, D) \cap H^-_n(p') \subseteq f_{in}(h_i, D)$, and hence $s^*_n \in D_n(h_n)$ for every $h_n \in (H_n(s^*_n)\backslash H^-_n(p')) \cup (f_{in}(h'_i, D) \cap H^-_n(p'))$. In particular, $s^*_n \in D_n(h_n)$ for every $h_n \in f_{in}(h'_i, D) \cap H_n(s^*_n)$. Moreover, as $s^*_n|_{H^-_n(p')} = s_n|_{H^-_n(p')}$ and s_n selects all player n actions in p', we see that s^*_n selects all player n actions in p' as well. As this holds for all $n \neq i$ we conclude that $(s_n^*)_{n \neq i} \in S_{-i}(h_i')$. Altogether, we conclude that $(s_n^*)_{n \neq i} \in D_{-i}^{+*}(h_i', f)$. Since we have seen above that $D_{-i}^{+*}(h_i', f) = D_{-i}^{+}(h_i', f)$, we then know that $(s_n^*)_{n \neq i} \in D_{-i}^{+}(h_i', f)$. We thus see that $s_i^* \in D_i^+(h_i', f)$ and $(s_n^*)_{n \neq i} \in D_{-i}^{+}(h_i', f)$ and hence $h_i' \in H_i^e(D, f)$.

Case 3. Suppose that p' weakly follows p and $j \neq i$. We will show that $s_j \in D_j^+(h_j', f)$ and $(s_n)_{n \neq j} \in D_{-j}^{+}(h_j', f)$. Since $(s_n)_{n \neq i} \in D_{-i}^{+}(h_i, f)$ and $j \neq i$ we know, by definition, that $s_j \in D_j(h_j)$ for every $h_j \in f_{ij}(h_i, D) \cap H_j(s_j)$. As h_j' weakly follows h_i and f is collectively forward decreasing we know that $f_{jj}(h_j', D) \subseteq f_{ij}(h_i, D)$. As such, $s_j \in D_j(h_j)$ for every $h_j \in f_{jj}(h_j', D) \cap H_j(s_j)$. Since $(s_j, (s_n)_{n \neq j})$ reaches p' we also know that $s_j \in S_j(h_j')$. Therefore, $s_j \in D_j^+(h_j', f)$.

Take some $n \neq j$. We distinguish two subcases: (3.1) $n = i$, and (3.2) $n \neq i$.

Case 3.1. Suppose that $n = i$. As $s_i \in D_i^+(h_i, f)$ we know, by definition, that $s_i \in D_i(h_i'')$ for every $h_i'' \in f_{ii}(h_i, D) \cap H_i(s_i)$. As h_j' weakly follows h_i and f is collectively forward decreasing we know that $f_{ji}(h_j', D) \subseteq f_{ii}(h_i, D)$. As such, $s_i \in D_i(h_i'')$ for every $h_i'' \in f_{ji}(h_j', D) \cap H_i(s_i)$.

Case 3.2. Suppose that $n \neq i$. As $(s_n)_{n \neq i} \in D_{-i}^{+}(h_i, f)$ we know, by definition, that $s_n \in D_n(h_n)$ for every $h_n \in f_{in}(h_i, D) \cap H_n(s_n)$. As h_j' weakly follows h_i and f is collectively forward decreasing we know that $f_{jn}(h_j', D) \subseteq f_{in}(h_i, D)$. As such, $s_n \in D_n(h_n)$ for every $h_n \in f_{jn}(h_j', D) \cap H_n(s_n)$.

By Cases 3.1 and 3.2 we conclude, for every $n \neq j$, that $s_n \in D_n(h_n)$ for every $h_n \in f_{jn}(h_j', D) \cap H_n(s_n)$. Since $(s_j, (s_n)_{n \neq j})$ reaches p' we also know that $(s_n)_{n \neq j} \in S_{-j}(h_j')$. We therefore conclude that $(s_n)_{n \neq j} \in D_{-j}^{+*}(h_j', f)$. It can be shown in the same was as above that $D_{-j}^{+*}(h_j', f) = D_{-j}^{+}(h_j', f)$, and hence $(s_n)_{n \neq j} \in D_{-j}^{+}(h_j', f)$. Thus, we see that $s_j \in D_j^+(h_j', f)$ and $(s_n)_{n \neq j} \in D_{-j}^{+}(h_j', f)$, and hence $h_j' \in H_j^e(D, f)$.

Case 4. Suppose that p' precedes p and $j \neq i$. We show that we can construct $s_j^* \in D_j^+(h_j', f)$ and $(s_n^*)_{n \neq j} \in D_{-j}^{+}(h_j', f)$. Since $(s_n)_{n \neq i} \in D_{-i}^{+}(h_i, f)$ and $j \neq i$ we know, by definition, that $s_j \in D_j(h_j)$ for every $h_j \in f_{ij}(h_i, D) \cap H_j(s_j)$. By Lemma 8.1 we know that $D_j^*(h_j) = D_j(h_j)$, and hence

$$s_j \in D_j^*(h_j) \text{ for every } h_j \in f_{ij}(h_i, D) \cap H_j(s_j). \tag{8.11}$$

Recall from above that D is dynamically consistent. Then, it follows from (8.11) and Lemma 8.4, setting $H_j^* := f_{ij}(h_i, D) \cap H_j^-(p')$, that there is some s_j^* with $s_j^*|_{H_j^-(p')} = s_j|_{H_j^-(p')}$ such that $s_j^* \in D_j^*(h_j)$ for every $h_j \in (H_j(s_j^*) \backslash H_j^-(p')) \cup (f_{ij}(h_i, D) \cap H_j^-(p'))$. By Lemma 8.1 we know that $D_j^*(h_j) = D_j(h_j)$, and hence $s_j^* \in D_j(h_j)$ for every $h_j \in (H_j(s_j^*) \backslash H_j^-(p')) \cup (f_{ij}(h_i, D) \cap H_j^-(p'))$.

As h_j' precedes h_i and f collectively preserves focus on past information sets, we conclude that $f_{jj}(h_j', D) \cap H_j^-(p') \subseteq f_{ij}(h_i, D)$, and hence $s_j^* \in D_j(h_j)$ for every $h_j \in (H_j(s_j^*) \backslash H_j^-(p')) \cup (f_{jj}(h_j', D) \cap H_j^-(p'))$. In particular, $s_j^* \in D_j(h_j)$ for every $h_j \in f_{jj}(h_j', D) \cap H_j(s_j^*)$. Moreover, as $s_j^*|_{H_j^-(p')} = s_j|_{H_j^-(p')}$ and s_j selects all player j actions in p', we see that s_j^* selects all player j actions in p' as well, which means that $s_j^* \in S_j(h_j')$. Altogether, we conclude that $s_j^* \in D_j^+(h_j', f)$.

Take some $n \neq j$. We distinguish two subcases: (4.1) $n = i$, and (4.2) $n \neq i$.

Case 4.1. Suppose that $n = i$. As $s_i \in D_i^+(h_i, f)$ we know, by definition, that $s_i \in D_i(h_i'')$ for every $h_i \in f_{ii}(h_i, D) \cap H_i(s_i)$. Since we know, by Lemma 8.1, that $D_i^*(h_i'') = D_i(h_i'')$, it follows that

$$s_i \in D_i(h_i'') \text{ for every } h_i \in f_{ii}(h_i, D) \cap H_i(s_i). \tag{8.12}$$

Recall from above that D is dynamically consistent. But then, it follows from (8.12) and Lemma 8.4, setting $H_i^* := f_{ii}(h_i, D) \cap H_i^-(p')$, that there is some s_i^* with $s_i^*|_{H_i^-(p')} = s_i|_{H_i^-(p')}$ such that $s_i^* \in D_i^*(h_i'')$ for every $h_i'' \in (H_i(s_i^*) \backslash H_i^-(p')) \cup (f_{ii}(h_i, D) \cap H_i^-(p'))$. By Lemma 8.1 we know that $D_i^*(h_i'') = D_i(h_i'')$, and hence $s_i^* \in D_i(h_i'')$ for every $h_i'' \in (H_i(s_i^*) \backslash H_i^-(p')) \cup (f_{ii}(h_i, D) \cap H_i^-(p'))$.

As h_j' precedes h_i and f collectively preserves focus on past information sets, we conclude that $f_{ji}(h_j', D) \cap H_i^-(p') \subseteq f_{ii}(h_i, D)$, and hence $s_i^* \in D_i(h_i'')$ for every $h_i'' \in (H_i(s_i^*) \backslash H_i^-(p')) \cup (f_{ji}(h_j', D) \cap H_i^-(p'))$. In particular, $s_i^* \in D_i(h_i'')$ for every $h_i'' \in f_{ji}(h_j', D) \cap H_i(s_i^*)$.

Case 4.2. Suppose that $n \neq i$. As $(s_n)_{n \neq i} \in D_{-i}^+(h_i, f)$ we know, by definition, that $s_n \in D_n(h_n)$ for every $h_n \in f_{in}(h_i, D) \cap H_n(s_n)$. Since we know, by Lemma 8.1, that $D_n^*(h_n) = D_n(h_n)$, it follows that

$$s_n \in D_n^*(h_n) \text{ for every } h_n \in f_{in}(h_i, D) \cap H_n(s_n). \tag{8.13}$$

Recall from above that D is dynamically consistent. But then, it follows from (8.13) and Lemma 8.4, setting $H_n^* := f_{in}(h_i, D) \cap H_n^-(p')$, that there is some s_n^* with $s_n^*|_{H_n^-(p')} = s_n|_{H_n^-(p')}$ such that $s_n^* \in D_n^*(h_n)$ for every $h_n \in (H_n(s_n^*) \backslash H_n^-(p')) \cup (f_{in}(h_i, D) \cap H_n^-(p'))$. By Lemma 8.1 we know that $D_n^*(h_n) = D_n(h_n)$, and hence $s_n^* \in D_n(h_n)$ for every $h_n \in (H_n(s_n^*) \backslash H_n^-(p')) \cup (f_{in}(h_i, D) \cap H_n^-(p'))$.

As h_j' precedes h_i and f collectively preserves focus on past information sets, we conclude that $f_{jn}(h_j', D) \cap H_n^-(p') \subseteq f_{in}(h_i, D)$, and hence $s_n^* \in D_n(h_n)$ for every $h_n \in (H_n(s_n^*) \backslash H_n^-(p')) \cup (f_{jn}(h_j', D) \cap H_n^-(p'))$. In particular, $s_n^* \in D_n(h_n)$ for every $h_n \in f_{jn}(h_j', D) \cap H_n(s_n^*)$.

