Chapter 2

Decision Problems

In this chapter of the online appendix, we first provide an axiomatic characterization for expected utility in Sections 2.8, 2.9 and 2.10. That is, we will impose a list of conditions – or axioms – on a conditional preference relation that is both necessary and sufficient for the conditional preference relation having an expected utility representation. Like for the utility design procedure, we build up our characterization in three steps: In Section 2.8 we restrict to the case of two choices, and show that some very basic axioms – the *regularity axioms* – characterize those conditional preference relations that have an expected utility representation. In Section 2.9 we move to the case with more than two choices, but where there are preference reversals for every pair choices. We introduce two new axioms, *three choice linear preference intensity* and *four choice linear preference intensity,* and show that these axioms, together with the regularity axioms, characterize expected utility. The two new axioms reveal the idea that the intensity by which the DM prefers a choice to another choice changes *linearly* with the belief he holds. In Section 2.10 we move to the general case, where there may be no preference reversals for some pairs of choices. That is, some choices may be weakly or strictly dominated by other choices. The axioms above are extended to *signed* conditional preference relations, where we consider so-called *signed* beliefs which possibly assign negative "probabilities" to states. These extended axioms, together with some additional axioms that concern cases where there is *constant preference intensity* between two choices, are shown to characterize expected utility for the general case. In Section 2.11 we discuss some economic applications for decision problems. All proofs can be found in Section 2.12.

2.8 Case of Two Choices

In this section we explore the situation where there are only two choices. We start by presenting some axioms, called the *regularity* axioms, which are required for an expected utility representation. Subsequently, we present a theorem which shows that these axioms are not only necessary but also

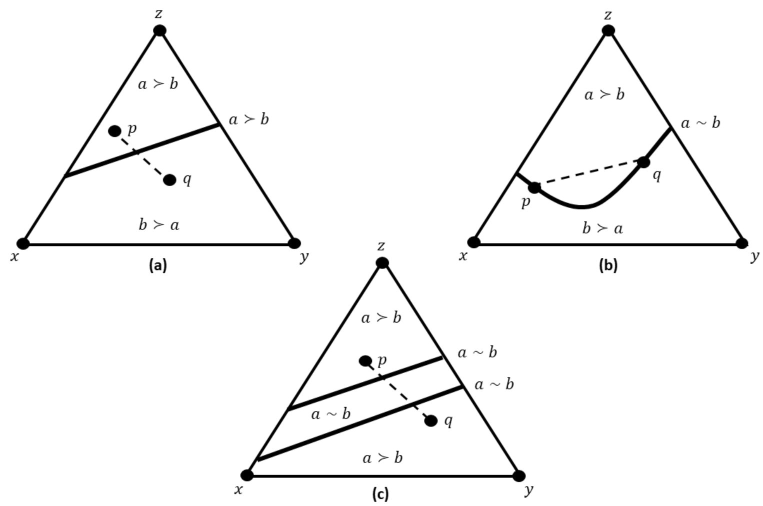

Figure 2.8.1 When an expected utility representation does not exist

sufficient for an expected utility representation. That is, if the conditional preference relation satisfies the regularity axioms, then we can find a utility matrix that represents it.

2.8.1 Regularity Axioms

Suppose the DM can only choose between two options, a and b. What properties should the conditional preference relation $\succsim$ have such that it can be represented by a utility matrix? Figure 2.8.1 presents a few instances where an expected utility representation does *not* exist, and these instances will give rise to our *regularity axioms.*

In each of the three instances we assume that the set of states is $S = \{x, y, z\}$. Consider first the conditional preference relation in panel (a). Hence, for all beliefs above the line the DM prefers a to b, for all beliefs on the line he still prefers a to b, whereas for all beliefs below the line he prefers b to a. Such a conditional preference relation cannot have an expected utility representation. For suppose it would be represented by a utility function. Then, at the belief p the expected utility for a must be greater than that for b, whereas at the belief q the expected utility for b must greater than that for a. But then, there must be a belief on the line segment between p and q where the two expected utilities are the same, and hence where the DM is indifferent between a and b. However, there is no belief where the DM is indifferent between a and b, and therefore we reach a contradiction. Thus, there is no expected utility representation. The reason is that this conditional preference relation violates the axiom of *continuity.*

Axiom 2.8.1 (Continuity) *If the beliefs p, q are such that $a \succ_p b$ and $b \succ_q a$, then there is a belief $r = (1-\lambda)p + \lambda q$ on the line segment between p and q with $\lambda \in (0,1)$, such that $a \sim_r b$.*

That is, if at the belief p the DM prefers a to b, and at the belief q he prefers b to a, then there must be a belief on the line segment between a and b where the DM is indifferent between a and b. In the formulation of the axiom, by $r = (1-\lambda)p + \lambda q$ we denote the belief that assigns to every state s the probability $r(s) = (1-\lambda)\cdot p(s) + \lambda \cdot q(s)$. For instance, if $p = (0.2, 0.3, 0.5)$, $q = (0.4, 0.4, 0.2)$ and $\lambda = 0.4$ then

$$r = (1-\lambda)p + \lambda q = (0.28, 0.34, 0.38).$$

Indeed, the probability that r assigns to x (the first state) is $(0.6)\cdot(0.2)+(0.4)\cdot(0.4) = 0.28$. Similarly for the other two probabilities. Please verify this.

Geometrically, $r = (1-\lambda)p + \lambda q$ is the belief on the line between p and q where the ratio between the distance to p and the distance to q is $\lambda/(1-\lambda)$. In particular, when $\lambda = 0.5$ then r is exactly halfway between p and q, and if $\lambda < 0.5$ then r is closer to p than to q.

Consider now the conditional preference relation in panel (b) of Figure 2.8.1. Also here an expected utility representation will not be possible. To see why, suppose the conditional preference relation would be represented by a utility function. Then, at the beliefs p and q the DM is indifferent between a and b, and hence the expected utilities for a and b must be the same at p and q. But then, the expected utilities for a and b must also be the same for every belief on the line segment between a and b, which is not the case. Hence, we reach a contradiction, and thus an expected utility representation is not possible. The reason is that the conditional preference relation violates the axiom *preservation of indifference.*

Axiom 2.8.2 (Preservation of indifference) *If the beliefs p, q are such that $a \sim_p b$ and $a \sim_q b$, and $r = (1-\lambda)p + \lambda q$ is a belief on the line segment between p and q with $\lambda \in (0,1)$, then $a \sim_r b$.*

In other words, if the DM is indifferent between a and b at two beliefs p and q, then the indifference is preserved if we vary the belief on the line segment between p and q. To formally see why preservation of indifference must hold if we have an expected utility representation, assume that $\succsim$ is represented by a utility function u. Then, at the beliefs p and q we must have that $u(a,p) = u(b,p)$ and $u(a,q) = u(b,q)$. Consider the belief $r = (1-\lambda)p + \lambda q$, for some $\lambda \in (0,1)$. Then,

$$\begin{aligned}
u(a,r) - u(b,r) &= u(a,(1-\lambda)p + \lambda q) - u(b,(1-\lambda)p + \lambda q) \\
&= (1-\lambda)\cdot u(a,p) + \lambda \cdot u(a,q) - (1-\lambda)\cdot u(b,p) - \lambda \cdot u(b,q) \\
&= (1-\lambda)\cdot (u(a,p) - u(b,p)) + \lambda \cdot (u(a,q) - u(b,q)) \\
&= (1-\lambda)\cdot 0 + \lambda \cdot 0 = 0,
\end{aligned}$$

and hence $u(a,r) = u(b,r)$. That is, the DM must be indifferent between a and b at r.

Finally, consider the conditional preference relation in panel (c) of Figure 2.8.1. An expected utility representation will also not be possible here. To see this, suppose there would be an expected utility representation. Then, at the beliefs p and q the expected utility for a must be larger than for b. As a consequence, the expected utility for a must be larger than that for b for all beliefs on the line segment between p and q, and as such the DM must prefer a to b for all beliefs on that line segment. This, however, is not the case, as there are beliefs on this line where the DM is indifferent between a and b. This conditional preference relation violates the axiom *preservation of strict preference.*

Axiom 2.8.3 (Preservation of strict preference) *If the beliefs p, q are such that $a \succsim_p b$ and $a \succ_q b$, and $r = (1-\lambda)p + \lambda q$ is a belief on the line segment between p and q with $\lambda \in (0,1)$, then $a \succ_r b$.*

Thus, if the DM weakly prefers a to b at the belief p, and prefers a to b at the belief q, then the preference between a and b will be preserved if we vary the belief on the line segment between p and q. Similarly as above, we can show that preservation of strict preference must necessarily hold if we have an expected utility representation. Indeed, suppose that $\succsim$ is represented by a utility function u. Then, we must have that $u(a,p) \geq u(b,p)$ and $u(a,q) > u(b,q)$. Consider the belief $r = (1-\lambda)p + \lambda q$, for some $\lambda \in (0,1)$. Then,

$$\begin{aligned} u(a,r)-u(b,r) &= u(a,(1-\lambda)p+\lambda q)-u(b,(1-\lambda)p+\lambda q) \\ &= (1-\lambda)\cdot u(a,p)+\lambda\cdot u(a,q)-(1-\lambda)\cdot u(b,p)-\lambda\cdot u(b,q) \\ &= (1-\lambda)\cdot(u(a,p)-u(b,p))+\lambda\cdot(u(a,q)-u(b,q)) \\ &> (1-\lambda)\cdot 0+\lambda\cdot 0=0, \end{aligned}$$

and hence $u(a,r) > u(b,r)$. That is, the DM must prefers a to b at r.

The three axioms above are called the *regularity axioms.* By our arguments above we know that every conditional preference relation with an expected utility representation must necessarily satisfy the three regularity axioms. However, the other direction is also true if there are only two choices: In the case of two choices, every conditional preference relation that satisfies the three regularity axioms will have an expected utility representation. We thus obtain the following characterization.

Theorem 2.8.1 (Expected utility for two choices) *Suppose there are only two choices. Then, a conditional preference relation has an expected utility representation, if and only if, it satisfies continuity, preservation of indifference and preservation of strict preference.*

With our arguments above we have already proven one direction of the theorem: If the conditional preference relation has an expected utility representation, then it must satisfy the three regularity axioms. In the remainder of this section we will sketch how to prove the other direction.

2.8.2 Why Axioms are Sufficient

We will now discuss, on an intuitive level, why the regularity axioms are enough to guarantee an expected utility representation. In Section 2.4 of the book we have seen the utility design procedure for two choices with preference reversals. With this procedure we can derive a utility function u, provided there are preference reversals between the two choices. Suppose now that the conditional preference relation satisfies the regularity axioms, and that there are preference reversals between the two choices. We will explain, without diving too much into technical details, why the utilities generated by this procedure will represent the conditional preference relation at hand.

To make our argument easier and more visual, let us assume there are three states, x, y and z. Suppose, moreover, that $a \succ_x b$, $b \succ_y a$ and $b \succ_z a$. Then, the conditional preference relation will look like the one in Figure 2.8.2 if it satisfies the regularity axioms.

Now, suppose we apply the utility design procedure, resulting in a utility function u. Recall that we set $u(b,y) > u(a,y)$, and that we obtain $u(b,x)$ and $u(b,z)$ by the utility difference property with respect to the beliefs p_1 and p_2. This will make sure that $u(a,p_1) = u(b,p_1)$ and $u(a,p_2) = u(b,p_2)$. But then, $u(a,p) = u(b,p)$ for every belief p on the line segment L between p_1 and p_2. Thus, the beliefs at which a and b yield the same expected utility are precisely the beliefs where the DM is indifferent between a and b.

Moreover, since $u(b,y) > u(a,y)$, the expected utility for b will be higher than the expected utility for a in the area to the right of L. This is precisely the area of beliefs for which the DM prefers b to a. Similarly, the expected utility for a will be higher than the expected utility for b in the area to the left of L. This, in turn, is precisely the area of beliefs for which the DM prefers a to b. As such, the utilities generated by the utility design procedure represent the conditional preference relation at hand.