From Cases 4.1 and 4.2 we conclude, for every $n \neq j$, that $s_n^* \in D_n(h_n)$ for every $h_n \in f_{jn}(h_j', D) \cap H_n(s_n^*)$. Moreover, as for all $n \neq j$ we have that $s_n^*|_{H_n^-(p')} = s_n|_{H_n^-(p')}$ and s_n selects all player n actions in p', we see that s_n^* selects all player n actions in p' as well. As this holds for all $n \neq j$ we conclude that $(s_n^*)_{n \neq j} \in S_{-j}(h_j')$. Altogether, we conclude that $(s_n^*)_{n \neq j} \in D_{-j}^{+*}(h_j', f)$. In the same was as above it can be shown that $D_{-j}^{+*}(h_j', f) = D_{-j}^+(h_j', f)$, and hence $(s_n^*)_{n \neq j} \in D_{-j}^+(h_j', f)$. We thus see that $s_j^* \in D_j^+(h_j', f)$ and $(s_n^*)_{n \neq j} \in D_{-j}^+(h_j', f)$ and hence $h_j' \in H_j^e(D, f)$.

From Cases 1–4 we thus conclude, for all j, that $h_j' \in H_j^e(D, f)$ for all $h_j' \in H(D_i^+(h_i, f) \times D_{-i}^+(h_i, f)) \cap H_j$. The proof of (a) is hereby complete.

(b) Take a collection of decision problems $D = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$, a player i, an information set $h_i \in H_i^e(D, f)$ and a predecessor $h_i' \in H_i^-(h_i)$. Take some $s_i \in D_i^+(h_i, f)$ and some $s_{-i} \in D_{-i}^+(h_i, f)$. Since $s_i \in S_i(h_i)$ and $s_{-i} \in S_{-i}(h_i)$, we conclude that (s_i, s_{-i}) reaches h_i. As h_i' precedes h_i and the game satisfies perfect recall, the strategy combination (s_i, s_{-i}) must reach h_i' as well. Recall that $s_i \in D_i^+(h_i, f)$ and $s_{-i} \in D_{-i}^+(h_i, f)$, which implies that $h_i' \in H(D_i^+(h_i, f) \times D_{-i}^+(h_i, f))$. By (a) we then conclude that $h_i' \in H_i^e(D, f)$. As such, $H_i^e(D, f)$ is closed under predecessors.

(c) Suppose that either $D = (rf^{k+1} \circ rg^m)(D^{full})$ and $E = (rf^k \circ rg^{m+1})(D^{full})$, or $D = (rf^k \circ rg^{m+1})(D^{full})$ and $E = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$, and let $D \subseteq_{\sigma,f,D'} E$ for some σ and D'.

Take a player i. To show that $H_i^e(D, f) \subseteq H_i^e(E, f)$ take some $h_i \in H_i^e(D, f)$. Then, $D_i^+(h_i, f) \neq \emptyset$ and $D_{-i}^+(h_i, f) \neq \emptyset$. Take some $s_i \in D_i^+(h_i, f)$ and some $(s_j)_{j \neq i} \in D_{-i}^+(h_i, f)$. We will show that $\sigma_i(s_i) \in E_i^+(h_i, f)$ and $(\sigma_j(s_j))_{j \neq i} \in E_{-i}^+(h_i, f)$, which would imply that $h_i \in H_i^e(E, f)$.

We start by showing that $\sigma_i(s_i) \in E_i^+(h_i, f)$. As $h_i \in H_i^e(D, f)$ and, by (b), $H_i^e(D, f)$ is closed under predecessors, it follows that $H_i^-(h_i) \subseteq H_i^e(D, f)$. Note that either $D = rf(D')$ or $D' = D^{full}$. In either case we have that $D \subseteq D'$, which implies that $H_i^e(D, f) \subseteq H_i^e(D', f)$, and hence $H_i^-(h_i) \subseteq H_i^e(D', f)$. By property (T1) of σ it then follows that $\sigma_i(s_i)|_{H_i^-(h_i)} = s_i|_{H_i^-(h_i)}$. Since $s_i \in S_i(h_i)$ we conclude that $\sigma_i(s_i) \in S_i(h_i)$ as well. We thus conclude that $h_i \in H_i^e(D, f) \cap H_i(\sigma_i(s_i))$. Since $H_i^e(D, f) \subseteq H_i^e(D', f)$ we have that $h_i \in H_i^e(D', f) \cap H_i(\sigma_i(s_i))$. As $s_i \in D_i^+(h_i, f)$, it follows by property (T2) of σ that $\sigma_i(s_i) \in E_i^+(h_i, f)$.

We next show that $(\sigma_j(s_j))_{j \neq i} \in E_{-i}^+(h_i, f)$. Recall that $(s_j)_{j \neq i} \in D_{-i}^+(h_i, f)$. By Lemma 8.9 we know that $D_{-i}^+(h_i, f) = D_{-i}^{++}(h_i, f)$, and thus $(s_j)_{j \neq i} \in D_{-i}^{++}(h_i, f)$. Hence, for every $j \neq i$ we have that $s_j \in D_j^+(h_j, f)$ for all $h_j \in f_{ij}(h_i, D) \cap H_j(s_j)$. By properties (T2) and (T3) of σ we then conclude that $\sigma_j(s_j) \in E_j(h_j)$ for all $h_j \in f_{ij}(h_i, D) \cap H_j(\sigma_j(s_j))$.

Recall that either $D = (rf^{k+1} \circ rg^m)(D^{full})$ and $E = (rf^k \circ rg^{m+1})(D^{full})$, or $D = (rf^k \circ rg^{m+1})(D^{full})$ and $E = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$. As f is monotone with respect to g we know that $f_{ij}(h_i, E) \subseteq f_{ij}(h_i, D)$, and hence

$$\sigma_j(s_j) \in E_j(h_j) \text{ for all } h_j \in f_{ij}(h_i, E) \cap H_j(\sigma_j(s_j)). \tag{8.14}$$

To show that $(\sigma_j(s_j))_{j \neq i} \in E_{-i}^{+*}(h_i, f)$ we need to verify that $(\sigma_j(s_j))_{j \neq i} \in S_{-i}(h_i)$. Take some $h_j \in H_j(\sigma_j(s_j))$ preceding h_i such that $(s_i, (s_n)_{n \neq i})$ reaches h_j. As $s_i \in D_i^+(h_i, f)$ and $(s_n)_{n \neq i} \in D_{-i}^+(h_i, f)$ we conclude that $h_j \in H(D_i^+(h_i, f) \times D_{-i}^+(h_i, f)) \cap H_j$. As $h_i \in H_i^e(D, f)$ it follows from (a) that $h_j \in H_j^e(D, f)$. Moreover, as $D \subseteq D'$ we have that $H_j^e(D, f) \subseteq H_j^e(D', f)$, and hence it follows that $h_j \in H_j^e(D', f)$. By property (T1) of σ it follows that for every $h_j \in H_j(\sigma_j(s_j))$ preceding h_i such that $(s_i, (s_n)_{n \neq i})$ reaches h_j we have that $(\sigma_j(s_j))(h_j) = s_j(h_j)$. Since this holds for every $j \neq i$, and since $(s_i, (s_j)_{j \neq i})$ reaches h_i, we conclude that $(s_i, (\sigma_j(s_j))_{j \neq i})$ reaches h_i as well. As such, $(\sigma_j(s_j))_{j \neq i} \in S_{-i}(h_i)$. Together with (8.14) it follows that $(\sigma_j(s_j))_{j \neq i} \in E_{-i}^{+*}(h_i, f)$. Recall that either $E = (rf^k \circ rg^{m+1})(D^{full})$ or $E = (rf^k \circ rg^m)(D^{full})$. In the same way as in the proof of (a) it can be shown that $E_{-i}^{+*}(h_i, f) = E_{-i}^+(h_i, f)$, and hence $(\sigma_j(s_j))_{j \neq i} \in E_{-i}^+(h_i, f)$. We thus see that $\sigma_i(s_i) \in E_i^+(h_i, f)$ and $(\sigma_j(s_j))_{j \neq i} \in E_{-i}^+(h_i, f)$, which implies that $E_i^+(h_i, f) \neq \emptyset$ and $E_{-i}^+(h_i, f) \neq \emptyset$. As such, $h_i \in H_i^e(E, f)$. As this holds for every $h_i \in H_i^e(D, f)$ it follows that $H_i^e(D, f) \subseteq H_i^e(E, f)$, which completes the proof. ■

Consider two profiles of transformation mappings $\tau = (\tau_i)_{i \in I}$ and $\rho = (\rho_i)_{i \in I}$ where $\tau_i, \rho_i : S_i \to S_i$ for every player i. Then, $\rho \circ \tau$ denotes the profile of transformation mappings that defines for every player i the transformation mapping $\rho_i \circ \tau_i : S_i \to S_i$.

Lemma 8.11 (Transitivity of inclusion operator) *Consider two focus functions f, g with $g \subseteq f$ that are monotone, individually forward decreasing and individually preserving focus on past information sets. Assume moreover that f is collectively forward decreasing, collectively preserving focus on past information* *sets, transitively closed and monotone with respect to g. Consider three collections of decision problems D, E, F where $D = (rf^k \circ rg^m)(D^{full})$ for some $k, m \geq 0$ such that $D \subseteq_{\tau,f,D'} E \subseteq_{\rho,f,E'} F$ for some τ, ρ and D', E' with $H_i^e(D', f) = H_i^e(E', f)$ for all players i. Then, $D \subseteq_{\rho\circ\tau,f,D'} F$.*

Proof. Let $\sigma := \rho \circ \tau$. Hence, $\sigma_i = \rho_i \circ \tau_i$ for every player i. In order to prove that $D \subseteq_{\sigma,f,D'} F$ we will now show that σ_i satisfies the properties (T1), (T2) and (T3).

(T1) As $D \subseteq_{\tau,f,D'} E \subseteq_{\rho,f,E'} F$ we have, for every player i and every strategy s_i, that $\tau_i(s_i)|_{H_i^e(D',f)} = s_i|_{H_i^e(D',f)}$ and $\rho_i(s_i)|_{H_i^e(E',f)} = s_i|_{H_i^e(E',f)}$. Since $H_i^e(D', f) = H_i^e(E', f)$, it follows that $(\rho_i \circ \tau_i)(s_i)|_{H_i^e(D',f)} = s_i|_{H_i^e(D',f)}$ for every strategy s_i, which implies property (T1) for σ_i.

(T2) Take some player i, some strategy s_i, and some $h_i \in H_i^e(D', f) \cap H_i(\sigma_i(s_i))$ with $s_i \in D_i^+(h_i)$. Since, by Lemma 8.10 (b), the collection $H_i^e(D', f)$ is closed under predecessors, we have that $H_i^-(h_i) \subseteq H_i^e(D', f)$. Since $D \subseteq_{\tau,f,D'} E \subseteq_{\rho,f,E'} F$ we conclude that $\tau_i(s_i)|_{H_i^e(D',f)} = s_i|_{H_i^e(D',f)}$ and $\rho_i(s_i)|_{H_i^e(E',f)} = s_i|_{H_i^e(E',f)}$. As $H_i^e(D', f) = H_i^e(E', f)$ it follows that $\sigma_i(s_i)|_{H_i^e(D',f)} = \tau_i(s_i)|_{H_i^e(D',f)}$ and hence $\sigma_i(s_i)|_{H_i^-(h_i)} = \tau_i(s_i)|_{H_i^-(h_i)}$. Since $h_i \in H_i(\sigma_i(s_i))$ we conclude that $h_i \in H_i(\tau_i(s_i))$ as well.

Thus, $h_i \in H_i^e(D', f) \cap H_i(\tau_i(s_i))$ and $s_i \in D_i^+(h_i)$. Since, by the assumption, $D \subseteq_{\tau,f,D'} E$, it follows from (T2) of τ that $\tau_i(s_i) \in E_i^+(h_i)$. Hence, we see that $h_i \in H_i^e(D', f) \cap H_i(\sigma_i(s_i))$ and $\tau_i(s_i) \in E_i^+(h_i)$. As $\sigma_i = \rho_i \circ \tau_i$ it follows that $h_i \in H_i^e(D', f) \cap H_i(\rho_i(\tau_i(s_i)))$ and $\tau_i(s_i) \in E_i^+(h_i)$. Recall that $H_i^e(D', f) = H_i^e(E', f)$, and hence $h_i \in H_i^e(E', f) \cap H_i(\rho_i(\tau_i(s_i)))$ and $\tau_i(s_i) \in E_i^+(h_i)$. Since, by the assumption, $E \subseteq_{\rho,f,E'} F$, it follows from (T2) of ρ that $\rho_i(\tau_i(s_i)) \in F_i^+(h_i)$. That is, $\sigma_i(s_i) \in F_i^+(h_i)$. As this holds for every strategy s_i, and every $h_i \in H_i^e(D', f) \cap H_i(\sigma_i(s_i))$ with $s_i \in D_i^+(h_i)$, property (T2) for σ_i follows.

(T3) Take some player i, some strategy s_i, and some $h_i \in H_i(\sigma_i(s_i)) \backslash H_i^e(D', f)$. Since $\sigma_i = \rho_i \circ \tau_i$ and $H_i^e(D', f) = H_i^e(E', f)$, we have that $h_i \in H_i(\rho_i(\tau_i(s_i))) \backslash H_i^e(E', f)$. As $E \subseteq_{\rho,f,E'} F$ it follows from (T3) of ρ that $\rho_i(\tau_i(s_i)) \in F_i(h_i)$. That is, $\sigma_i(s_i) \in F_i(h_i)$. This holds for every strategy s_i and every $h_i \in H_i(\sigma_i(s_i)) \backslash H_i^e(D', f)$, and as such (T3) holds.

Thus, (T1), (T2) and (T3) hold for σ, which implies that $D \subseteq_{\sigma,f,D'} F$. ■

Lemma 8.12 (Monotonicity on explicable information sets) *Consider two focus functions f, g with $g \subseteq f$ that are monotone, individually forward decreasing and individually preserving focus on past information sets. Assume moreover that f is collectively forward decreasing, collectively preserving focus on past information sets, transitively closed and monotone with respect to g. Suppose that $E = (rf^k \circ rg^m)$ for some $k, m \geq 0$, that D is a (g, f)-semi reduction of E, and let*

$$rf(E) \subseteq_{\tau,f,E} D \subseteq_{\rho,f,D'} E$$

for some τ, ρ and D'. Then,

$$rf(D) \subseteq_{\sigma,f,D} rf(E)$$

for some σ.

Proof. Set $D^{rf} := rf(D)$ and $E^{rf} := rf(E)$. Hence, we must show that $D^{rf} \subseteq_{\sigma,f,D} E^{rf}$ for some σ. We first show that E^{rf} is dynamically consistent. To see this, recall that $E = (rf^k \circ rg^m)(D^{full})$. Then, $E^{rf} = (rf^{k+1} \circ rg^m)(D^{full})$. Let $m^* \leq m$ be the smallest number such that $rg^{m^*}(D^{full}) = rg^m(D^{full})$. Then, $E^{rf} = (g^{\leq m}, f)^{m^*+k+1}(D^{full})$. As f, g are monotone and $g \subseteq f$ it follows by Lemma 8.8 that $(g^{\leq m}, f)$ is monotone as well. Moreover, $(g^{\leq m}, f)$ is individually forward decreasing and individually preserves focus on past information sets since it inherits these properties from g and f. By Lemma 8.6 it then follows that E^{rf} is dynamically consistent.

For every player i we construct the mapping $\sigma_i : S_i \to S_i$ as follows. Since E^{rf} is dynamically consistent, we know by Lemma 8.5 (b) that there is for every information set $h_i \in (H_i \backslash H_i^e(D, f))^{first}$ a strategy $s_i^E[h_i]$ such that $s_i^E[h_i] \in E_i^{rf*}(h_i')$ for all $h_i' \in H_i(s_i^E[h_i]) \backslash H_i^-(h_i)$. Hence, in particular, $s_i^E[h_i] \in E_i^{rf*}(h_i')$ for all $h_i' \in H_i(s_i^E[h_i]) \cap H_i^+(h_i)$. Moreover, as $E^{rf} = (g^{\leq m}, f)^{m^*+k+1}(D^{full})$ we know from Lemma 8.1 that $E_i^{rf*}(h_i') = E_i^{rf}(h_i')$ for all $h_i' \in H_i$. Thus, $s_i^E[h_i] \in E_i^{rf}(h_i')$ for all $h_i' \in H_i(s_i^E[h_i]) \cap H_i^+(h_i)$.

For every strategy $s_i \in S_i$ we define $\sigma_i(s_i)$ to be the unique strategy where

$$\sigma_i(s_i)|_{H_i^e(D,f)} = s_i|_{H_i^e(D,f)} \text{ and} \tag{8.15}$$

$$\sigma_i(s_i)|_{H_i^+(h_i)} = s_i^E[h_i]|_{H_i^+(h_i)} \text{ for all } h_i \in (H_i \backslash H_i^e(D, f))^{first}. \tag{8.16}$$

This construction is well-defined since, by Lemma 8.10 (b), the collection $H_i^e(D, f)$ is closed under predecessors, and hence the collection $H_i \backslash H_i^e(D, f)$ is closed under weak followers.

In view of (8.15) we conclude that σ satisfies property (T1). It remains to prove properties (T2) and (T3).

(T3) Take a strategy s_i and an information set $h_i^* \in H_i(\sigma_i(s_i)) \backslash H_i^e(D, f)$. We must show that $\sigma_i(s_i) \in E_i^{rf}(h_i^*)$. Let h_i be the information set in $(H_i \backslash H_i^e(D, f))^{first}$ that weakly precedes h_i^*. Then, by (8.16), $\sigma_i(s_i)|_{H_i^+(h_i)} = s_i^E[h_i]|_{H_i^+(h_i)}$. As h_i^* weakly follows h_i we know that $\sigma_i(s_i)|_{H_i^+(h_i^*)} = s_i^E[h_i]|_{H_i^+(h_i^*)}$. Moreover, by construction, $s_i^E[h_i] \in E_i^{rf}(h_i')$ for all $h_i' \in H_i(s_i^E[h_i]) \cap H_i^+(h_i)$, which implies, in particular, that $s_i^E[h_i] \in E_i^{rf}(h_i^*)$. As, by Lemma 8.1, $E_i^{rf}(h_i^*) = E_i^{rf*}(h_i^*)$ we know that $s_i^E[h_i] \in E_i^{rf*}(h_i^*)$. That is, $s_i^E[h_i]$ is not strictly dominated in $(S_i(h_i^*), E_{-i}^{rf}(h_i^*))$. By Lemma 2.1 we then know that $s_i^E[h_i]$ is optimal in $S_i(h_i^*)$ for some belief $b_i \in \Delta(E_{-i}^{rf}(h_i^*))$. Since the expected utility induced by $s_i^E[h_i]$ under the belief b_i only depends on the behavior of $s_i^E[h_i]$ on $H_i^+(h_i^*)$, and $\sigma_i(s_i)|_{H_i^+(h_i^*)} = s_i^E[h_i]|_{H_i^+(h_i^*)}$, we conclude that also $\sigma_i(s_i)$ is optimal in $S_i(h_i^*)$ for the belief b_i. As $b_i \in \Delta(E_{-i}^{rf}(h_i^*))$, it follows by Lemma 2.1 that $\sigma_i(s_i)$ is not strictly dominated in $(S_i(h_i^*), E_{-i}^{rf}(h_i^*))$. That is, $\sigma_i(s_i) \in E_i^{rf*}(h_i^*)$. Since, by Lemma 8.1, $E_i^{rf}(h_i^*) = E_i^{rf*}(h_i^*)$, it follows that $\sigma_i(s_i) \in E_i^{rf}(h_i^*)$. As this holds for all strategies s_i and all information sets $h_i^* \in H_i(\sigma_i(s_i)) \backslash H_i^e(D, f)$, we see that (T3) holds.

(T2) Take a strategy s_i and some $h_i^* \in H_i^e(D, f) \cap H_i(\sigma_i(s_i))$ with $s_i \in D_i^{rf+}(h_i^*, f)$. We must show that $\sigma_i(s_i) \in E_i^{rf+}(h_i^*, f)$. Assume, on the contrary, that $\sigma_i(s_i) \notin E_i^{rf+}(h_i^*, f)$. Then, $\sigma_i(s_i) \notin E_i^{rf}(h_i)$ for some $h_i \in f_{ii}(h_i^*, E^{rf}) \cap H_i(\sigma_i(s_i))$. Let $h_i \in f_{ii}(h_i^*, E^{rf}) \cap H_i(\sigma_i(s_i))$ be such $\sigma_i(s_i) \notin E_i^{rf}(h_i)$, and $\sigma_i(s_i) \in E_i^{rf}(h_i')$ for all $h_i' \in f_{ii}(h_i^*, E^{rf})$ preceding h_i. As $\sigma_i(s_i) \notin E_i^{rf}(h_i)$ it follows by (T3) that $h_i \in H_i^e(D, f)$.