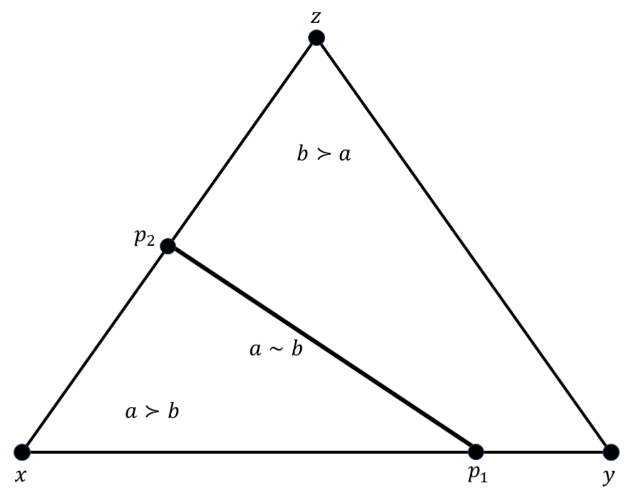

Figure 2.8.2 Why the utility design procedure works

2.9 Case of Preference Reversals

In this section we move from two choices to three choices or more. We start by showing that the regularity axioms from the previous section are no longer sufficient to guarantee an expected utility representation if there are three choices or more. In response, we introduce two additional axioms, *three choice linear preference intensity* and *four choice linear preference intensity,* which both reveal the idea that the intensity by which you prefer a choice to another choice must change linearly with the belief. We show that these new axioms, together with the regularity axioms, guarantee an expected utility representation if there are preference reversals for every pair of choices.

2.9.1 Why Regularity Axioms are Not Sufficient

In the previous section we have seen that for the case of two choices, the regularity axioms were sufficient to guarantee an expected utility representation. However, if we move to three choices or more, this is no longer true. To see this, consider the conditional preference relation in Figure 2.9.1 for the example "The birthday party". At first sight, there seems nothing wrong with this conditional preference relation. It may be verified that all the regularity axioms are satisfied, and that for every belief p the preference relation $\succsim_p$ on choices is transitive. Such conditional preference relations, where the preference relation $\succsim_p$ on choices is transitive for every belief p, are said to satisfy the *transitivity axiom.*

Axiom 2.9.1 (Transitivity) *For every belief* p*, the preference relation* $\succsim_p$ *on choices is transitive.*

However, as we will demonstrate, the conditional preference relation in Figure 2.9.1 does not have an expected utility representation.

To see this, consider a general conditional preference relation $\succsim$ with an expected utility representation u, and a line of beliefs l. For an example of a line of beliefs, see the grey line in Figure 2.9.2. Consider three choices a, b and c, and suppose that p_{ab}, p_{ac} and p_{bc} are beliefs on that line where the DM is indifferent between a and b, between a and c, and between b and c, respectively. Assume

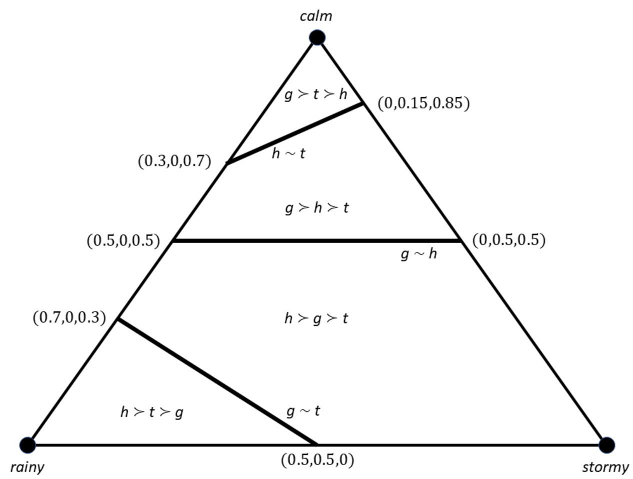

Figure 2.9.1 Regularity axioms are not sufficient for an expected utility representation

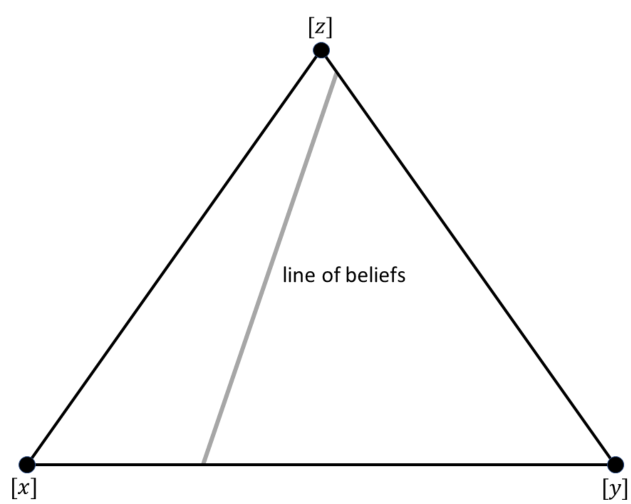

Figure 2.9.2 Line of beliefs

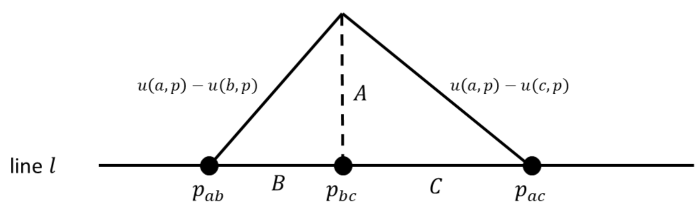

Figure 2.9.3 Expected utility difference on a line of beliefs with three choices

moreover that p_{bc} is in between p_{ab} and p_{ac}, and that the DM prefers a to b, and a to c, at the belief p_{bc}. Then, the expected utility differences between a and b, and between a and c, on the line l must behave as in Figure 2.9.3.

Note that at the belief p_{ab}, the expected utility difference between a and b must be 0. Similarly, at the belief p_{ac} the expected utility difference between a and c must be 0. At the belief p_{bc}, the expected utility of b must be the same as the expected utility of c. Hence, at the belief p_{bc} the expected utility difference between a and b must be the same as the expected utility difference between a and c. This results in the two triangles in Figure 2.9.3.

From the first triangle we can derive the constant rate at which the expected utility difference between a and b changes on the line l. Indeed,

$$\frac{\Delta(u(a,p)-u(b,p))}{\Delta p}=\frac{A}{B}, \tag{2.9.1}$$

where Δp is an arbitrary change of the belief p on the line l, and $\Delta(u(a,p)-u(b,p))$ is the induced change in expected utility difference between a and b. The lengths A and B are as given in the figure.

Similarly, from the second triangle we can derive the change rate of the expected utility difference between a and c through the equation

$$\frac{\Delta(u(a,p)-u(c,p))}{\Delta p}=-\frac{A}{C}. \tag{2.9.2}$$

Indeed, note that $u(a,p)-u(c,p)$ gets smaller if we move to the right on the line l. For that reason, the change rate above is negative. By combining (2.9.1) and (2.9.2) we get

$$\frac{\Delta(u(a,p)-u(b,p))}{\Delta(u(a,p)-u(c,p))}=\frac{\Delta(u(a,p)-u(b,p))/\Delta p}{\Delta(u(a,p)-u(c,p))/\Delta p}=\frac{A/B}{-A/C}=-\frac{C}{B}. \tag{2.9.3}$$

Here, the ratio

$$\frac{\Delta(u(a,p)-u(b,p))}{\Delta(u(a,p)-u(c,p))}$$

expresses how quickly the expected utility difference between a and b changes, compared to the change rate of the expected utility difference between a and c.

Fix a state s. Then, the length C in the figure is proportional to the difference $p_{ac}(s)-p_{bc}(s)$, whereas the length B is proportional to $p_{bc}(s)-p_{ab}(s)$. As such,

$$-\frac{C}{B}=\frac{p_{ac}(s)-p_{bc}(s)}{p_{ab}(s)-p_{bc}(s)}.$$

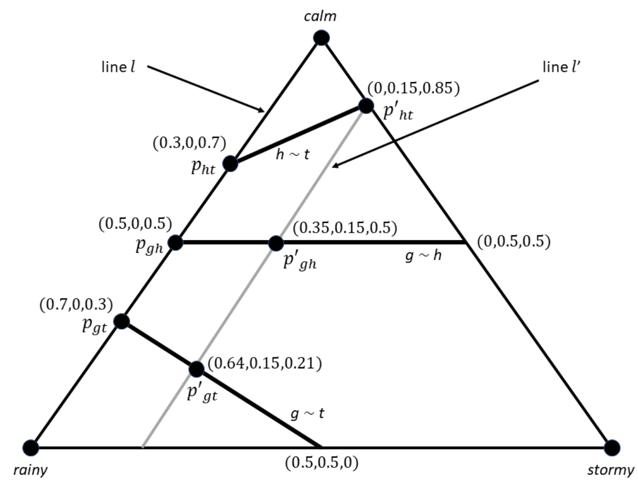

Figure 2.9.4 Why regularity axioms are not sufficient for an expected utility representation

Together with (2.9.3) this leads to

$$\frac{\Delta(u(a,p)-u(b,p))}{\Delta(u(a,p)-u(c,p))} = \frac{p_{ac}(s)-p_{bc}(s)}{p_{ab}(s)-p_{bc}(s)}. \tag{2.9.4}$$

Now, consider a line l' of beliefs that is parallel to the line l, like in Figure 2.9.4.

Let p'_{ab}, p'_{ac} and p'_{bc} be beliefs on the new line l' where the DM is indifferent between a and b, between a and c, and between b and c, respectively. Then, (2.9.4) also applies to this new line and these new beliefs.

Since the expected utility difference between any two choices changes linearly with the beliefs, the change rate ratio

$$\frac{\Delta(u(a,p)-u(b,p))}{\Delta(u(a,p)-u(c,p))}$$

must be the same on the line l as on the parallel line l'. In view of (2.9.4) we then conclude that

$$\frac{p_{ac}(s)-p_{bc}(s)}{p_{ab}(s)-p_{bc}(s)} = \frac{p'_{ac}(s)-p'_{bc}(s)}{p'_{ab}(s)-p'_{bc}(s)}. \tag{2.9.5}$$

However, equation (2.9.5) is violated for the conditional preference relation in Figure 2.9.1. To see that, consider the parallel lines l and l' in Figure 2.9.4. Indeed, if we choose *rainy* for the state s, then we have that

$$\frac{p_{gt}(r)-p_{gh}(r)}{p_{ht}(r)-p_{gh}(r)} = \frac{0.7-0.5}{0.3-0.5} = -1$$

whereas

$$\frac{p'_{gt}(r)-p'_{gh}(r)}{p'_{ht}(r)-p'_{gh}(r)} = \frac{0.64-0.35}{0-0.35} \neq -1.$$

As such, the conditional preference relation from Figure 2.9.1 cannot have an expected utility representation.

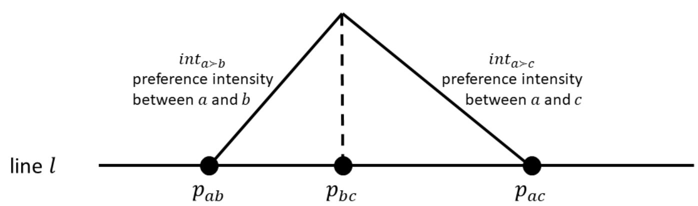

Figure 2.9.5 Preference intensities on a line of beliefs for three choices

2.9.2 Three Choice Linear Preference Intensity

This raises the question: What is wrong with the conditional preference relation in Figure 2.9.1? Consider again the parallel lines of beliefs l and l' in Figure 2.9.4. Along those lines, the DM changes his belief in exactly the same direction, while departing from different initial beliefs. Intuitively, changing the belief in the same direction should change the intensity by which the DM prefers one choice to another by the same rate. But, as we will show, the conditional preference relation in Figure 2.9.1 is not compatible with this principle.

To see this, consider an arbitrary conditional preference relation, a line l of beliefs, and three choices a, b and c. Assume that p_{ab}, p_{ac} and p_{bc} are beliefs on this line where the DM is indifferent between a and b, between a and c, and between b and c, respectively. As before, suppose that p_{bc} is in between p_{ab} and p_{ac}, and that the DM prefers a to b, and a to c, at the belief p_{bc}. If we assume that the preference intensity between any two choices changes linearly on the line l, then the preference intensities must behave as in Figure 2.9.5.

The picture looks exactly the same as the one in Figure 2.9.3. This should not come as a surprise, since we have seen that expected utility differences can be identified with preference intensities. By using the same arguments as above for expected utility differences, we can then conclude, for any state s, that

$$\frac{\Delta(int_{a \succ b}(p))}{\Delta(int_{a \succ c}(p))} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)}. \tag{2.9.6}$$

Here, $int_{a \succ b}(p)$ denotes the intensity by which the DM prefers choice a to choice b at the belief p, and similarly for $int_{a \succ c}(p)$. As such, the ratio

$$\frac{\Delta(int_{a \succ b}(p))}{\Delta(int_{a \succ c}(p))}$$

describes how quickly the preference intensity between a and b changes on the line l, compared to the change rate of the preference intensity between a and c.