We will show that $s_i \in D_i^{rf+}(h_i, f)$. To prove this, we first show that $f_{ii}(h_i^*, E^{rf}) \subseteq f_{ii}(h_i^*, D^{rf})$. Take some $h_i' \in f_{ii}(h_i^*, E^{rf})$. Then, there is some $j \neq i$ and some $h_j \in f_{ij}(h_i^*, E^{rf})$ such that $h_i' \in f_{ji}(h_j, E^{rf})$. Recall that $E = (rf^k \circ rg^m)(D^{full})$ and that D is a (g, f)-semi reduction of E. It may be verified that D^{rf} is then a (g, f)-semi reduction of E^{rf}. Since f is monotone with respect to g we know that $f_{ij}(h_i^*, E^{rf}) \subseteq f_{ij}(h_i^*, D^{rf})$ and $f_{ji}(h_j, E^{rf}) \subseteq f_{ji}(h_j, D^{rf})$. Thus, $h_j \in f_{ij}(h_i^*, D^{rf})$ and $h_i' \in f_{ji}(h_j, D^{rf})$, which implies that $h_i' \in f_{ii}(h_i^*, D^{rf})$. Thus, $f_{ii}(h_i^*, E^{rf}) \subseteq f_{ii}(h_i^*, D^{rf})$. In particular, since $h_i \in f_{ii}(h_i^*, E^{rf})$, we conclude that $h_i \in f_{ii}(h_i^*, D^{rf})$.

To show that $s_i \in D_i^{rf+}(h_i, f)$, take some $h_i' \in f_{ii}(h_i, D^{rf}) \cap H_i(s_i)$. Hence, either (i) $h_i' = h_i$ or (ii) there is some player $j \neq i$ and some $h_j \in f_{ij}(h_i, D^{rf})$ such that $h_i' \in f_{ji}(h_j, D^{rf})$. If $h_i' = h_i$ we see that $s_i \in D_i^{rf}(h_i')$, since $s_i \in D_i^{rf+}(h_i^*, f)$ and $h_i \in f_{ii}(h_i^*, D^{rf})$. Now, suppose that $h_i' \neq h_i$. Then, by (ii) and the fact that $h_i \in f_{ii}(h_i^*, D^{rf})$ we conclude that h_j is reachable from h_i^* under D^{rf} and f. Since f is transitively closed we know that $h_j \in f_{ij}(h_i^*, D^{rf})$. Since $h_i' \in f_{ji}(h_j, D^{rf})$ we conclude that $h_i' \in f_{ii}(h_i^*, D^{rf})$. From the fact that $s_i \in D_i^{rf+}(h_i^*, f)$ it follows that $s_i \in D_i^{rf}(h_i')$. As this holds for every $h_i' \in f_{ii}(h_i, D^{rf}) \cap H_i(s_i)$ we conclude that $s_i \in D_i^{rf+}(h_i, f)$.

As $h_i \in f_{ii}(h_i, D^{rf})$ it follows that $s_i \in D_i^{rf}(h_i)$. Thus, by definition, $s_i \in D_i(h_i)$ and s_i is not strictly dominated in $(D_i(h_i), D_{-i}^{rf}(h_i))$. As $h_i \in H_i^e(D, f)$ we know that $D_{-i}^+(h_i, f) \neq \emptyset$ and thus, by definition, $D_{-i}^{rf}(h_i) = D_{-i}^+(h_i, f)$. Moreover, by Lemma 8.9 we know that $D_{-i}^+(h_i, f) = D_{-i}^{++}(h_i, f)$, and hence s_i is not strictly dominated in $(D_i(h_i), D_{-i}^{++}(h_i, f))$. Thus, by Lemma 2.1 there is a belief $b_i^D \in \Delta(D_{-i}^{++}(h_i, f))$ such that s_i is optimal in $D_i(h_i)$ for b_i^D.

Recall that that $D \subseteq_{\rho, f, D'} E$ for some ρ and D', where $\rho = (\rho_i)_{i \in I}$. For every opponents' strategy combination $s_{-i} = (s_j)_{j \neq i}$ we define $\rho_{-i}(s_{-i}) := (\rho_j(s_j))_{j \neq i}$. Define the belief b_i^E by

$$b_i^E(s_{-i}) := \sum_{s'_{-i} \in S_{-i} : \rho_{-i}(s'_{-i}) = s_{-i}} b_i^D(s'_{-i}). \tag{8.17}$$

We show that $b_i^E \in \Delta(E_{-i}^+(h_i, f))$, and that $\sigma_i(s_i)$ is optimal in $S_i(h_i)$ for b_i^E.

To show that $b_i^E \in \Delta(E_{-i}^+(h_i, f))$, take some $s_{-i} \in S_{-i}$ with $b_i^E(s_{-i}) > 0$. By (8.17) there is some $s'_{-i} \in S_{-i}$ with $\rho_{-i}(s'_{-i}) = s_{-i}$ such that $b_i^D(s'_{-i}) > 0$. Since $b_i^D \in \Delta(D_{-i}^{++}(h_i, f))$ it must be that $s'_{-i} \in D_{-i}^{++}(h_i, f)$. Let $s'_{-i} = (s'_j)_{j \neq i}$. As $s'_{-i} \in D_{-i}^{++}(h_i, f)$ we have, for every $j \neq i$, that $s'_j \in D_j^+(h_j)$ for all $h_j \in f_{ij}(h_i, D) \cap H_j(s'_j)$. By properties (T2) and (T3) of ρ it follows that $\rho_j(s'_j) \in E_j(h_j)$ for all $h_j \in f_{ij}(h_i, D) \cap H_j(\rho_j(s'_j))$.

Recall that D is a (g, f)-semi reduction of E. Since f is monotone with respect to g we have that $f_{ij}(h_i, E) \subseteq f_{ij}(h_i, D)$, and hence we conclude, for all $j \neq i$, that

$$\rho_j(s'_j) \in E_j(h_j) \text{ for all } h_j \in f_{ij}(h_i, E) \cap H_j(\rho_j(s'_j)). \tag{8.18}$$

As $(s'_j)_{j \neq i} \in S_{-i}(h_i)$ and $h_i \in H_i^e(D, f) \subseteq H_i^e(D', f)$ it can be shown, on the basis of (T1) for ρ, that $(\rho_j(s'_j))_{j \neq i} \in S_{-i}(h_i)$. Together with (8.18) we conclude that $(\rho_j(s'_j))_{j \neq i} \in E_{-i}^{+*}(h_i, f)$. Recall that $E = r(g^{\leq m}, f)^{m^* + k}(D^{full})$ for some m^* and that $(g^{\leq m}, f)$ is monotone. By Lemma 8.2 we then know that

$$E_{-i}^{+*}(h_i, f) = E_{-i}^{+*}(h_i, (g^{\leq m}, f)) = E_{-i}^{+}(h_i, (g^{\leq m}, f)) = E_{-i}^{+}(h_i, f),$$

and hence $(\rho_j(s'_j))_{j\neq i} \in E^+_{-i}(h_i, f)$. Since $s_{-i} = \rho_{-i}(s'_{-i}) = (\rho_j(s'_j))_{j\neq i}$ it follows that $s_{-i} \in E^+_{-i}(h_i, f)$. As this holds for every $s_{-i} \in S_{-i}$ with $b_i^E(s_{-i}) > 0$, we see that $b_i^E \in \Delta(E^+_{-i}(h_i, f))$.

Recall that $\sigma_i(s_i) \notin E_i^{rf}(h_i)$. By Lemma 8.1 we know that $E_i^{rf}(h_i) = E_i^{rf*}(h_i)$, and hence $\sigma_i(s_i) \notin E_i^{rf*}(h_i)$. As $h_i \in H_i(\sigma_i(s_i))$ it follows that $\sigma_i(s_i) \in S_i(h_i)$. Since $\sigma_i(s_i) \notin E_i^{rf*}(h_i)$ it must be that $\sigma_i(s_i)$ is strictly dominated in $(S_i(h_i), E^{rf}_{-i}(h_i))$. As we have seen above that $b_i^E \in \Delta(E^+_{-i}(h_i, f))$ it follows in particular that $E^+_{-i}(h_i, f) \neq \emptyset$, and hence $E^{rf}_{-i}(h_i) = E^+_{-i}(h_i, f)$. Thus, $\sigma_i(s_i)$ is strictly dominated in $(S_i(h_i), E^+_{-i}(h_i, f))$. By Lemma 2.1 we then know that $\sigma_i(s_i)$ is not optimal in $S_i(h_i)$ for any belief $b_i \in \Delta(E^+_{-i}(h_i, f))$. As we have seen that $b_i^E \in \Delta(E^+_{-i}(h_i, f))$, we conclude that $\sigma_i(s_i)$ is not optimal in $S_i(h_i)$ for b_i^E.

Recall that $\sigma_i(s_i) \in E_i^{rf}(h'_i)$ for all $h'_i \in f_{ii}(h_i^*, E^{rf})$ preceding h_i. As $h_i \in f_{ii}(h_i^*, E^{rf})$ we know that h_i is reachable from h_i^* under E^{rf} and f. Hence, every $h'_i \in f_{ii}(h_i, E^{rf})$ is reachable from h_i^* under E^{rf} and f. Since f is transitively closed it follows that $f_{ii}(h_i, E^{rf}) \subseteq f_{ii}(h_i^*, E^{rf})$. As such, we conclude that $\sigma_i(s_i) \in E_i^{rf}(h'_i)$ for all $h'_i \in f_{ii}(h_i, E^{rf})$ preceding h_i.

By Lemma 8.1 we know that $E_i^{rf}(h'_i) = E_i^{rf*}(h'_i)$ for all $h'_i \in H_i$, and hence $\sigma_i(s_i) \in E_i^{rf*}(h'_i)$ for all $h'_i \in f_{ii}(h_i, E^{rf})$ preceding h_i. Set $H_i^* := f_{ii}(h_i, E^{rf}) \cap H_i^-(h_i)$ and choose some history $p \in h_i$. Then, by perfect recall, $H_i^-(h_i) = H_i^-(p)$, which implies that $H_i^* \subseteq H_i^-(p)$, and $\sigma_i(s_i) \in E_i^{rf*}(h'_i)$ for all $h'_i \in H_i^*$. Recall that E^{rf} is dynamically consistent, and that $b_i^E \in \Delta(E^+_{-i}(h_i, f)) = \Delta(E^{rf}_{-i}(h_i))$. By Lemma 8.4 there is thus a strategy s_i^* such that $s_i^*|_{H_i^-(p)} = \sigma_i(s_i)|_{H_i^-(p)}$ and $s_i^* \in E_i^{rf*}(h'_i)$ for all $h'_i \in H_i^* \cup (H_i(s_i^*)\backslash H_i^-(p))$, and such that s_i^* is optimal for b_i^E on $S_i(h_i)$. By the definition of H_i^* we thus conclude that

$$s_i^* \in E_i^{rf*}(h'_i) \text{ for all } h'_i \in f_{ii}(h_i, E^{rf}) \cap H_i(s_i^*). \tag{8.19}$$

As, by Lemma 8.1, $E_i^{rf*}(h'_i) = E_i^{rf}(h'_i)$ for all h'_i, it follows that $s_i^* \in E_i^{rf}(h'_i)$ for all $h'_i \in f_{ii}(h_i, E^{rf}) \cap H_i(s_i^*)$. Moreover, since $h_i \in H_i(\sigma_i(s_i))$ we know that $\sigma_i(s_i)$ selects all player i actions in p. As $s_i^*|_{H_i^-(p)} = \sigma_i(s_i)|_{H_i^-(p)}$ we see that s_i^* selects all player i actions in p as well, which implies that $s_i^* \in S_i(h_i)$. Together with (8.19) we conclude that $s_i^* \in E_i^{rf+}(h_i, f)$.

We have thus found a strategy $s_i^* \in E_i^{rf+}(h_i, f)$ that is optimal in $S_i(h_i)$ for b_i^E. As $\sigma_i(s_i)$ is not optimal in $S_i(h_i)$ for b_i^E we conclude that

$$u_i(\sigma_i(s_i), b_i^E) < u_i(s_i^*, b_i^E). \tag{8.20}$$

Recall that $D \subseteq_{\rho,f,D'} E$. By Lemma 8.10 (c) it then follows that $H_i^e(D, f) \subseteq H_i^e(E, f)$. Since we have seen above that $h_i \in H_i^e(D, f)$ we conclude that $h_i \in H_i^e(E, f)$. Recall also that $E^{rf} \subseteq_{\tau,f,E} D$. As $h_i \in H_i^e(E, f)$ and $s_i^* \in E_i^{rf+}(h_i, f)$ it follows by property (T2) of τ that

$$\tau_i(s_i^*) \in D_i^+(h_i, f). \tag{8.21}$$

We next show that

$$u_i(s_i^*, b_i^E) = u_i(\tau_i(s_i^*), b_i^D). \tag{8.22}$$

By definition,

$$\begin{aligned} u_i(s_i^*, b_i^E) &= \sum_{s_{-i} \in S_{-i}} b_i^E(s_{-i}) \cdot u_i(z(s_i^*, s_{-i})) \\ &= \sum_{s_{-i} \in S_{-i}} \sum_{s'_{-i} \in S_{-i} : \rho_{-i}(s'_{-i}) = s_{-i}} b_i^D(s'_{-i}) \cdot u_i(z(s_i^*, s_{-i})), \end{aligned} \tag{8.23}$$

where the second equality follows from the definition of b_i^E in (8.17). Take some $s_{-i}, s'_{-i} \in S_{-i}$ with $\rho_{-i}(s'_{-i}) = s_{-i}$ and $b_i^D(s'_{-i}) > 0$. Since $b_i^D \in \Delta(D_{-i}^{++}(h_i, f))$ we know that $s'_{-i} \in D_{-i}^{++}(h_i, f)$. As, by definition, $D_{-i}^{++}(h_i, f) \subseteq D_{-i}^{+}(h_i, f)$ we conclude that $s'_{-i} \in D_{-i}^{+}(h_i, f)$. Moreover, we have seen in (8.21) that $\tau_i(s_i^*) \in D_i^+(h_i, f)$. Hence, $(\tau_i(s_i^*), s'_{-i}) \in D_i^+(h_i, f) \times D_{-i}^+(h_i, f)$. As $h_i \in H_i^e(D, f)$ it follows from Lemma 8.10 (a) that $H(D_i^+(h_i, f) \times D_{-i}^+(h_i, f)) \cap H_j \subseteq H_j^e(D, f)$ for every player j, and thus

$$(\tau_i(s_i^*), s'_{-i}) \text{ only reaches information sets in } H_j^e(D, f) \text{ for every player } j. \tag{8.24}$$

As $E^{rf} \subseteq_{\tau, f, E} D$ we know by property (T1) of τ that $\tau_i(s_i^*)|_{H_i^e(E,f)} = s_i^*|_{H_i^e(E,f)}$. Since $D \subseteq_{\rho, f, D'} E$ for some D' it follows from Lemma 8.10 (c) that $H_i^e(D, f) \subseteq H_i^e(E, f)$. Hence, we conclude that

$$\tau_i(s_i^*)|_{H_i^e(D,f)} = s_i^*|_{H_i^e(D,f)}. \tag{8.25}$$

Recall that $D \subseteq_{\rho, f, D'} E$. Let $s'_{-i} = (s'_j)_{j \neq i}$. By property (T1) of ρ it follows that $\rho_j(s'_j)|_{H_i^e(D',f)} = s'_j|_{H_i^e(D',f)}$ for every $j \neq i$. Recall that $D = rf(D')$ or $D' = D^{full}$. In either case we have that $D \subseteq D'$, which implies that $H_i^e(D, f) \subseteq H_i^e(D', f)$, and hence

$$\rho_j(s'_j)|_{H_i^e(D,f)} = s'_j|_{H_i^e(D,f)} \text{ for every } j \neq i. \tag{8.26}$$

By combining (8.24), (8.25) and (8.26), we see that

$$z(\tau_i(s_i^*), s'_{-i}) = z(s_i^*, s_{-i}) \text{ whenever } \rho_{-i}(s'_{-i}) = s_{-i} \text{ and } b_i^D(s'_{-i}) > 0. \tag{8.27}$$

By combining (8.23) and (8.27) we get

$$\begin{aligned} u_i(s_i^*, b_i^E) &= \sum_{s_{-i} \in S_{-i}} \sum_{s'_{-i} \in S_{-i} : \rho_{-i}(s'_{-i}) = s_{-i}} b_i^D(s'_{-i}) \cdot u_i(z(\tau_i(s_i^*), s'_{-i})) \\ &= \sum_{s'_{-i} \in S_{-i}} b_i^D(s'_{-i}) \cdot u_i(z(\tau_i(s_i^*), s'_{-i})) = u_i(\tau_i(s_i^*), b_i^D), \end{aligned}$$

which was to show.

We next prove that

$$u_i(\sigma_i(s_i), b_i^E) = u_i(s_i, b_i^D). \tag{8.28}$$

By definition,

$$\begin{aligned} u_i(\sigma_i(s_i), b_i^E) &= \sum_{s_{-i} \in S_{-i}} b_i^E(s_{-i}) \cdot u_i(z(\sigma_i(s_i), s_{-i})) \\ &= \sum_{s_{-i} \in S_{-i}} \sum_{s'_{-i} \in S_{-i} : \rho_{-i}(s'_{-i}) = s_{-i}} b_i^D(s'_{-i}) \cdot u_i(z(\sigma_i(s_i), s_{-i})), \end{aligned} \tag{8.29}$$

where the second equality follows from the definition of b_i^E in (8.17). Take some $s_{-i}, s'_{-i} \in S_{-i}$ with $\rho_{-i}(s'_{-i}) = s_{-i}$ and $b_i^D(s'_{-i}) > 0$. In the same way as above we then conclude that $s'_{-i} \in D_{-i}^+(h_i, f)$. Moreover, we have seen above that $s_i \in D_i^{rf+}(h_i, f)$. As $D^{rf} = rf(D) \subseteq D$ it then follows that $s_i \in D_i^+(h_i, f)$. Hence, $(s_i, s'_{-i}) \in D_i^+(h_i, f) \times D_{-i}^+(h_i, f)$. As $h_i \in H_i^e(D, f)$ it follows from Lemma 8.10 (a) that $H(D_i^+(h_i, f) \times D_{-i}^+(h_i, f)) \cap H_j \subseteq H_j^e(D, f)$ for all players j, and thus

$$(s_i, s'_{-i}) \text{ only reaches information sets in } H_j^e(D, f) \text{ for all players } j. \tag{8.30}$$

By (8.15) we know that

$$\sigma_i(s_i)|_{H_i^e(D,f)} = s_i|_{H_i^e(D,f)}. \tag{8.31}$$

Let $s'_{-i} = (s'_j)_{j \neq i}$. In the same way as above we conclude that

$$\rho_j(s'_j)|_{H_i^e(D,f)} = s'_j|_{H_i^e(D,f)} \text{ for every } j \neq i. \tag{8.32}$$

By combining (8.30), (8.31) and (8.32), we see that

$$z(s_i, s'_{-i}) = z(\sigma_i(s_i), s_{-i}) \text{ whenever } \rho_{-i}(s'_{-i}) = s_{-i} \text{ and } b_i^D(s'_{-i}) > 0. \tag{8.33}$$

By combining (8.29) and (8.33) we get

$$\begin{aligned} u_i(\sigma_i(s_i), b_i^E) &= \sum_{s_{-i} \in S_{-i}} \sum_{s'_{-i} \in S_{-i} : \rho_{-i}(s'_{-i}) = s_{-i}} b_i^D(s'_{-i}) \cdot u_i(z(s_i, s'_{-i})) \\ &= \sum_{s'_{-i} \in S_{-i}} b_i^D(s'_{-i}) \cdot u_i(z(s_i, s'_{-i})) = u_i(s_i, b_i^D), \end{aligned}$$

which was to show.

By combining (8.20), (8.22) and (8.28) we conclude that

$$u_i(s_i, b_i^D) = u_i(\sigma_i(s_i), b_i^E) < u_i(s_i^*, b_i^E) = u_i(\tau_i(s_i^*), b_i^D).$$

However, since we know by (8.21) that $\tau_i(s_i^*) \in D_i^+(h_i, f)$ and $D_i^+(h_i, f) \subseteq D_i(h_i)$, this contradicts the assumption that s_i is optimal in $D_i(h_i)$ for the belief b_i^D. Hence, we must conclude that $\sigma_i(s_i) \in E_i^{rf+}(h_i^*, f)$.