Now, consider a second line of beliefs l' that is parallel to l. Suppose that p'_{ab}, p'_{ac} and p'_{bc} are beliefs on the new line l' where the DM is indifferent between a and b, between a and c, and between b and c, respectively. If we assume that the preference intensities change linearly with the belief then, in particular, the change rate of the preference intensities on l' must be the same as on l. We can then conclude, by (2.9.6), that

$$\frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)} = \frac{p'_{ac}(s) - p'_{bc}(s)}{p'_{ab}(s) - p'_{bc}(s)} \tag{2.9.7}$$

for every state s. By cross-multiplication, this is equivalent to

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)). \tag{2.9.8}$$

This property, which is a consequence of the preference intensity between three choices changing linearly with the belief, will be called *three choice linear preference intensity.*

To define the property formally, we must first explain more precisely what we mean by *parallel lines of beliefs*. Consider two beliefs p_1, p_2 and a number λ such that $(1-\lambda) \cdot p_1 + \lambda \cdot p_2$ is again a belief. Then, the belief

$$p = (1-\lambda) \cdot p_1 + \lambda \cdot p_2$$

lies on the line that goes through p_1 and p_2.

A *line of beliefs* is a set of beliefs l for which there are two beliefs p_1, p_2 such that

$$l = \{p \in \Delta(S) \mid p = (1-\lambda) \cdot p_1 + \lambda \cdot p_2 \text{ for some number } \lambda\}.$$

That is, l contains exactly those beliefs that are on the line through p_1 and p_2. In this case, we say that l is the line that goes through p_1 and p_2. In Figure 2.9.4, for instance, the line l goes through the beliefs p_{gt} and p_{gh}.

Now, consider two lines of beliefs l and l', where l goes through the beliefs p_1 and p_2, and l' goes through the beliefs p'_1 and p'_2. We say that the lines l and l' are *parallel* if there is a number λ such that $p_1 - p_2 = \lambda \cdot (p'_1 - p'_2)$.

We are now ready to introduce the new axiom *three choice linear preference intensity.*

Axiom 2.9.2 (Three choice linear preference intensity) *For every three choices a, b, c, every two parallel lines of beliefs l and l' containing beliefs where the DM is not indifferent between any of these three choices, every triple of beliefs p_{ab}, p_{ac}, p_{bc} on l where the DM is indifferent between the respective choices, and every triple of beliefs $p'_{ab}, p'_{ac}, p'_{bc}$ on l' where the DM is indifferent between the respective choices, it holds for every state s that*

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)).$$

We have thus verified above that the conditional preference relation in Figure 2.9.1 violates the axiom of three choice linear preference intensity. In particular, this means that the conditional preference relation cannot be based on preference intensities that change linearly with the belief.

2.9.3 Geometric Characterization

In a decision problem with three states or more, it may be quite demanding to verify that three choice linear preference intensity holds. In this case, we would have to check the formula above for *every* two parallel lines l and l'. And if there are at least three states, there are many – indeed, infinitely many – of such parallel lines. In this light, it would be nice to find an easier way of checking for linear preference intensity. This is precisely what we will do in this subsection.

Consider the conditional preference relation in Figure 2.9.6, which is a copy from Figure 2.3.1 in the book. It turns out that it satisfies three choice linear preference intensity. The easiest argument is to use the fact that it has an expected utility representation, as we have seen in Section 2.3 of the book. Therefore, the preference intensities are given by the differences in expected utility, and will thus vary linearly with the belief. This, in turn, guarantees three choice linear preference intensity, as we have seen above.

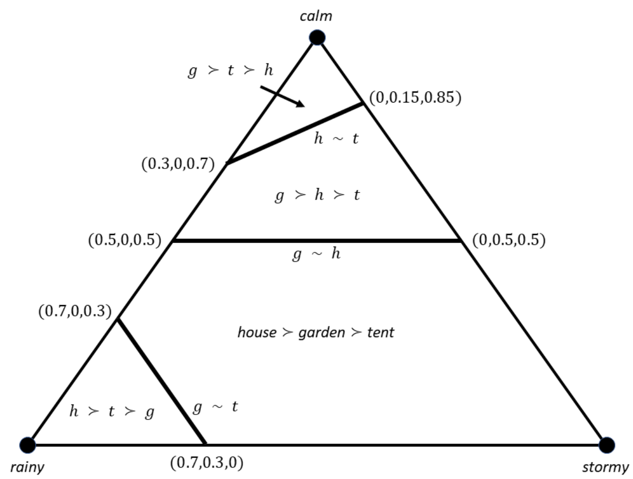

Figure 2.9.6 Conditional preference relation that satisfies three choice linear preference intensity

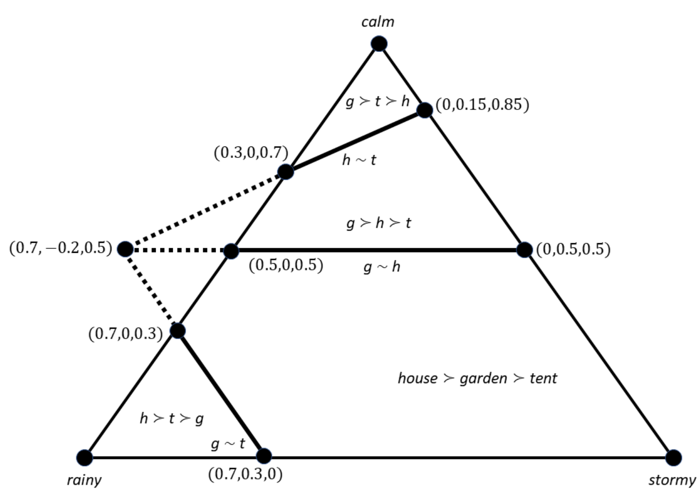

Figure 2.9.7 Geometric characterization of three choice linear preference intensity

But suppose we would not know whether the conditional preference relation in Figure 2.9.6 has an expected utility representation or not. How would we then show that it satisfies three choice linear preference intensity? The key lies in extending the sets of beliefs where you are indifferent between two choices to the area outside the belief triangle. Consider, for instance, the set of beliefs where you are indifferent between *house* and *tent.* This is a line segment that goes through the beliefs $(0.3, 0, 0, 7)$ and $(0, 0.15, 0.85)$. If we would extend this line segment outside the belief triangle, we would get the dotted line in Figure 2.9.7.

Note that this line goes through the point $(0.7, -0.2, 0.5)$ outside the belief triangle. To see this, observe that

$$(0.7, -0.2, 0.5) = (1 - \lambda) \cdot (0.3, 0, 0.7) + \lambda \cdot (0, 0.15, 0.85)$$

for $\lambda = -4/3$. The reason that $(0.7, -0.2, 0.5)$ is outside the belief triangle is that its second coordinate is negative. Hence, these three numbers do not correspond to the probabilities in a belief, because probabilities must always be non-negative.

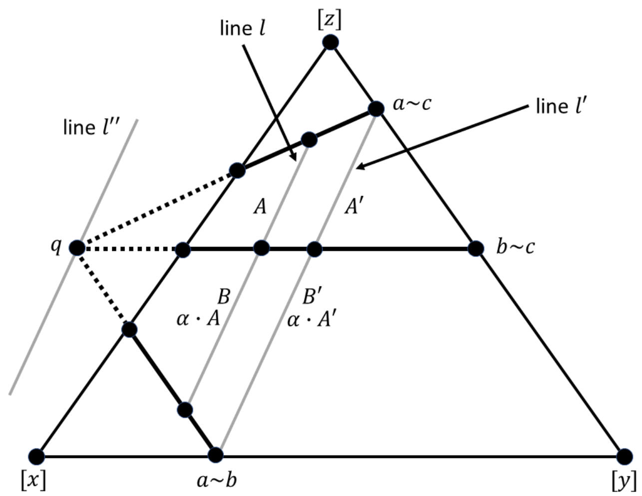

Figure 2.9.8 Geometric characterization of three choice linear preference intensity

Similarly, we can also extend the line segment of beliefs where you are indifferent between *garden* and *house,* and the line segment of beliefs where you are indifferent between *garden* and *tent,* outside the belief triangle. This results in the other two dotted lines in Figure 2.9.7.

As can be observed, the three dotted lines meet at the same point $(0.7, -0.2, 0.5)$ outside the belief triangle. This, as we will demonstrate now, is not a coincidence if three choice linear preference intensity is satisfied.

Consider a conditional preference relation with three choices a, b and c, as shown in Figure 2.9.8. Suppose that it satisfies three choice linear preference intensity. Then, equation (2.9.7) guarantees that the ratio between the lengths A and B on the line l is the same as the ratio between A' and B' on any parallel line l'. This means that there must be some fixed number α such that $B' = \alpha \cdot A'$ on *every* line l' that is parallel to l.

Consider now the dotted line through the beliefs where the DM is indifferent between a and c, and the dotted line through the beliefs where the DM is indifferent between b and c. Suppose that these two lines meet at some point q, possibly outside the belief triangle, and consider the line l'' that is parallel to l and goes through this point q. See Figure 2.9.8 for an illustration. Then, on the line l'' it must be that the corresponding length A'' is equal to 0. Since we have seen that $B'' = \alpha \cdot A''$ for every line l'' that is parallel to l, we then know that also B'' must be equal to 0. This, however, means that the line through the beliefs where the DM is indifferent between b and c must meet the line through the beliefs where the DM is indifferent between a and b at this point q. In other words, the three dotted lines must meet at the same point q.

The other direction is also true: If the three dotted lines meet at the same point, then three choice linear preference intensity must be satisfied. To see this, consider two arbitrary parallel lines of beliefs, like the lines l and l' in Figure 2.9.8. Then, it must be that the ratio between A and B is the same as the ratio between A' and B'. This, in turn, implies that three choice linear preference intensity must hold at the lines l and l'.

Thus, we see that for the case of three states, three choice linear preference intensity is satisfied precisely when the linear extensions of the three indifference sets – that is, the sets of beliefs where the DM is indifferent between two choices – all meet at the same point, possibly outside the belief triangle.

A similar result can be shown for an arbitrary number of states. To state this result formally, we must first define more precisely what we mean by the *linear extension* of a set of beliefs. Consider the set of beliefs $P_{a\sim b}$ where the DM is indifferent between the choices a and b. Take a point q, possibly outside the set of beliefs. Then, we say that q is in the *linear extension* of $P_{a\sim b}$ if there are beliefs p_1, p_2 in $P_{a\sim b}$, and a number λ, such that

$$q = (1-\lambda)\cdot p_1 + \lambda \cdot p_2.$$

In other words, the point q is on the line through two beliefs in $P_{a\sim b}$. By $\langle P_{a\sim b}\rangle$ we denote the set of all points q that are in the linear extension of $P_{a\sim b}$, and $\langle P_{a\sim b}\rangle$ is simply called the *linear extension* of $P_{a\sim b}$.

In Figure 2.9.7, for instance, $\langle P_{g\sim t}\rangle$ contains all the points, inside and outside the belief triangle, that are on the line through the beliefs $(0.7, 0, 0.3)$ and $(0.7, 0.3, 0)$ in $P_{g\sim t}$. In particular, the point $(0.7, -0.2, 0.5)$ outside the belief triangle is in $\langle P_{g\sim t}\rangle$.

By using an argument like the one above, one can show that if three choice linear preference intensity is satisfied, then every point that is in both $\langle P_{a\sim b}\rangle$ and $\langle P_{b\sim c}\rangle$ will also be in $\langle P_{a\sim c}\rangle$. In Figure 2.9.7, for instance, the point $(0.7, -0.2, 0.5)$, which is in both $\langle P_{g\sim h}\rangle$ and $\langle P_{h\sim t}\rangle$, is also in $\langle P_{g\sim t}\rangle$.

Proposition 2.9.1 (Geometric characterization of three choice linear preference intensity) *Consider a conditional preference relation $\succsim$ that has preference reversals on every pair of choices, and satisfies the regularity axioms and transitivity. Then, $\succsim$ satisfies three choice linear preference intensity, if and only if, for every three choices a, b, c, every point that is both in $\langle P_{a\sim b}\rangle$ and $\langle P_{b\sim c}\rangle$ will also be in $\langle P_{a\sim c}\rangle$.*

The latter condition, that every point which is in $\langle P_{a\sim b}\rangle$ and $\langle P_{b\sim c}\rangle$ will also be in $\langle P_{a\sim c}\rangle$, is in general easy to check. By the result above, verifying this condition is sufficient for checking whether three choice linear preference intensity is satisfied. From a practical viewpoint this is a very useful result.

Proposition 2.9.1 can be applied to show more easily that the conditional preference relation in Figure 2.9.1 violates three choice linear preference intensity. To see this, consider Figure 2.9.9. It can be seen that the linear extensions of $P_{g\sim t}$, $P_{g\sim h}$ and $P_{h\sim t}$ do not meet at the same point. Indeed, the point q, outside the belief triangle, is in both $\langle P_{g\sim t}\rangle$ and $\langle P_{g\sim h}\rangle$, but not in $\langle P_{h\sim t}\rangle$.