Since this holds for every player i, every strategy s_i, and every $h_i^* \in H_i^e(D, f) \cap H_i(\sigma_i(s_i))$ with $s_i \in D_i^{rf+}(h_i^*, f)$, we see that (T2) is satisfied. As such, (T1), (T2) and (T3) hold, which implies that $D^{rf} \subseteq_{\sigma,f,D} E^{rf}$. This completes the proof. ■

Lemma 8.13 (Sandwich lemma) *Consider two focus functions f, g with $g \subseteq f$ that are monotone, individually forward decreasing and individually preserving focus on past information sets. Assume moreover that f is collectively forward decreasing, collectively preserving focus on past information sets, transitively closed and monotone with respect to g. For a given $m \geq 0$ let $(E^k)_{k=0}^{\infty}$ be the f-rationalizability procedure that starts at $rg^m(D^{full})$, and $(D^k)_{k=0}^{\infty}$ be the f-rationalizability procedure that starts at $rg^{m+1}(D^{full})$. Then,*

$$E^1 \subseteq_{\tau,f,E^0} D^0 \subseteq_{\rho,f,D^{full}} E^0$$

for some τ and ρ, and for every $k \geq 1$ we have that

$$D^k \subseteq_{\sigma,f,D^{k-1}} E^k \text{ and } E^{k+1} \subseteq_{\tilde{\sigma},f,E^k} D^k$$

for some σ and $\tilde{\sigma}$.

Proof. Fix the number m.

(a) We start by showing that $D^0 \subseteq_{\rho,f,D^{full}} E^0$ for some ρ. Note that $D^0 = rg(E^0) \subseteq E^0$, which implies that $D_i^{0+}(h_i, f) \subseteq E_i^{0+}(h_i, f)$ for every player i and information set $h_i \in H_i$. We can thus use, for every player i, the identity mapping $\rho_i : S_i \to S_i$ with $\rho_i(s_i) := s_i$ for every strategy s_i. Then, for every strategy $s_i \in S_i$ we have that properties (T1), (T2) and (T3) hold. In fact, (T3) is trivially satisfied since $H_i^e(D^{full}, f) = H_i$, and hence $H_i(\sigma_i(s_i)) \backslash H_i^e(D^{full}, f)$ is always empty. Thus, $D^0 \subseteq_{\rho,f,D^{full}} E^0$, which was to show.

(b) We next show that $E^1 \subseteq_{\tau,f,E^0} D^0$ for some τ. Recall that $E^1 = rf(E^0)$ and $D^0 = rg(E^0)$. Set $E := E^0$, $E^{rf} := rf(E)$ and $E^{rg} := rg(E)$. Hence, we must show that $E^{rf} \subseteq_{\tau,f,E} E^{rg}$.

We first show that E^{rg} is dynamically consistent. To see this, recall that $E^0 = rg^m(D^{full})$, and hence $E^{rg} = rg(E^0) = rg^{m+1}(D^{full})$. As g is monotone, individually forward decreasing and individually preserves focus on past information sets, it follows from Lemma 8.6 that E^{rg} is dynamically consistent.

For every player i we construct the mapping $\tau_i : S_i \to S_i$ as follows. Since E^{rg} is dynamically consistent we know by Lemma 8.5 (b) that there is for every information set $h_i \in (H_i \backslash H_i^e(E, f))^{first}$ a strategy $s_i[h_i]$ such that $s_i[h_i] \in E_i^{rg*}(h_i')$ for all $h_i' \in H_i(s_i[h_i]) \backslash H_i^-(h_i)$. Hence, in particular, $s_i[h_i] \in E_i^{rg*}(h_i')$ for all $h_i' \in H_i(s_i[h_i]) \cap H_i^+(h_i)$. Moreover, we know from Lemma 8.1 that $E_i^{rg*}(h_i') = E_i^{rg}(h_i')$ for all $h_i' \in H_i$. Thus, $s_i[h_i] \in E_i^{rg}(h_i')$ for all $h_i' \in H_i(s_i[h_i]) \cap H_i^+(h_i)$.

For every strategy $s_i \in S_i$ we define $\tau_i(s_i)$ to be the unique strategy where

$$\tau_i(s_i)|_{H_i^e(E,f)} = s_i|_{H_i^e(E,f)} \text{ and} \tag{8.34}$$

$$\tau_i(s_i)|_{H_i^+(h_i)} = s_i[h_i]|_{H_i^+(h_i)} \text{ for all } h_i \in (H_i \backslash H_i^e(E, f))^{first}. \tag{8.35}$$

This construction is well-defined since, by Lemma 8.10 (b), the collection $H_i^e(E, f)$ is closed under predecessors, and hence the collection $H_i \backslash H_i^e(E, f)$ is closed under weak followers.

In view of (8.34) we conclude that τ satisfies property (T1). It remains to prove properties (T2) and (T3).

(T3) Take a strategy s_i and an information set $h_i^* \in H_i(\tau_i(s_i))\backslash H_i^e(E,f)$. We must show that $\tau_i(s_i) \in E_i^{rg}(h_i^*)$. Let h_i be the information set in $(H_i\backslash H_i^e(E,f))^{first}$ that weakly precedes h_i^*. Then, by (8.35), $\tau_i(s_i)|_{H_i^+(h_i)} = s_i[h_i]|_{H_i^+(h_i)}$. As h_i^* weakly follows h_i we know that $\tau_i(s_i)|_{H_i^+(h_i^*)} = s_i[h_i]|_{H_i^+(h_i^*)}$. Moreover, by construction, $s_i[h_i] \in E_i^{rg}(h_i')$ for all $h_i' \in H_i(s_i[h_i]) \cap H_i^+(h_i)$. Recall that $h_i^* \in H_i(\tau_i(s_i)) \cap H_i^+(h_i)$. As $\tau_i(s_i)|_{H_i^+(h_i)} = s_i[h_i]|_{H_i^+(h_i)}$ we conclude that $h_i^* \in H_i(s_i[h_i]) \cap H_i^+(h_i)$ as well, and hence $s_i[h_i] \in E_i^{rg}(h_i^*)$. As, by Lemma 8.1, $E_i^{rg}(h_i^*) = E_i^{rg*}(h_i^*)$ we know that $s_i[h_i] \in E_i^{rg*}(h_i^*)$. That is, $s_i[h_i]$ is not strictly dominated in $(S_i(h_i^*), E_{-i}^{rg}(h_i^*))$. By Lemma 2.1 we then know that $s_i[h_i]$ is optimal in $S_i(h_i^*)$ for some belief $b_i \in \Delta(E_{-i}^{rg}(h_i^*))$. Since the expected utility induced by $s_i[h_i]$ under the belief b_i only depends on the behavior of $s_i[h_i]$ on $H_i^+(h_i^*)$, and $\tau_i(s_i)|_{H_i^+(h_i^*)} = s_i[h_i]|_{H_i^+(h_i^*)}$, we conclude that also $\tau_i(s_i)$ is optimal in $S_i(h_i^*)$ for the belief b_i. As $b_i \in \Delta(E_{-i}^{rg}(h_i^*))$ it follows by Lemma 2.1 that $\tau_i(s_i)$ is not strictly dominated in $(S_i(h_i^*), E_{-i}^{rg}(h_i^*))$. That is, $\tau_i(s_i) \in E_i^{rg*}(h_i^*)$. Since, by Lemma 8.1, $E_i^{rg}(h_i^*) = E_i^{rg*}(h_i^*)$ it follows that $\tau_i(s_i) \in E_i^{rg}(h_i^*)$. As this holds for all strategies s_i and all information sets $h_i^* \in H_i(\tau_i(s_i))\backslash H_i^e(E,f)$, we see that (T3) holds.

(T2) Take some strategy s_i and some information set $h_i^* \in H_i^e(E,f) \cap H_i(\tau_i(s_i))$ with $s_i \in E_i^{rf+}(h_i^*, f)$. We must show that $\tau_i(s_i) \in E_i^{rg+}(h_i^*, f)$. Assume, on the contrary, that $\tau_i(s_i) \notin E_i^{rg+}(h_i^*, f)$. Then, $\tau_i(s_i) \notin E_i^{rg}(h_i)$ for some $h_i \in f_{ii}(h_i^*, E^{rg}) \cap H_i(\tau_i(s_i))$. It follows by (T3) that $h_i \in H_i^e(E,f)$.

We will now show that $s_i \in E_i^{rf+}(h_i, f)$. Recall that $E^{rg} = rg(E)$ and $E^{rf} = rf(E)$. Then, E^{rf} is a (g,f)-semi reduction of E^{rg}. As f is monotone with respect to g it follows that $f_{ii}(h_i^*, E^{rg}) \subseteq f_{ii}(h_i^*, E^{rf})$. Since $h_i \in f_{ii}(h_i^*, E^{rg})$ we conclude that $h_i \in f_{ii}(h_i^*, E^{rf})$. To show that $s_i \in E_i^{rf+}(h_i, f)$ take some $h_i' \in f_{ii}(h_i, E^{rf}) \cap H_i(s_i)$. Hence, either (i) $h_i' = h_i$, or (ii) there is some player $j \neq i$ and some $h_j \in f_{ij}(h_i, E^{rf})$ such that $h_i' \in f_{ji}(h_j, E^{rf})$. If $h_i' = h_i$ then $s_i \in E_i^{rf}(h_i')$ since $s_i \in E_i^{rf+}(h_i^*, f)$ and $h_i \in f_{ii}(h_i^*, E^{rf})$. If $h_i' \neq h_i$ then we conclude, from (ii) and the fact that $h_i \in f_{ii}(h_i^*, E^{rf})$, that h_j is reachable from h_i^* under E^{rf} and f. Since f is transitively closed we know that $h_j \in f_{ij}(h_i^*, E^{rf})$. As such, $h_i' \in f_{ii}(h_i^*, E^{rf})$. From the fact that $s_i \in E_i^{rf+}(h_i^*, f)$ it follows that $s_i \in E_i^{rf}(h_i')$. As this holds for every $h_i' \in f_{ii}(h_i, E^{rf}) \cap H_i(s_i)$ we conclude that $s_i \in E_i^{rf+}(h_i, f)$.

As $E_i^{rf+}(h_i, f) \subseteq E_i^{rf}(h_i)$ we see, in particular, that $s_i \in E_i^{rf}(h_i)$. By Lemma 8.1 we know that $E_i^{rf}(h_i) = E_i^{rf*}(h_i)$, and hence $s_i \in E_i^{rf*}(h_i)$. Thus, by definition, s_i is not strictly dominated in $(S_i(h_i), E_{-i}^{rf}(h_i))$. By Lemma 2.1 there is thus a belief $b_i \in \Delta(E_{-i}^{rf}(h_i))$ such that s_i is optimal in $S_i(h_i)$ for b_i.