Conceptually, the condition in Proposition 2.9.1 may be viewed as an extension of *transitivity* outside the set of beliefs. Indeed, by Question 2.1.2 (d) in the book we know that if the conditional preference relation is transitive, then every belief that is in both $P_{a\sim b}$ and $P_{b\sim c}$ will also be in $P_{a\sim c}$. The condition in Proposition 2.9.1 states that this must also be true for the linear extensions of these three indifference sets.

2.9.4 Four Choice Linear Preference Intensity

Suppose now that there are at least four choices at the disposal of the DM. If the DM's preference intensities between these four choices change linearly with the belief, then this has another consequence which is different from three choice linear preference intensity. This new property will be called *four choice linear preference intensity.*

To see what it says, consider a line of beliefs l, and four choices a, b, c and d. Suppose that $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}$ and p_{cd} are beliefs on the line l where the DM is indifferent between the respective

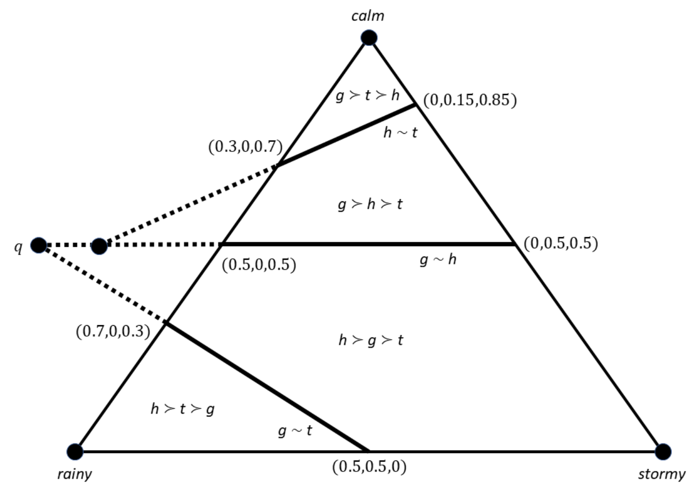

Figure 2.9.9 Violation of three choice linear preference intensity

choices. If the preference intensities change linearly with the beliefs, then we have seen in the previous section that

$$\frac{\Delta(int_{a\succ b}(p))}{\Delta(int_{a\succ c}(p))} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)} \tag{2.9.9}$$

In fact, the same holds for the triples of choices a, b, d and a, c, d, and thus we know that

$$\frac{\Delta(int_{a\succ b}(p))}{\Delta(int_{a\succ d}(p))} = \frac{p_{ad}(s) - p_{bd}(s)}{p_{ab}(s) - p_{bd}(s)} \tag{2.9.10}$$

and

$$\frac{\Delta(int_{a\succ c}(p))}{\Delta(int_{a\succ d}(p))} = \frac{p_{ad}(s) - p_{cd}(s)}{p_{ac}(s) - p_{cd}(s)} \tag{2.9.11}$$

as well. Clearly,

$$\frac{\Delta(int_{a\succ b}(p))}{\Delta(int_{a\succ d}(p))} = \frac{\Delta(int_{a\succ b}(p))}{\Delta(int_{a\succ c}(p))} \cdot \frac{\Delta(int_{a\succ c}(p))}{\Delta(int_{a\succ d}(p))}.$$

If we combine this equation with (2.9.9), (2.9.10) and (2.9.11) we obtain

$$\frac{p_{ad}(s) - p_{bd}(s)}{p_{ab}(s) - p_{bd}(s)} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)} \cdot \frac{p_{ad}(s) - p_{cd}(s)}{p_{ac}(s) - p_{cd}(s)}.$$

By cross-multiplication, this leads to the condition

$$\begin{aligned} & (p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{cd}(s)) \cdot (p_{ad}(s) - p_{bd}(s)) \\ = \; & (p_{ab}(s) - p_{bd}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) \cdot (p_{ad}(s) - p_{cd}(s)). \end{aligned}$$

This property is called *four choice linear preference intensity.*

Axiom 2.9.3 (Four choice linear preference intensity) *For every line of beliefs l, for every four choices a, b, c, d such that there is a belief on this line where the DM is not indifferent between any pair of choices in $\{a, b, c, d\}$, and for every six beliefs $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}$ and p_{cd} on the line l where the DM is indifferent between the respective choices, it holds for every state s that*

$$\begin{aligned} & (p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{cd}(s)) \cdot (p_{ad}(s) - p_{bd}(s)) \\ = \; & (p_{ab}(s) - p_{bd}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) \cdot (p_{ad}(s) - p_{cd}(s)). \end{aligned}$$

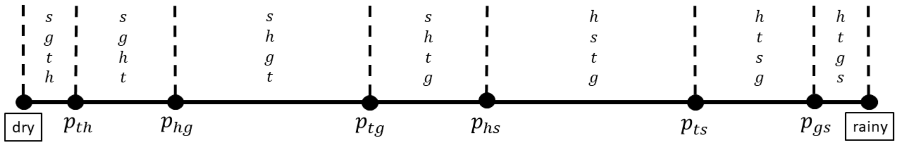

Figure 2.9.10 The birthday party with two states and four choices

To illustrate this axiom, suppose that in the example "The birthday party" you replace the states *stormy* and *calm* by the "summary" state *dry,* like we did in Figure 2.4.7 of the book. At the same time, you thought about another possible location for your party, which is a nice *square* close to your house. Assume that your conditional preference relation is given by Figure 2.9.10, which should be read in the same way as Figure 2.4.7 of the book.

It may be verified that this conditional preference relation satisfies the regularity axioms, transitivity and three choice linear preference intensity. In fact, three choice linear preference intensity is satisfied trivially, as there are no two distinct parallel lines of beliefs in this scenario. The only possible line of beliefs is the one that goes through the beliefs $[d]$ and $[r]$, assigning probability 1 to *dry* and *rainy,* respectively.

However, depending on the precise probabilities we specify for the beliefs $p_{th}, ..., p_{gs}$, the axiom of four choice linear preference intensity may be satisfied or violated. If we take, for instance, $p_{th}(r) = 0.05$, $p_{hg}(r) = 0.12$, $p_{tg}(r) = 0.4$, $p_{hs}(r) = 0.5$, $p_{ts}(r) = 0.8$ and $p_{gs}(r) = 0.88$, like we did in Figure 2.4.7 of the book, then it may be verified that *four choice linear preference intensity* is satisfied. Please verify this. If we change $p_{gs}(r)$ into 0.9, while leaving the other indifference beliefs the same, then it turns out that *four choice linear preference intensity* is violated. Also verify this, please.

Note that a conditional preference relation with an expected utility representation will always satisfy four choice linear preference intensity. The reason is that in this case, the preference intensity between any two choices will always change linearly with the belief, as this preference intensity corresponds to the difference in expected utility between these two choices. As such, the second conditional preference relation above, where $p_{gs}(r) = 0.9$, cannot have an expected utility representation.

It turns out that the axioms we have gathered until now, which are the regularity axioms, transitivity, three choice linear preference intensity and four choice linear preference intensity, are sufficient for guaranteeing an expected utility representation, provided there are *preference reversals* for all pairs of choices. This is the content of the following result.

Theorem 2.9.1 (Expected utility with preference reversals) *Consider a conditional preference relation $\succsim$ where there are preference reversals for all pairs of choices. Then, $\succsim$ has an expected utility representation, if and only if, $\succsim$ satisfies continuity, preservation of indifference, preservation of strict preference, transitivity, three choice linear preference intensity and four choice linear preference intensity,*

Recall that the last two axioms indicate that the preference intensities must change linearly with the belief. In that light, the result above shows that the conditional preference relations with an expected utility representation are precisely those that are based on preference intensities that change linearly with the belief, of course assuming the regularity axioms and transitivity in the background.

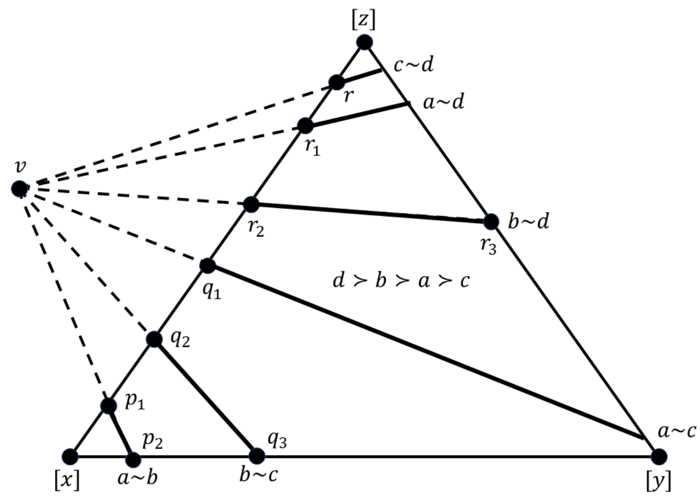

Figure 2.9.11 Intuition for why utility design procedure works

We will now provide an intuition for why the axioms above are sufficient to guarantee an expected utility representation. Consider a conditional preference relation $\succsim$ with preference reversals for all pairs of choices which satisfies all of the axioms above. Recall that we can use the utility design procedure from Section 2.4.2 of the book to compute a utility function u. We will intuitively explain why this utility function u *represents* the conditional preference relation $\succsim$.

As an illustration, consider a conditional preference relation with four choices and three states, as given in Figure 2.9.11. We have only indicated the preference between the four choices in one of the seven regions, as to not crowd the picture too much. The preferences in the other six regions can be derived form these preferences, using the lines of beliefs where the DM becomes indifferent between two choices.

It may first be verified that this conditional preference relation satisfies the regularity axioms and transitivity. To see why it satisfies three choice linear preference intensity, note that the lines of beliefs $P_{a\sim b}$, $P_{a\sim c}$ and $P_{b\sim c}$, when extended linearly outside the belief triangle, all meet at the same point v. Therefore, we know by Proposition 2.9.1 that three choice linear preference intensity is satisfied for the choices a, b and c. In a similar way, it can also be verified that three choice linear preference intensity holds for the triple of choices a, b, d, the triple a, c, d and the triple b, c, d.

Moreover, we assume that the beliefs p_1, q_2, q_1, r_2, r_1 and r are chosen such that they satisfy the equation for four choice linear preference intensity on the line between $[x]$ and $[z]$. It may be verified that this implies that the conditional preference relation satisfies four choice linear preference intensity.

Finally, note that there are preference reversals for all pairs of choices, and that there are beliefs where the DM is indifferent between some, but not all, choices. Hence, all the conditions for applying the utility design procedure are satisfied.

The utility design procedure would work as follows. We start by choosing the utilities for a arbitrarily, and by choosing an arbitrary utility for $u(b, z)$ larger than $u(a, z)$. Subsequently, we apply the utility difference property to the beliefs p_1 and p_2, for the choices a and b, to determine $u(b, x)$ and $u(b, y)$. The utility difference property makes sure that the expected utility of a will be equal to the expected utility of b at the beliefs p_1 and p_2. As such, the expected utility of a will be the same as for b at all beliefs where the DM is indifferent between a and b.

Next, to determine the utilities for c, we apply the utility difference property to the belief q_1 for the choices c and a, and to the beliefs q_2 and q_3 for the choices c and b. This guarantees that at the beliefs q_2 and q_3, the expected utility for b will be same as for c. As such, at all beliefs where the DM is indifferent between b and c, the expected utility for b will be the same as for c.

Note that the lines of beliefs $P_{a\sim b}$ and $P_{b\sim c}$, when extended linearly, meet at the same point v. As the expected utility for a is the same as for b on $P_{a\sim b}$, and the expected utility for b is the same as for c on $P_{b\sim c}$, it follows that the expected utilities for a, b and c will all be the same at the point v. In particular, the expected utility for a will be the same as for c at the point v.

Now, the utility difference property at q_1 for the choices c and a makes sure that at q_1, the expected utility for a is the same as for c. But then, the expected utility for a must be the same as for c at all points on the line through v and q_1. In particular, the expected utility for a must be the same as for c at all beliefs where the DM is indifferent between a and c. Hence, we have seen so far that the utilities for a, b and c get the three indifference lines $P_{a\sim b}$, $P_{a\sim c}$ and $P_{b\sim c}$ right.

Subsequently, we generate the utilities for d by applying the utility difference property to the belief r_1 for the choices d and a, and to the beliefs r_2 and r_3 for the choices d and b. In the same way as above for a, b, c, it then follows that the utilities for a, b and d will get the indifference lines $P_{a\sim d}$ and $P_{b\sim d}$ right. It remains to show that it will also get the last indifference line $P_{c\sim d}$ right.

Note that the indifference lines $P_{a\sim d}$ and $P_{a\sim c}$, when extended linearly, meet at the same point v. Since the expected utility for a is the same as for d at all beliefs in $P_{a\sim d}$, and the expected utility for a is the same as for c at all beliefs in $P_{a\sim c}$, it follows that the expected utilities for a, c and d are all the same at the point v. In particular, at the point v the expected utility for c is the same as for d.

Consider the line between $[x]$ and $[z]$. Since four choice linear preference intensity is satisfied on this line, the belief r on this line where the DM is indifferent between c and d is uniquely given by the other five indifference beliefs p_1, q_2, q_1, r_2 and r_1 on this line, through the equation in four choice linear preference intensity.

On the other hand, the conditional preference relation induced by the utilities generated for a, b, c and d will also satisfy four choice linear preference intensity on this line. Since we have seen that the utilities get the indifference beliefs p_1, q_2, q_1, r_2 and r_1 right, and r is uniquely given by p_1, q_2, q_1, r_2 and r_1 through four choice linear preference intensity, the utilities must also get the indifference belief r right. That is, at the belief r the expected utility for c must be the same as for d.

Altogether, we see that the expected utility for c must be the same as for d at the points v and r, and hence they must be the same at all beliefs on the line through v and r. As such, at all beliefs where the DM is indifferent between c and d, the expected utility for c will be the same as for d. Hence, the utilities also get the last line of indifference beliefs $P_{c\sim d}$ right.

Not only this, the utilities will also get all the preferences between the choices right for all possible beliefs. In other words, the utility matrix generated by the utility design procedure will indeed *represent* the conditional preference relation at hand.

2.10 *General Case

In the previous section we have considered scenarios in which there are preference reversals for all pairs of choices. We will now move towards the general scenario where there may be no preference reversals for some pairs of choices, because some choices may be weakly, or even strictly, dominated by other choices. For those scenarios, we investigate what additional axioms need to be imposed on a conditional preference relation such that it allows for an expected utility representation.

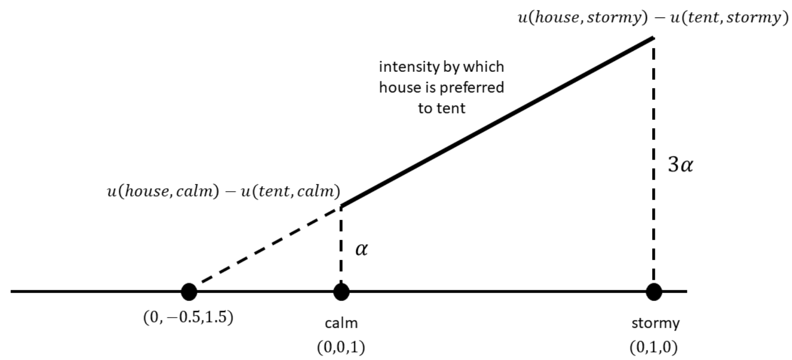

Figure 2.10.1 Signed indifference belief

As we will see, the key is to look at so-called *signed* beliefs, which allow for "probabilities" that are negative, or larger than one. Although signed beliefs cannot be interpreted directly as likelihoods, they may be used to measure how the preference intensity between two choices changes when we move from one state to another.

Every conditional preference relation can be extended to a *signed* conditional preference relation, which does not only assign a preference relation over choices to every belief, but also to every *signed* belief. The axioms we have seen so far for conditional preference relations can be generalized to signed conditional preference relations, and we will argue that these generalized axioms make intuitive sense if we assume that the preference intensity between two choices changes linearly with the belief. Besides, we need two new axioms. Our main result shows that a conditional preference relation has an expected utility representation precisely when it can be extended to a signed conditional preference relation that satisfies each of these axioms.

2.10.1 Signed Beliefs

In the previous section we restricted ourselves to scenarios where there are preference reversals for all pairs of choices. We will now extend our analysis to cases where there may be no preference reversals between some pairs of choices, because some choices are weakly, or strictly, dominated by other choices.

Consider again the birthday party example and let us focus, for the moment, on your choices *house* and *tent.* Suppose that you always prefer *house* to *tent,* irrespective of your belief about the weather, because you had a terrible camping experience last year. That is, your choice *house* strictly dominates your choice *tent.* Or, in other words, the intensity by which you prefer *house* to *tent* is always positive, no matter what belief you have.

Still, it seems plausible that the intensity by which you prefer *house* to *tent* is greater at the state *stormy* than at the state *calm,* because in the former case there is a chance that the tent will be blown away. Assume that the intensity by which you prefer *house* to *tent* at the state *stormy* is three times higher than at the state *calm.* See Figure 2.10.1. Then, on the line of beliefs through the states *calm* and *stormy,* we could linearly extend this preference intensity *outside* the set of beliefs. This is depicted by the dashed line in Figure 2.10.1. If we do so, the preference intensity between *house* and *tent* would become zero at the vector $(0, -0.5, 1.5)$ outside the belief triangle.

This vector, which assigns 0 to state *rainy,* -0.5 to state *stormy* and 1.5 to state *calm,* cannot really be interpreted in the way we understand a belief. The reason is that it involves numbers which are negative, or larger than 1, which cannot be interpreted as likelihoods. But the indifference vector $(0, -0.5, 1.5)$ does have a meaning: It states that the preference intensity between *house* and *tent* at the state *stormy* is three times higher than at the state *calm.* Such vectors, which may include numbers less than 0 or larger than 1, but where the sum is equal to 1, are called *signed beliefs.*

Definition 2.10.1 (Signed belief) *For a given set of states S, a* ***signed belief*** *q assigns to every state s a (possibly negative) number $q(s)$ such that $\sum_{s \in S} q(s) = 1$.*

Thus, every belief is a signed belief, but not *vice versa.* Let us now go back to the example above, with the choices *house* and *tent.* As we have seen in earlier sections, the preference intensity between *house* and *tent* at a given belief, or state, can be identified with the expected utility difference between the choices. Thus, we conclude that

$$\frac{u(\textit{house, stormy}) - u(\textit{tent, stormy})}{u(\textit{house, calm}) - u(\textit{tent, calm})} = 3,$$

as visualized in Figure 2.10.1. Moreover, at the signed indifference belief $q = (0, -0.5, 1.5)$ we have that $q(\textit{stormy}) = -0.5$ and $q(\textit{calm}) = 1.5$, which implies that $q(\textit{calm})/q(\textit{stormy}) = -3$. If we combine this with the equality above, we obtain

$$\frac{u(\textit{house, stormy}) - u(\textit{tent, stormy})}{u(\textit{tent, calm}) - u(\textit{house, calm})} = \frac{q(\textit{calm})}{q(\textit{stormy})}.$$

Note that this involves the same expression as the *utility difference property* in Section 2.4 of the book. In fact, it may be viewed as a generalization of the utility difference property to *signed* indifference beliefs.

To see why, in general, the utility difference property also holds for signed "indifference" beliefs, consider two choices, a and b, two states x and y, and a signed "indifference" belief q on the line through $[x]$ and $[y]$ where the DM is "indifferent" between a and b. See Figure 2.10.2 for an illustration. Similarly to Section 2.4 of the book, we conclude that

$$\frac{u(a,x) - u(b,x)}{u(a,y) - u(b,y)} = \frac{A}{B}.$$

Moreover, it can be seen from the figure that

$$\frac{A}{B} = \frac{q(x) - 1}{q(x) - 0} = -\frac{1 - q(x)}{q(x)} = -\frac{q(y)}{q(x)}.$$

By combining these two equations we obtain

$$\frac{u(a,x) - u(b,x)}{u(b,y) - u(a,y)} = \frac{q(y)}{q(x)},$$

which may be viewed as the *utility difference property for signed beliefs.*

Let us return again to our example above, where you always prefer *house* to *tent* for every belief. We assumed that the intensity by which you prefer *house* to *tent* when it is *stormy* is three times

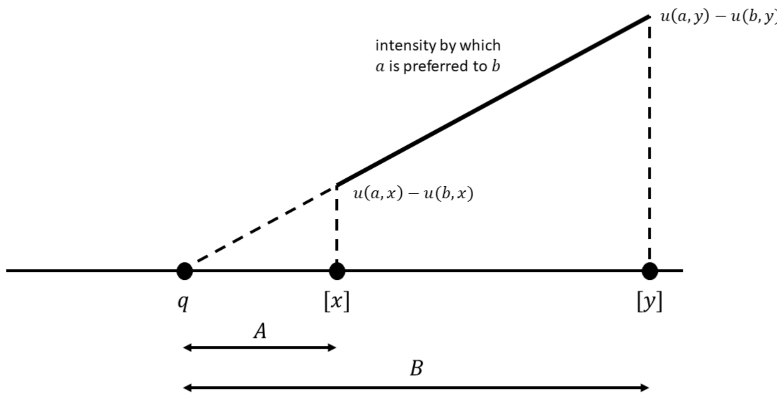

Figure 2.10.2 Utility difference property for signed "indifference" beliefs

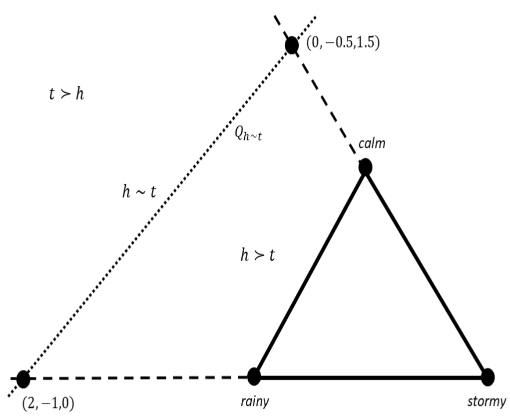

Figure 2.10.3 Signed conditional preference relation

higher than when it is *calm.* Now suppose, additionally, that the intensity by which you prefer *house* to *tent* when it is *stormy* is two times higher than when it is *rainy.* If we assume that your preference intensity between *house* and *tent* changes linearly with your belief, your preferences for all possible signed beliefs can visualized by Figure 2.10.3.

This picture should be read as follows: The dotted line, going through the signed beliefs $(2, -1, 0)$ and $(0, -0.5, 1.5)$, represents all the signed beliefs where you are "indifferent" between *house* and *tent.* Consider, for instance, the signed belief $q = (2, -1, 0)$ on the line through *rain* and *stormy.* By the utility difference property above, we have that

$$\frac{u(\textit{house, stormy}) - u(\textit{tent, stormy})}{u(\textit{tent, rainy}) - u(\textit{house, rainy})} = \frac{q(\textit{rainy})}{q(\textit{stormy})} = \frac{2}{-1} = -2.$$

Thus, the intensity by which you prefer *house* to *tent* when it is *stormy* is twice as high as when it is *rainy,* as required. The set of signed beliefs where you are "indifferent" between *house* and *tent* is denoted by $Q_{h\sim t}$ in the figure.

Moreover, the figure indicates that for all signed beliefs to the right of the dotted line you "prefer" *house* to *tent,* whereas on the other side of the dotted line you "prefer" *tent* to *house.* Since all (traditional) beliefs are to the right of the dotted line, you prefer *house* to *tent* for all beliefs, as it should be.

Such an object, which specifies a preference relation over choices for every *signed* belief, is called a *signed conditional preference relation.*

Definition 2.10.2 (Signed conditional preference relation) *A **signed conditional preference relation** $\succsim^*$ assigns to every signed belief q a preference relation $\succsim^*_q$ over the choices.*

Similarly to "normal" conditional preference relations, we can also define what it means for a *signed* conditional preference relation to be represented by a utility function. Formally, we say that a signed conditional preference relation $\succsim^*$ is *represented* by a utility function u if for every signed belief q and every two choices a and b we have that

$$a \succsim^*_q b \text{ if and only if } u(a, q) \geq u(b, q).$$

Here,

$$u(a, q) := \sum_{s \in S} q(s) \cdot u(a, s)$$

represents the "expected utility" induced by the choice a at the signed belief q, and similarly for $u(b, q)$. We write "expected utility" between quotes since q need not necessarily be a belief.

In Figure 2.10.3 we can say that the "normal" conditional preference relation $\succsim$, where you always prefer *house* to *tent* for every belief, can be *extended* to the signed conditional preference relation $\succsim^*$ depicted in that figure. In a sense, the signed conditional preference relation reveals more information, as it also states how the intensities by which you prefer *house* to *tent* compare at the three states of weather. In general, extending a conditional preference relation to a signed conditional preference relation can be defined as follows.