We now show that $b_i \in \Delta(E_{-i}^{rg}(h_i))$. To show this we prove that $E_{-i}^{rf}(h_i) \subseteq E_{-i}^{rg}(h_i)$. Take some $(s_j)_{j\neq i} \in E_{-i}^{rf}(h_i)$. As we have seen above that $h_i \in H_i^e(E,f)$ we know that $E_{-i}^+(h_i, f) \neq \emptyset$, which means that $E_{-i}^{rf}(h_i) = E_{-i}^+(h_i, f)$. Thus, $(s_j)_{j\neq i} \in E_{-i}^+(h_i, f)$, which means that $(s_j)_{j\neq i} \in E_{-i}(h_i)$, and for every $j \neq i$ it holds that $s_j \in E_j(h_j)$ for all $h_j \in f_{ij}(h_i, E) \cap H_j(s_j)$. Since $g \subseteq f$ it follows, for every $j \neq i$, that $s_j \in E_j(h_j)$ for all $h_j \in g_{ij}(h_i, E) \cap H_j(s_j)$. But then, $(s_j)_{j\neq i} \in E_{-i}^{rg}(h_i)$. Hence, it follows that $E_{-i}^{rf}(h_i) \subseteq E_{-i}^{rg}(h_i)$. Since $b_i \in \Delta(E_{-i}^{rf}(h_i))$ we conclude that $b_i \in \Delta(E_{-i}^{rg}(h_i))$.

Recall that $\tau_i(s_i) \notin E_i^{rg}(h_i)$. By Lemma 8.1 we know that $E_i^{rg}(h_i) = E_i^{rg*}(h_i)$, and hence $\tau_i(s_i) \notin E_i^{rg*}(h_i)$. As $h_i \in H_i(\tau_i(s_i))$ it follows that $\tau_i(s_i) \in S_i(h_i)$. Since $\tau_i(s_i) \notin E_i^{rg*}(h_i)$, it must be that $\tau_i(s_i)$ is strictly dominated in $(S_i(h_i), E_{-i}^{rg}(h_i))$. By Lemma 2.1 we then know that $\tau_i(s_i)$ is not optimal in $S_i(h_i)$ for any belief $b_i' \in \Delta(E_{-i}^{rg}(h_i))$. As we have seen that $b_i \in \Delta(E_{-i}^{rg}(h_i))$, we conclude that $\tau_i(s_i)$ is not optimal in $S_i(h_i)$ for b_i. Hence, there is some strategy s_i^* such that

$$u_i(\tau_i(s_i), b_i) < u_i(s_i^*, b_i). \tag{8.36}$$

We now prove that

$$u_i(\tau_i(s_i), b_i) = u_i(s_i, b_i). \tag{8.37}$$

Recall that $b_i \in \Delta(E_{-i}^{rf}(h_i))$ and that $h_i \in H_i^e(E, f)$. Hence, $E_{-i}^+(h_i, f) \neq \emptyset$ and thus, by definition, $E_{-i}^{rf}(h_i) = E_{-i}^+(h_i, f)$. We therefore conclude that $b_i \in \Delta(E_{-i}^+(h_i, f))$.

By definition,

$$u_i(\tau_i(s_i), b_i) = \sum_{s_{-i} \in S_{-i}} b_i(s_{-i}) \cdot u_i(z(\tau_i(s_i), s_{-i})). \tag{8.38}$$

Take some $s_{-i} \in S_{-i}$ with $b_i(s_{-i}) > 0$. As $b_i \in \Delta(E_{-i}^+(h_i, f))$ it follows that $s_{-i} \in E_{-i}^+(h_i, f)$. Moreover, we know from above that $s_i \in E_i^{rf+}(h_i, f) \subseteq E_i^+(h_i, f)$. Hence, $(s_i, s_{-i}') \in E_i^+(h_i, f) \times E_{-i}^+(h_i, f)$. As $h_i \in H_i^e(E, f)$ it follows from Lemma 8.10 (a) that $H(E_i^+(h_i, f) \times E_{-i}^+(h_i, f)) \cap H_i \subseteq H_i^e(E, f)$, and thus

$$(s_i, s_{-i}) \text{ only reaches information sets in } H_i^e(E, f). \tag{8.39}$$

By (8.34) we know that

$$\tau_i(s_i)|_{H_i^e(E,f)} = s_i|_{H_i^e(E,f)}. \tag{8.40}$$

By combining (8.39) and (8.40) we see that

$$z(s_i, s_{-i}) = z(\tau_i(s_i), s_{-i}) \text{ whenever } b_i(s_{-i}) > 0. \tag{8.41}$$

By combining (8.38) and (8.41) we get

$$u_i(\tau_i(s_i), b_i) = \sum_{s_{-i} \in S_{-i}} b_i(s_{-i}) \cdot u_i(z(s_i, s_{-i})) = u_i(s_i, b_i),$$

which was to show.

By combining (8.36) and (8.37) we conclude that

$$u_i(s_i, b_i) = u_i(\tau_i(s_i), b_i) < u_i(s_i^*, b_i).$$

However, this contradicts the fact that s_i is optimal in $S_i(h_i)$ for the belief b_i. Hence, we must conclude that $\tau_i(s_i) \in E_i^{rg+}(h_i^*, f)$.

Since this holds for every player i, every strategy s_i, and every information set $h_i^* \in H_i^e(E, f) \cap H_i(\tau_i(s_i))$ with $s_i \in E_i^{rf+}(h_i^*, f)$, we see that (T2) is satisfied. By the properties (T1), (T2) and (T3) above we thus conclude that $E^{rf} \subseteq_{\tau,f,E} E^{rg}$, and hence $E^1 \subseteq_{\tau,f,E^0} D^0$, which was to show.

(c) We finally show, by induction on k, that properties

$$(\mathbf{k.1})\ D^k \subseteq_{\sigma,f,D^{k-1}} E^k \text{ and } (\mathbf{k.2})\ E^{k+1} \subseteq_{\tilde{\sigma},f,E^k} D^k$$

hold for every $k \geq 1$.

Induction start. We start with property (1.1). Recall that $E^1 = rf(E^0)$. In (a) and (b) we have shown that $rf(E^0) \subseteq_{\tau,f,E^0} D^0 \subseteq_{\rho,f,D^{full}} E^0$ for some τ and ρ. It then follows from Lemma 8.12 that $rf(D^0) \subseteq_{\sigma,f,D^0} rf(E^0)$ for some σ. Hence, $D^1 \subseteq_{\sigma,f,D^0} E^1$ for some σ, which was to show.

We next show property (1.2). Recall that $D^1 = rf(D^0)$. By (1.1) and (b) we know that $rf(D^0) \subseteq_{\sigma,f,D^0} E^1 \subseteq_{\tau,f,E^0} D^0$. It then follows by Lemma 8.12 that $rf(E^1) \subseteq_{\tilde{\sigma},f,E^1} rf(D^0)$ for some $\tilde{\sigma}$. Hence, $E^2 \subseteq_{\tilde{\sigma},f,E^1} D^1$ for some $\tilde{\sigma}$, which was to show.

Induction step. Suppose that $k \geq 2$ and that properties $(k-1.1)$ and $(k-1.2)$ hold. We first show that $(k.1)$ holds. Recall that $E^k = rf(E^{k-1})$. By $(k-1.2)$ and $(k-1.1)$ we know that $rf(E^{k-1}) \subseteq_{\tau,f,E^{k-1}} D^{k-1} \subseteq_{\rho,f,D^{k-2}} E^{k-1}$ for some τ, ρ. By Lemma 8.12 it then follows that $rf(D^{k-1}) \subseteq_{\sigma,f,D^{k-1}} rf(E^{k-1})$ for some σ. Hence, $D^k \subseteq_{\sigma,f,D^{k-1}} E^k$, which was to show.

We next show that $(k.2)$ holds. Recall that $D^k = rf(D^{k-1})$. By $(k.1)$ and $(k-1.2)$ we know that $rf(D^{k-1}) \subseteq_{\tau,f,D^{k-1}} E^k \subseteq_{\rho,f,E^{k-1}} D^{k-1}$ for some τ, ρ. By Lemma 8.12 it then follows that $rf(E^k) \subseteq_{\sigma,f,E^k} rf(D^{k-1})$ for some σ. Hence, $E^{k+1} \subseteq_{\sigma,f,E^k} D^k$ for some σ, which was to show. By induction on k we then conclude that $(k.1)$ and $(k.2)$ hold for every $k \geq 1$. This completes the proof. ■

8.2.2 Proof of Theorem 5.1

Proof of Theorem 5.1. (a) Let M be such that $rg^M(D^{full}) = rg^{M+1}(D^{full})$. For every $m \in \{0, ..., M\}$ let $(D^{m,k})_{k=0}^{\infty}$ be the f-rationalizability procedure that starts at $rg^m(D^{full})$. Let K be such that $D^{m,K} = D^{m,K+1}$ for all $m \in \{0, ..., M\}$. Then, by construction, $D^{m,K} = r(g^{\leq m}, f)^{\infty}(D^{full})$ for every $m \in \{0, ..., M\}$ and $D^{M,K} = r(g, f)^{\infty}(D^{full})$.

By Lemma 8.13 we know, for every $m \in \{0, ..., M-1\}$, that $D^{m,K+1} \subseteq_{\sigma^m,f,D^{m,K}} D^{m+1,K}$ and $D^{m+1,K+1} \subseteq_{\tau^m,f,D^{m+1,K}} D^{m,K+1}$ for some σ^m, τ^m. As $D^{m,K+1} = D^{m,K}$ and $D^{m+1,K+1} = D^{m+1,K}$ it follows that

$$D^{m,K} \subseteq_{\sigma^m,f,D^{m,K}} D^{m+1,K} \subseteq_{\tau^m,f,D^{m+1,K}} D^{m,K} \tag{8.42}$$

for some σ^m, τ^m. By Lemma 8.10 (c) we then know that $H_i^e(D^{m,K}) \subseteq H_i^e(D^{m+1,K}) \subseteq H_i^e(D^{m,K})$ for all players i. That is, $H_i^e(D^{m,K}) = H_i^e(D^{m+1,K})$ for all $m \in \{0, 1, ..., M-1\}$ and all players i. By (8.42) and a repeated application of Lemma 8.11 we then conclude that

$$D^{0,K} \subseteq_{(\sigma^{M-1} \circ ... \circ \sigma^0),f,D^{0,K}} D^{M,K} \text{ and } D^{M,K} \subseteq_{(\tau^0 \circ ... \circ \tau^{M-1}),f,D^{0,K}} D^{0,K}. \tag{8.43}$$

Moreover, we have seen in the proof of Lemma 8.12 that we can construct these transformation mappings $\sigma^0, ..., \sigma^{M-1}$ and $\tau^0, ..., \tau^{M-1}$ on the basis of (8.15) and (8.16). Suppose that the transformation mappings above have this specific construction.