Definition 2.10.3 (Extension to signed conditional preference relation) *A signed conditional preference relation $\succsim^*$ **extends** a conditional preference relation $\succsim$ if for every belief $p \in \Delta(S)$ the preference relations $\succsim_p$ and $\succsim^*_p$ coincide.*

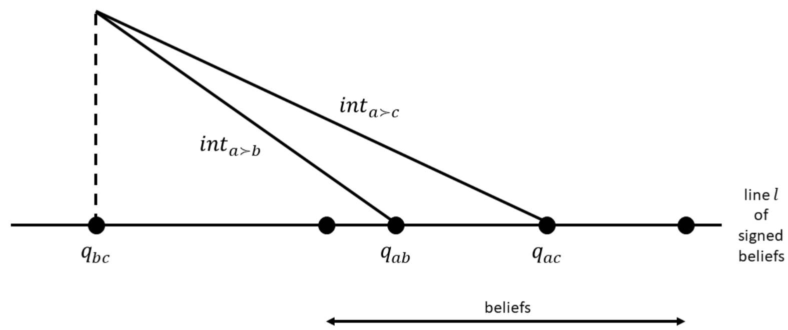

Figure 2.10.4 Relative change of preference intensities on a line

Clearly, if a conditional preference relation $\succsim$ can be extended to a signed conditional preference relation $\succsim^*$ with an expected utility representation u, then the utility function u will also represent the conditional preference relation $\succsim$. Thus, if we are given a conditional preference relation $\succsim$, where some choices possibly weakly or strictly dominate other choices, and ask whether it has an expected utility representation, then it all boils down to the following question: Can $\succsim$ be extended to a signed conditional preference relation $\succsim^*$ with an expected utility representation? If the answer is "yes", then $\succsim$ will have an expected utility representation as well. If the answer is "no", then there is no expected utility representation for $\succsim$.

In a sense, the question whether an expected utility representation exists is thus shifted to the domain of *signed* conditional preference relations. As we will see in the following two subsections, a signed conditional preference relation will have an expected utility representation precisely when it satisfies (a generalized version of) the regularity axioms, transitivity, three choice linear preference intensity and four choice linear preference intensity, together with two additional axioms that are related to cases of "constant preference intensity".

2.10.2 Extending the Previous Axioms

As we have seen above, a signed conditional preference relation reveals for every two choices a and b how the intensities by which the DM prefers a to b at the various states relate to each other. But if there are more than two choices, then a signed conditional preference relation will also tell us, for every line of signed beliefs, and every three choices a, b, c, how quickly the preference intensity between a and b changes on this line, as compared to the speed with which the preference intensity between a and c changes.

To see this, consider Figure 2.10.4 where we have depicted the preference intensities between a and b and between a and c on a line of signed beliefs. Here, q_{ab}, q_{ac} and q_{bc} are the signed beliefs on the line l where the DM is "indifferent" between the respective choices. In the figure, we assume that q_{ab} and q_{ac} are "real" beliefs whereas q_{bc} is not, but this is not relevant for our argument. As in Section 2.9.2, $int_{a \succ b}$ denotes the intensity by which the DM prefers a to b, and similarly for $int_{a \succ c}$. Note that the two preference intensities must be equal at the signed belief q_{bc}, where the DM is "indifferent" between b and c.

Similarly to Section 2.9.2, we conclude that the relative change rates of these two preference intensities satisfy

$$\frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ c}(q))} = \frac{q_{ac}(s) - q_{bc}(s)}{q_{ab}(s) - q_{bc}(s)} \tag{2.10.1}$$

for every state s where the probability of s is not constant on the line. That is, the three signed indifference beliefs q_{ab}, q_{ac} and q_{bc} determine the relative speed at which the preference intensities between a and b and between a and c change on the line.

If we assume that the preference intensity between any two choices changes linearly with the belief, then the relative change rates of the preference intensities must always be the same on two parallel lines of signed beliefs l and l'. Hence, if we consider a line l' of signed beliefs parallel to the line l in Figure 2.10.4, then on the line l' the ratio $\Delta(int_{a\succ b}(q))/\Delta(int_{a\succ c}(q))$ must be the same as on the line l. If q'_{ab}, q'_{ac} and q'_{bc} denote the signed "indifference" beliefs on the line l' then, in view of (2.10.1), we must have that

$$\frac{q_{ac}(s) - q_{bc}(s)}{q_{ab}(s) - q_{bc}(s)} = \frac{q'_{ac}(s) - q'_{bc}(s)}{q'_{ab}(s) - q'_{bc}(s)},$$

and hence

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q'_{ac}(s) - q'_{bc}(s)) = (q'_{ab}(s) - q'_{bc}(s)) \cdot (q_{ac}(s) - q_{bc}(s)). \tag{2.10.2}$$

This may be viewed as an extension of *three choice linear preference intensity* to signed conditional preference relations. Although this property includes signed beliefs, it follows solely from the assumption that the preference intensity between any two choices of the *original* conditional preference relation – which does *not* include signed beliefs – changes linearly with the belief.

Now, consider four choices a, b, c, d, and a line l of signed beliefs. Similarly to Section 2.9.4, it must hold that

$$\frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ d}(q))} = \frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ c}(q))} \cdot \frac{\Delta(int_{a\succ c}(q))}{\Delta(int_{a\succ d}(q))}.$$

Together with (2.10.1), and using the same arguments as in Section 2.9.4, this leads to

$$\begin{aligned} & (q_{ab}(s) - q_{bc}(s)) \cdot (q_{ac}(s) - q_{cd}(s)) \cdot (q_{ad}(s) - q_{bd}(s)) \\ = \; & (q_{ab}(s) - q_{bd}(s)) \cdot (q_{ac}(s) - q_{bc}(s)) \cdot (q_{ad}(s) - q_{cd}(s)). \end{aligned} \tag{2.10.3}$$

Here, $q_{ab}, ..., q_{cd}$ denote the signed "indifference" beliefs for the six pairs of choices from $\{a, b, c, d\}$. This formula can be viewed as an extension of *four choice linear preference intensity* to signed conditional preference relations. Again, the formula follows from the assumption that the preference intensities of the *original* conditional preference relation – which does *not* include signed beliefs – change linearly with the belief.

Summarizing, we see that if the preference intensities of the original conditional preference relation $\succsim$ change linearly with the belief, then it must be possible to extend $\succsim$ to a signed conditional preference relation $\succsim^*$ that satisfies three choice and four choice linear preference intensity.

But we can say even more: In this case, the signed conditional preference relation $\succsim^*$ must also satisfy (generalizations of) the regularity axioms and transitivity.

To see why it satisfies the regularity axioms, we consider Figure 2.10.5 as a reference point. It depicts the conditional preferences between two choices, a and b, with three states x, y and z. Moreover, for every belief it shows the intensity by which the DM prefers a to b, along the vertical axis. In particular, the DM always prefers a to b for every belief, but this preference intensity varies in a linear fashion with the belief.

Note that these preference intensities are captured by the signed conditional preference relation shown in the same figure. Indeed, if we linearly extend the preference intensity levels outside the

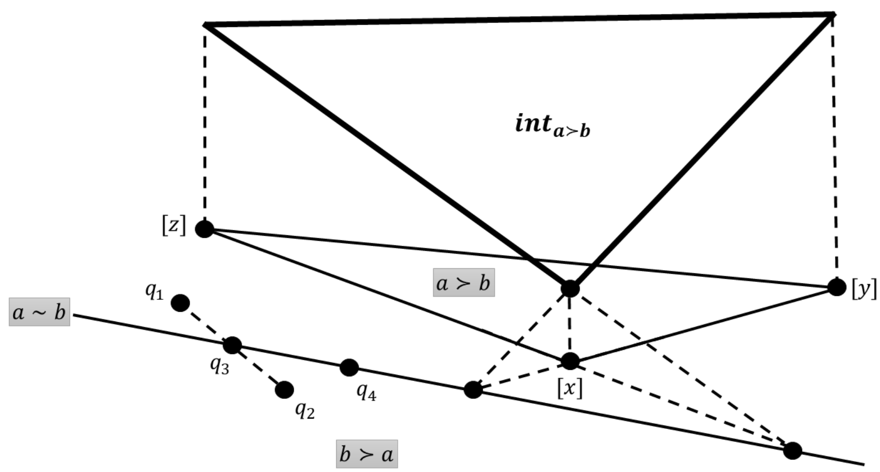

Figure 2.10.5 Regularity axioms for a signed conditional preference relation

belief triangle, then the signed beliefs where the DM is "indifferent" are exactly those points where the preference intensity is zero.

On the basis of this figure, we will now argue that the signed conditional preference relation $\succsim^*$ must satisfy the three regularity axioms. We will start with *continuity.* Take two signed beliefs, q_1 and q_2, where $a \succ^*_{q_1} b$ and $b \succ^*_{q_2} a$. See Figure 2.10.5 for an illustration. Then, it can be seen that there is a signed belief q_3 on the line segment between q_1 and q_2 where the DM is "indifferent". This establishes continuity.

We next consider *preservation of indifference.* Take two signed beliefs, q_3 and q_4, where the DM is "indifferent". See Figure 2.10.5 for an illustration. Then, for every signed belief on the line segment between q_3 and q_4, the DM would also be "indifferent". Thus, preservation of indifference holds.

Now, consider *preservation of strict preference.* Consider a belief q_3 where $a \succsim^*_{q3} b$ and a belief q_1 with $a \succ^*_{q_1} b$. See Figure 2.10.5 for an illustration. Then, it can be seen that at every belief on the line segment between q_3 and q_1 (excluding q_3 and q_1) the DM strictly prefers a to b. This yields preservation of strict preference.

We finally turn to *transitivity.* Consider three choices, a, b and c, and a signed belief q such that $a \succsim^*_q b$ and $b \succsim^*_q c$. We will see that it then naturally follows that $a \succsim^*_q c$. Take some "real" beliefs p_1, p_2 such that q is on the line through p_1 and p_2. See Figure 2.10.6 for an illustration in case the preference intensity between a and b is linear in the belief. Let q_{ab} be the signed belief on that line where the DM is "indifferent" between a and b. Then, according to Figure 2.10.6 we must have that

$$\frac{int_{a\succ b}(p_1)}{int_{a\succ b}(p_2)} = \frac{A}{B}.$$

Let $q_{ab} = (1-\lambda)p_1 + \lambda p_2$ for some number λ. Then,

$$\frac{A}{B} = \frac{0-\lambda}{1-\lambda} = \frac{\lambda}{\lambda - 1}.$$

By combining the two equalities above we get that

$$q_{ab} = (1-\lambda)p_1 + \lambda p_2 \text{ with } \frac{\lambda}{\lambda-1} = \frac{int_{a\succ b}(p_1)}{int_{a\succ b}(p_2)}.$$

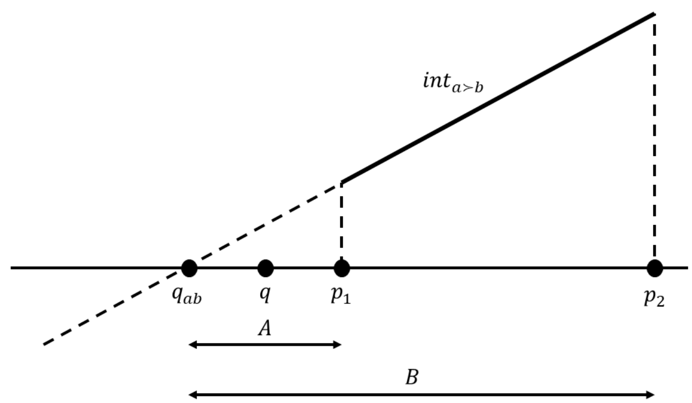

Figure 2.10.6 Continuity for signed conditional preference relations

Since $a \succsim_q^* b$, we must have that q is either equal to q_{ab}, or to the right of q_{ab}, and hence

$$q = (1-\lambda)p_1 + \lambda p_2 \text{ with } \frac{\lambda}{\lambda - 1} \leq \frac{int_{a\succ b}(p_1)}{int_{a\succ b}(p_2)}.$$

As at the signed belief q both λ and $\lambda - 1$ are negative, this can be rewritten as

$$q = (1-\lambda)p_1 + \lambda p_2 \text{ with } \lambda \cdot int_{a\succ b}(p_2) \geq (\lambda - 1) \cdot int_{a\succ b}(p_1). \tag{2.10.4}$$

As $b \succsim_q^* c$, it follows in a similar way that

$$q = (1-\lambda)p_1 + \lambda p_2 \text{ with } \lambda \cdot int_{b\succ c}(p_2) \geq (\lambda - 1) \cdot int_{b\succ c}(p_1). \tag{2.10.5}$$

By adding (2.10.4) and (2.10.5) we get

$$\begin{gathered} q = (1-\lambda)p_1 + \lambda p_2 \text{ with} \\ \lambda \cdot (int_{a\succ b}(p_2) + int_{b\succ c}(p_2)) \geq (\lambda - 1) \cdot (int_{a\succ b}(p_1) + int_{b\succ c}(p_1)). \end{gathered} \tag{2.10.6}$$

Intuitively, $int_{a\succ b}(p_1) + int_{b\succ c}(p_1)$ represents the intensity by which the DM prefers a to c. Indeed, it makes sense to view preference intensity as an *additive* concept, which means that the intensity by which the DM prefers a to c can be written as the sum of the intensity by which he prefers a to b and the intensity by which he prefers b to c. Thus, $int_{a\succ b}(p_1) + int_{b\succ c}(p_1) = int_{a\succ c}(p_1)$, and similarly for p_2.

If we substitute this into (2.10.6) we obtain that

$$q = (1-\lambda)p_1 + \lambda p_2 \text{ with } \lambda \cdot int_{a\succ c}(p_2) \geq (\lambda - 1) \cdot int_{a\succ c}(p_1),$$

and hence

$$q = (1-\lambda)p_1 + \lambda p_2 \text{ with } \frac{\lambda}{\lambda - 1} \leq \frac{int_{a\succ c}(p_1)}{int_{a\succ c}(p_2)}.$$

In view of Figure 2.10.6, this means that $a \succsim_q^* c$. This establishes transitivity.

Summarizing, we have seen that if the preference intensities between the choices vary linearly with the belief, then every signed conditional preference relation $\succsim^*$ that extends $\succsim$ must satisfy the regularity axioms. Moreover, as shown above, the signed conditional preference relation $\succsim^*$ will also satisfy three choice and four choice linear preference intensity. Finally, it must also satisfy transitivity. For completeness, we now summarize these axioms for signed conditional preference relations.

Definition 2.10.4 (Axioms for signed conditional preference relations) *A signed conditional preference relation $\succsim^*$ satisfies*

(a) ***continuity*** *if for every two choices a, b and every two signed beliefs q_1 and q_2 with $a \succ^*_{q_1} b$ and $b \succ^*_{q_2} a$, there is a $\lambda \in (0,1)$ such that the DM is "indifferent" between a and b at the signed belief $(1-\lambda)q_1 + \lambda q_2$;*

(b) ***preservation of indifference*** *if for every two choices a, b, for every two signed beliefs q_1, q_2 with $a \sim^*_{q_1} b$ and $a \sim^*_{q_2} b$, and for every $\lambda \in (0,1)$, the DM is "indifferent" between a and b at the signed belief $(1-\lambda)q_1 + \lambda q_2$;*

(c) ***preservation of strict preference*** *if for every two choices a, b, for every two signed beliefs q_1, q_2 with $a \succsim^*_{q_1} b$ and $a \succ^*_{q_2} b$, and for every $\lambda \in (0,1)$, the DM "prefers" a to b at the signed belief $(1-\lambda)q_1 + \lambda q_2$;*

(d) ***transitivity*** *if the preference relation $\succsim^*_q$ is transitive for every signed belief q;*

(e) ***three choice linear preference intensity*** *if for every three choices a, b, c, every two parallel lines of signed beliefs l and l' containing signed beliefs where the DM is not "indifferent" between any of these three choices, every triple of signed beliefs q_{ab}, q_{ac}, q_{bc} on l where the DM is "indifferent" between the respective choices, and every triple of signed beliefs $q'_{ab}, q'_{ac}, q'_{bc}$ on l' where the DM is "indifferent" between the respective choices, it holds for every state s that*

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q'_{ac}(s) - q'_{bc}(s)) = (q'_{ab}(s) - q'_{bc}(s)) \cdot (q_{ac}(s) - q_{bc}(s));$$

(f) ***four choice linear preference intensity*** *if for every line of signed beliefs l, for every four choices a, b, c, d such that there is a signed belief on this line where the DM is not "indifferent" between any pair of choices in $\{a, b, c, d\}$, and for every six signed beliefs $q_{ab}, q_{ac}, q_{ad}, q_{bc}, q_{bd}$ and q_{cd} on the line l where the DM is "indifferent" between the respective choices, it holds for every state s that*

$$\begin{aligned} & (q_{ab}(s) - q_{bc}(s)) \cdot (q_{ac}(s) - q_{cd}(s)) \cdot (q_{ad}(s) - q_{bd}(s)) \\ = \; & (q_{ab}(s) - q_{bd}(s)) \cdot (q_{ac}(s) - q_{bc}(s)) \cdot (q_{ad}(s) - q_{cd}(s)). \end{aligned}$$

However, as we will see in the next subsection, these axioms will not suffice to guarantee an expected utility representation for a signed conditional preference relation. The reason is that so far we have not examined cases where the preference intensity between two choices is *constant*. These scenarios will be studied in the following subsection.

2.10.3 Axioms for Constant Preference Intensity

So far we have only studied cases where for every pair of choices a and b, the signed conditional preference relation admits a signed belief q where the DM is "indifferent" between a and b. As we have seen in Figure 2.10.2, such signed "indifference" beliefs q measure how the preference intensity between a and b varies if we move from one state to another.

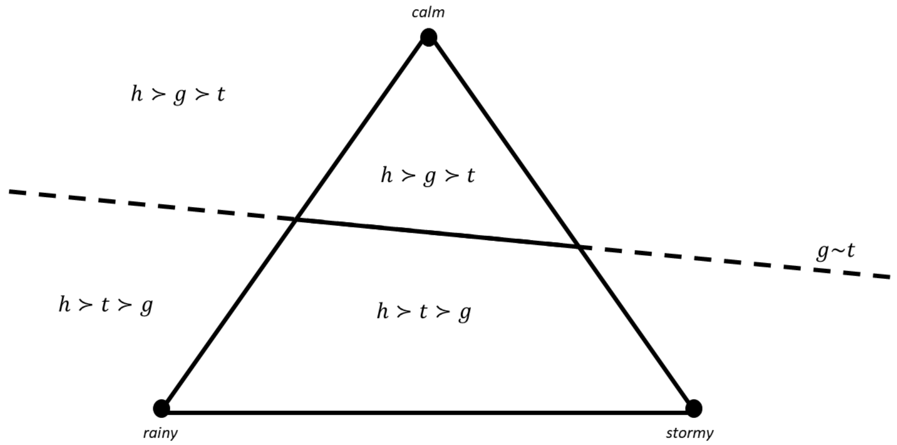

Figure 2.10.7 Constant preference intensity

But what if there are no signed beliefs q for which the DM is "indifferent" between a and b? This is precisely the case when the intensity by which the DM prefers a to b is always the same for every belief. Indeed, in such a scenario the preference intensity between a and b in Figure 2.10.2 would be horizontal, and hence would nowhere become zero on the line. In these cases, the DM would either always "'prefer" a to b for every signed belief, or always "prefer" b to a for every signed belief. We say that there is *constant preference intensity* between a and b.

Definition 2.10.5 (Constant preference intensity) *A signed conditional preference relation* $\succsim^*$ *reveals a* ***constant preference intensity*** *between choices* a *and* b *if either* $a \succ_q^* b$ *for every signed belief* q, *or* $b \succ_q^* a$ *for every signed belief* q, *or* $a \sim_q^* b$ *for every signed belief* q.

As an illustration, consider the signed conditional preference relation in Figure 2.10.7. Since you are never "indifferent" between *house* and *garden,* or between *house* and *tent,* for any signed belief, we conclude that you have a constant preference intensity between *house* and *garden,* and between *house* and *tent.*

It may be verified that this signed conditional preference relation $\succsim^*$ satisfies all the axioms from the previous subsection. Nevertheless, it does not allow for an expected utility representation. To see this, note that an expected utility representation u for $\succsim^*$ must necessarily have a constant utility difference between *house* and *garden,* and between *house* and *tent.* Indeed, otherwise there would be signed beliefs where you would become "indifferent" between *house* and *garden,* or between *house* and *tent.* But then, the utility difference between *garden* and *tent* must also be constant across all signed beliefs. This would mean, in turn, that you are either always indifferent between *garden* and *tent,* or that you always prefer one of these choices over the other. However, this would contradict the signed conditional preference relation at hand, and hence there is no expected utility representation for the signed conditional preference relation $\succsim^*$.

This raises the question: What is "wrong" with this signed conditional preference relation? Recall that the intensity by which you prefer *house* to *garden,* and the intensity by which you prefer *house* to *tent,* is constant across all signed beliefs. But then, the preference intensity between *garden* and *tent* should also be constant – a principle which is violated in Figure 2.10.7. This principle is called *transitive constant preference intensity.*

Axiom 2.10.1 (Transitive constant preference intensity) *If under the signed conditional preference relation $\succsim^*$ there is a constant preference intensity between choices a and b, and between choices b and c, then there must also be a constant preference intensity between choices a and c.*

Let us now go back to the axiom *four choice linear preference intensity.* As we have seen before, this axiom reveals that on a line of signed beliefs we have that

$$\frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ d}(q))} = \frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ c}(q))} \cdot \frac{\Delta(int_{a\succ c}(q))}{\Delta(int_{a\succ d}(q))}. \tag{2.10.7}$$

Suppose now that the preference intensity between c and d is constant. Then, the preference intensity between a and c and the preference intensity between a and d will only differ by a constant. In particular, the speed at which the preference intensity between a and c changes will be the same as the speed at which the preference intensity between a and d changes, which means that $\Delta(int_{a\succ c}(q))/\Delta(int_{a\succ d}(q)) = 1$. By (2.10.7) we then get that

$$\frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ d}(q))} = \frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ c}(q))}.$$

Since we have seen that

$$\frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ d}(q))} = \frac{q_{ad}(s) - q_{bd}(s)}{q_{ab}(s) - q_{bd}(s)} \text{ and } \frac{\Delta(int_{a\succ b}(q))}{\Delta(int_{a\succ c}(q))} = \frac{q_{ac}(s) - q_{bc}(s)}{q_{ab}(s) - q_{bc}(s)}$$

we conclude that

$$\frac{q_{ad}(s) - q_{bd}(s)}{q_{ab}(s) - q_{bd}(s)} = \frac{q_{ac}(s) - q_{bc}(s)}{q_{ab}(s) - q_{bc}(s)}.$$

This, in turn, yields the formula

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q_{ad}(s) - q_{bd}(s)) = (q_{ab}(s) - q_{bd}(s)) \cdot (q_{ac}(s) - q_{bc}(s)). \tag{2.10.8}$$

Suppose, in addition, that the preference intensity between a and b would also be constant. That is, the preference intensities between a and b, and between c and d, would both be constant. Then, on a line of signed beliefs the preference intensities between the various pairs of choices would yield a picture similar to that in Figure 2.10.8. Note that the preference intensities between a and b, and between c and d, correspond to horizontal lines, as these are constant on the line of signed beliefs. Moreover, at the signed belief q_{bd} where the preference intensity between a and d is equal to the preference intensity between a and b, it must be that the preference intensity between b and d is zero, and hence the DM is "indifferent" between b and d. Similarly, at the signed belief q_{ac} the preference intensity between a and d is equal to the preference intensity between c and d, and thus the DM is "indifferent" between a and c.

Moreover, the difference between the preference intensity between a and d on the one hand and the preference intensity between b and d on the other hand must always be equal to the constant preference intensity α between a and b. As such, the line of the preference intensity between b and d is parallel to the line of the preference intensity between a and d. In fact, the first line is obtained by the second line if we shift it downwards by the amount α. Finally, note that at the signed belief q_{bc} the preference intensity between b and d is equal to the preference intensity between c and d, and hence the DM is "indifferent" between b and c.

From the picture it can clearly be seen that

$$q_{ac}(s) - q_{bc}(s) = q_{ad}(s) - q_{bd}(s) \tag{2.10.9}$$

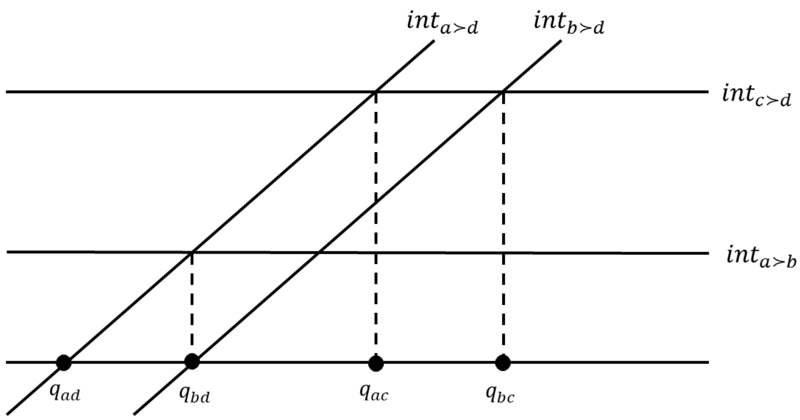

Figure 2.10.8 Four choice linear preference intensity with constant preference intensity

for every state s.