Set $\sigma^* := \sigma^{M-1} \circ ... \circ \sigma^0$ and $\tau^* := \tau^0 \circ ... \circ \tau^{M-1}$. We will show that σ^* transforms every combination $(s_i)_{i \in I}$ of f-rationalizable strategies into a combination $(\sigma_i^*(s_i))_{i \in I}$ of (g, f)-rationalizable strategies yielding the same outcome. Let $\sigma^* = (\sigma_i^*)_{i \in I}$. Then, by construction of the transformation mappings $\sigma^0, ..., \sigma^{M-1}$, for every player i the mapping $\sigma_i^* : S_i \to S_i$ transforms every strategy s_i into the unique strategy $\sigma_i^*(s_i)$ that satisfies

$$\sigma_i^*(s_i)|_{H_i^e(D^{0,K}, f)} = s_i|_{H_i^e(D^{0,K}, f)} \text{ and} \tag{8.44}$$

$$\sigma_i^*(s_i)|_{H_i^+(h_i)} = s_i^{M,K}[h_i]|_{H_i^+(h_i)} \text{ for all } h_i \in (H_i \backslash H_i^e(D^{0,K}, f))^{first}, \tag{8.45}$$

where $s_i^{M,K}[h_i] \in D_i^{M,K}(h_i')$ for all $h_i' \in H_i^+(h_i) \cap H_i(s_i^{M,K}[h_i])$.

Now, take for every player i an f-rationalizable strategy s_i. As $rf^\infty(D^{full}) = D^{0,K}$ we must have, for every player i, that $s_i \in D_i^{0,K}(h_i)$ for every $h_i \in H_i(s_i)$. Then, for every player j, every information set in H_j that is reached by $(s_i)_{i \in I}$ is in $H_j^e(D^{0,K}, f)$. To see this, consider an information set $h_j \in H_j$ that is reached by $(s_i)_{i \in I}$. As $s_j \in D_j^{0,K}(h_j')$ for every $h_j' \in H_j(s_j)$ we see that $s_j \in D_j^{0,K+}(h_j)$ and thus $D_j^{0,K+}(h_j, f) \neq \emptyset$. Moreover, for every $i \neq j$ we have that $s_i \in D_i^{0,K}(h_i)$ for every $h_i \in H_i(s_i)$ which implies that $(s_i)_{i \neq j} \in D_{-j}^{0,K+*}(h_j, f)$. Recall that $D^{0,K} = rf^K(D^{full})$. As f is monotone it follows from Lemma 8.2 that $D_{-j}^{0,K+*}(h_j, f) = D_{-j}^{0,K+}(h_j, f)$, and hence it follows that $(s_i)_{i \neq j} \in D_{-j}^{0,K+}(h_j, f)$. As such, $D_{-j}^{0,K+}(h_j, f) \neq \emptyset$. Since we have seen that $D_j^{0,K+}(h_j, f) \neq \emptyset$ also, we conclude that $h_j \in H_j^e(D^{0,K}, f)$. Thus, for every player j, every information set in H_j that is reached by $(s_i)_{i \in I}$ is in $H_j^e(D^{0,K}, f)$. By (8.44) it then follows that $z((s_i)_{i \in I}) = z(\sigma_i^*(s_i))_{i \in I}$.

We next show that $\sigma_i^*(s_i)$ is (g, f)-rationalizable for every player i. Take some player i. As s_i is f-rationalizable and $rf^\infty(D^{full}) = D^{0,K}$, we conclude that $s_i \in D_i^{0,K}(h_i)$ for every $h_i \in H_i(s_i)$. This implies that $s_i \in D_i^{0,K+}(h_i, f)$ for all $h_i \in H_i(s_i)$. Since, by (8.43), $D^{0,K} \subseteq_{\sigma^*, f, D^{0,K}} D^{M,K}$, we know that σ^* satisfies conditions (T2) and (T3). As $s_i \in D_i^{0,K+}(h_i, f)$ for all $h_i \in H_i(s_i)$ it follows by (T2) and (T3) that $\sigma_i^*(s_i) \in D_i^{M,K}(h_i)$ for all $h_i \in H_i(\sigma_i^*(s_i))$. As $D^{M,K} = r(g, f)^\infty(D^{full})$ we conclude that $\sigma_i^*(s_i)$ is (g, f)-rationalizable.

We thus see that σ^* transforms every combination of f-rationalizable strategies $(s_i)_{i \in I}$ into a combination of (g, f)-rationalizable strategies $(\sigma_i^*(s_i))_{i \in I}$ such that $z((s_i)_{i \in I}) = z((\sigma_i^*(s_i))_{i \in I})$. That is, both combinations of strategies yield the same outcome. In a similar fashion it can be shown that τ^* transforms every combination of (g, f)-rationalizable strategies $(s_i)_{i \in I}$ into a combination of f-rationalizable strategies $(\sigma_i^*(s_i))_{i \in I}$ yielding the same outcome. As such, the set of f-rationalizable outcomes is the same as the set of (g, f)-rationalizable outcomes.

(b) Take an f-rationalizable outcome z. Then, by (a), the outcome z is also (g, f)-rationalizable. That is, there is a combination $(s_i)_{i \in I}$ of (g, f)-rationalizable strategies with $z((s_i)_{i \in I}) = z$. Since, by construction, $r(g, f)^\infty(D^{full}) \subseteq rg^\infty(D^{full})$, every (g, f)-rationalizable strategy is also g-rationalizable, and thus the outcome z is g-rationalizable. As such, every f-rationalizable outcome is g-rationalizable.

(c) Suppose that f is future oriented. We first show that $f_{ii}(h_i, D^{0,K}) \subseteq H_i \backslash H_i^-(h_i)$ for all players i and $h_i \in H_i$. To see this, consider some $h_i' \in f_{ii}(h_i, D^{0,K})$. Then, there is some $j \neq i$ and $h_j \in f_{ij}(h_i, D^{0,K})$

such that $h_i' \in f_{ji}(h_j, D^{0,K})$. Since f is future oriented we have that h_i weakly precedes h_j and h_j weakly precedes h_i'. As the game has a cycle-free ordering of information sets, h_i' cannot precede h_i, and hence $h_i' \in H_i \backslash H_i^-(h_i)$. Thus, $f_{ii}(h_i, D^{0,K}) \subseteq H_i \backslash H_i^-(h_i)$.

Now consider, for every player i, the mapping $\sigma_i^* : S_i \to S_i$ we constructed in (a). We will show that σ_i^* is the identity mapping, by proving that $H_i^e(D^{0,K}, f) = H_i$. To see this, consider some $h_i \in H_i$. As f is monotone, individually forward decreasing and individually preserves focus on past information sets, we know from Lemma 8.6 that $D^{0,K}$ is dynamically consistent. Hence, we conclude from Lemma 8.5 (b) that there is a strategy $s_i^* \in S_i(h_i)$ such that $s_i^* \in D_i^{0,K*}(h_i')$ for all $h_i' \in H_i(s_i^*) \backslash H_i^-(h_i)$. Since we have seen above that $f_{ii}(h_i, D^{0,K}) \subseteq H_i \backslash H_i^-(h_i)$ we conclude that $s_i^* \in D_i^{0,K*}(h_i')$ for all $h_i' \in f_{ii}(h_i, D^{0,K}) \cap H_i(s_i^*)$. As, by Lemma 8.1, $D_i^{0,K*}(h_i') = D_i^{0,K}(h_i')$, we see that $s_i^* \in D_i^{0,K}(h_i')$ for all $h_i' \in f_{ii}(h_i, D^{0,K}) \cap H_i(s_i^*)$. Hence, $s_i^* \in D_i^{0,K+}(h_i, f)$, which implies that $D_i^{0,K+}(h_i, f) \neq \emptyset$.

To show that $D_{-i}^{0,K+}(h_i, f) \neq \emptyset$ choose some history $p \in h_i$. For a fixed player $j \neq i$ choose some strategy s_j that selects all player j actions in p. As $D^{0,K}$ is dynamically consistent we know, by Lemma 8.4, that there is some strategy s_j^* with $s_j^*|_{H_j^-(p)} = s_j|_{H_j^-(p)}$ such that $s_j^* \in D_j^{0,K*}(h_j, f)$ for all $h_j \in H_j(s_j^*) \backslash H_j^-(p)$. In particular, $s_j^* \in D_j^{0,K*}(h_j, f)$ for all $h_j \in H_j^+(h_i) \cap H_j(s_j^*)$. Since f is future oriented we must have that $f_{ij}(h_i, D^{0,K}) \subseteq H_j^+(h_i)$. Hence,

$$s_j^* \in D_j^{0,K*}(h_j, f) \text{ for all } h_j \in f_{ij}(h_i, D^{0,K}) \cap H_j(s_j^*). \tag{8.46}$$

Moreover, since $s_j^*|_{H_j^-(p)} = s_j|_{H_j^-(p)}$ and s_j selects all player j actions in p, we know that s_j^* selects all player j actions in p as well. Since this holds for all $j \neq i$ we conclude that $(s_j^*)_{j \neq i} \in S_{-i}(h_i)$. Together with (8.46) we see that $(s_j^*)_{j \neq i} \in D_{-i}^{0,K+*}(h_i, f)$. On the basis of Lemma 8.2 it can be shown, in a similar way as in the proof of Lemma 8.10 (a), that $D_{-i}^{0,K+*}(h_i, f) = D_{-i}^{0,K+}(h_i, f)$, and thus $(s_j^*)_{j \neq i} \in D_{-i}^{0,K+}(h_i, f)$. Hence, $D_{-i}^{0,K+}(h_i, f) \neq \emptyset$. As we have already seen that $D_i^{0,K+}(h_i, f) \neq \emptyset$ we conclude that $h_i \in H_i^e(D^{0,K}, f)$. As this holds for every $h_i \in H_i$ we see that $H_i^e(D^{0,K}, f) = H_i$.

But then, it follows from the construction of σ_i^* in (8.44) and (8.45) that σ_i^* is the identity mapping from S_i to S_i. Now, take some f-rationalizable strategy s_i for player i. By our proof of (a) we know that $s_i = \sigma_i^*(s_i)$ is (g, f)-rationalizable and hence, in particular, s_i is g-rationalizable. That is, every f-rationalizable strategy is g-rationalizable. This completes the proof. ■

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