The formulas (2.10.8) and (2.10.9) give rise to the following axiom, which we call *four choice linear preference intensity with constant preference intensity.*

Axiom 2.10.2 (Four choice linear preference intensity with contant preference intensity) *For every line of signed beliefs l, and for every four choices a, b, c, d such that there is a signed belief on this line where the DM is not "indifferent" between any pair of choices in $\{a, b, c, d\}$, the following holds:*

(a) if there is a constant preference intensity between c and d, but not between the other five pairs of choices, then for every five signed beliefs $q_{ab}, q_{ac}, q_{ad}, q_{bc}$ and q_{bd} on the line l where the DM is "indifferent" between the respective choices, it holds for every state s that

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q_{ad}(s) - q_{bd}(s)) = (q_{ab}(s) - q_{bd}(s)) \cdot (q_{ac}(s) - q_{bc}(s));$$

(b) if there is a constant preference intensity between a and b, and between c and d, but not between the other four pairs of choices, then for every four signed beliefs q_{ac}, q_{ad}, q_{bc} and q_{bd} on the line l where the DM is "indifferent" between the respective choices, it holds for every state s that

$$q_{ac}(s) - q_{bc}(s) = q_{ad}(s) - q_{bd}(s).$$

Now, take a conditional preference relation $\succsim$ with an expected utility representation. Then, as we have seen in Section 2.9, the induced preference intensity between every two choices will vary linearly with the belief. Consequently, on the basis of our arguments above, we can extend $\succsim$ to a signed conditional preference relation $\succsim^*$ that satisfies all of the axioms above.

However, it turns out that the opposite direction is also true: If we can extend the conditional preference relation $\succsim$ to a signed conditional preference relation $\succsim^*$ that satisfies all of the axioms above, then there will be an expected utility representation for $\succsim$. As such, the axioms above characterize precisely those conditional preference relations that admit an expected utility representation. We thus obtain the following result.

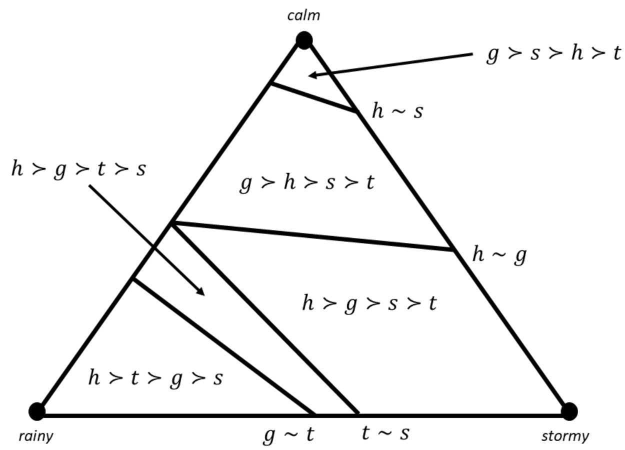

Figure 2.10.9 Verifying the axioms

Theorem 2.10.1 (Expected utility for the general case) *A conditional preference relation has an expected utility representation, if and only if, it can be extended to a signed conditional preference relation that satisfies continuity, preservation of indifference, preservation of strict preference, transitivity, three choice linear preference intensity, four choice linear preference intensity, transitive constant preference intensity and four choice linear preference intensity with constant preference intensity.*

As we have argued above, all of these axioms are consequences of assuming that the preference intensities between every two choices change linearly with the belief. When viewed in this light, the result above states that expected utility may be seen as an expression of linear preference intensity.

2.10.4 Verifying the Axioms

We have seen in Theorem 2.10.1 that a conditional preference relation has an expected utility representation precisely when it can be extended to a *signed* conditional preference relation that satisfies a list of axioms. This result can be used for two purposes: First, as we have done above, we can use it to show that a given conditional preference relation has an expected utility representation. But we can also use it to prove that a given conditional preference relation $\succsim$ does *not* have an expected utility representation. Indeed, if we show that every signed conditional preference relation $\succsim^*$ that extends $\succsim$ violates at least one of the axioms, then we know that $\succsim$ cannot have an expected utility representation. Moreover, the axiom, or axioms, that are violated tell us what is "wrong" with the conditional preference relation at hand.

As an example, consider the conditional preference relation $\succsim$ in Figure 2.10.9. We will show that every signed conditional preference relation $\succsim^*$ that extends $\succsim$ must necessarily violate some of the axioms.

To see why, note that the indifference sets in $\succsim^*$ between *house* and *garden,* between *house* and *square,* between *garden* and *tent* and between *tent* and *square* are uniquely given by the corresponding indifference sets in $\succsim$, inside the belief triangle. Hence, these four indifference sets in $\succsim^*$ must be given by the corresponding dashed lines in Figure 2.10.10. Now, consider the signed belief q_1 in Figure 2.10.10, where you are "indifferent" between *house* and *square,* and "indifferent" between *square* and *tent.* By transitivity of $\succsim^*$, you must then also be indifferent between *house* and *tent* at q_1. In a

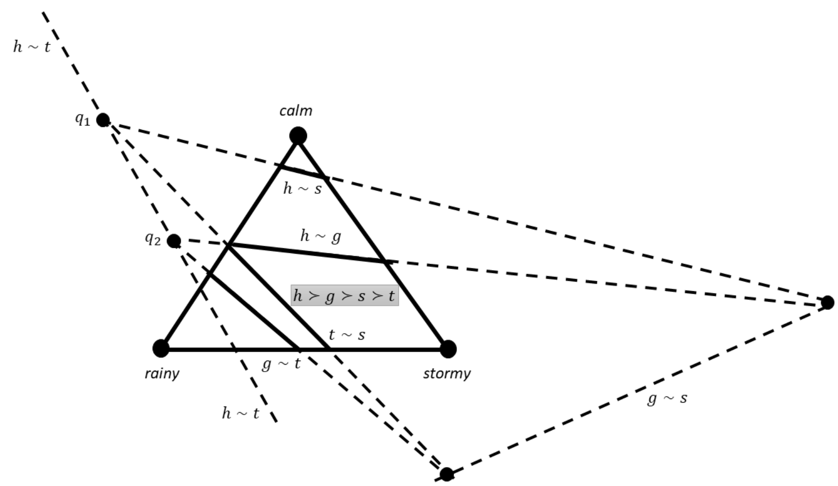

Figure 2.10.10 Verifying the axioms

similar way, we can conclude that at the signed belief q_2, you must also be "indifferent" between *house* and *tent.*

Hence, by preservation of indifference, the "indifference set" in $\succsim^*$ between *house* and *tent* must be a line that passes through the signed beliefs q_1 and q_2. But then, as can be seen from Figure 2.10.10, this indifference set will pass through the belief triangle. In other words, there must be "real" beliefs for which you are indifferent between *house* and *tent.* This, however, contradicts the conditional preference relation $\succsim$ we started from, since under $\succsim$ you will always prefer *house* to *tent* for every belief.

We thus see that there is no signed conditional preference relation $\succsim^*$ that extends $\succsim$ and satisfies all the axioms from Theorem 2.10.1. By the same theorem we can then conclude that the conditional preference relation in Figure 2.10.9 has no expected utility representation.

2.11 Economic Applications

In this section we discuss two economic applications of our approach to decision theory – one from consumer theory and one from producer theory.

2.11.1 Consumption under Uncertainty

Consider a consumer who must decide whether he wants to buy one unit from good a or one unit from good b. The consumer knows the quality of good a, because he has purchased this good before, but is uncertain about the quality of good b. Assume that the quality of good b can either be *poor, medium* or *good.* These are the three *states* in this scenario. The conditional preference relation of the consumer is depicted in Figure 2.11.1. Qualitatively speaking, the consumer would only consider buying good b if he assigns a sufficiently high probability to its quality being *good.*

We will now use the utility design procedure from Section 2.4.1 to verify whether the conditional preference relation has an expected utility representation, and if so, how such an expected utility

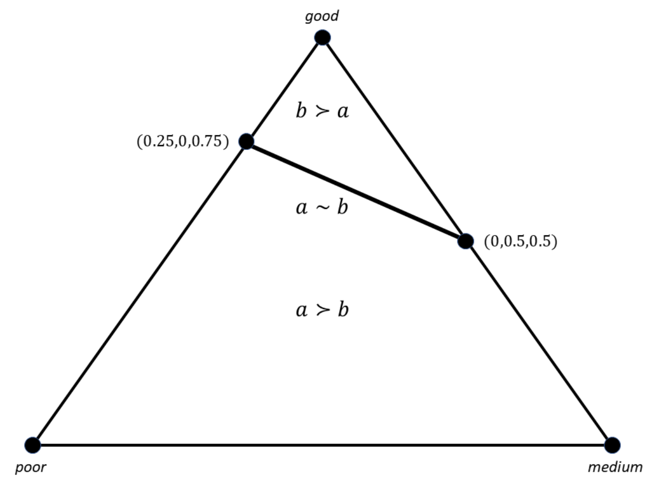

Figure 2.11.1 Conditional preference relation of consumer who is uncertain about quality of good b

representation would look like. Suppose we set all utilities for good a equal to 3, that is, $u(a,\, poor) = u(a,\, medium) = u(a,\, good) = 3$. Moreover, at the state *good,* where the consumer prefers b to a, we set $u(b,\, good) = 4$.

To compute the utility $u(b\,,\ poor)$, we apply the utility difference property to the belief $p_1 = (0.25, 0, 0.75)$ on the line segment between *poor* and *good,* and obtain that

$$\frac{u(a,\ poor) - u(b,\ poor)}{u(b,\ good) - u(a,\ good)} = \frac{p_1(good)}{p_1(poor)} = \frac{0.75}{0.25} = 3.$$

By filling in the utilities that have already been determined, we get

$$\frac{3 - u(b,\ poor)}{4 - 3} = 3 - u(b,\ poor) = 3,$$

and hence $u(b,\, poor) = 0$.

We finally compute the utility $u(b\,,\ medium)$ by applying the utility difference property to the belief $p_2 = (0, 0.5, 0.5)$ on the line segment between *medium* and *good,* and obtain that

$$\frac{u(a,\ medium) - u(b,\ medium)}{u(b,\ good) - u(a,\ good)} = \frac{p_2(good)}{p_2(medium)} = \frac{0.5}{0.5} = 1.$$

If we fill in the utilities that have already been determined, we get

$$\frac{3 - u(b,\ medium)}{4 - 3} = 3 - u(b,\ medium) = 1,$$

and hence $u(b,\, medium) = 2$.

We thus obtain the utility function u given by Table 2.11.1. It may be verified that this utility function indeed represents the consumer's conditional preference relation.

As there are preference reversals between the two choices a and b, we conclude from Theorem 2.5.1 in the book that the utility differences are unique up to a positive multiplicative constant. As a consequence, the consumer's relative preference intensities are unique, and these are given by the utility function from Table 2.11.1.

	poor	*medium*	*good*
good a	3	3	3
good b	0	2	4

Table 2.11.1 Expected utility representation for the consumer's conditional preference relation

Note that at the three states *poor, medium* and *good,* the utility differences between goods a and b are $3, 1$ and -1, respectively. As such, we conclude that the intensity by which the consumer prefers good a to good b is three times as large when the quality of good b is *poor*, compared to the case where the quality of good b is *medium.* This makes perfect intuitive sense. Also, the intensity by which the consumer prefers good a to good b when b's quality is *medium* is the same as the intensity by which he prefers good b to good a when b's quality is *good.*

2.11.2 Production under Uncertainty

In this chapter so far we have focused on scenarios where there are *finitely* many choices and states. As we will see, the notions of *conditional preference relation* and *expected utility representation* can naturally be extended to cases where there are infinitely many choices and states.

Consider a monopolist who must decide which price to charge for the good it is offering. The problem, however, is that the monopolist is uncertain about the *price elasticity of demand.* More precisely, if the monopolist chooses a price p, then the demand for the good is equal to $a - e \cdot p$, where $a > 0$ is known but e is unknown to the monopolist. The number e determines how quickly the demand for the good drops if the monopolist increases its price, and can thus be viewed as a measure for the price elasticity of demand. From now on, we will refer to e as the *elasticity parameter.* Suppose that there are no fixed costs, and the marginal cost of the monopolist is constant, and equal to $c > 0$.

As the monopolist can choose any price $p \geq 0$, the set of possible choices is infinite. Assume that it is known that the elasticity parameter e is in the interval $[e_1, e_2]$, where $e_1 > 0$, $e_2 > e_1$ and

$$e_2 < \frac{2a}{c + a/e_1}. \tag{2.11.1}$$

Then, the set of states is the interval $[e_1, e_2]$, which is also an infinite set. Still, the monopolist is able to form a belief β about the state.

Suppose that the monopolist holds the following conditional preference relation $\succsim$: For every belief β about the elasticity parameter, he prefers price p_1 to price p_2 precisely when the expected profit induced by the price p_1 and the belief β is greater than the expected profit induced by p_2 and β. What would then be the optimal price for the monopolist, for every possible belief β about the elasticity parameter?

To answer this question, let us first determine the expected profit $\pi(p, \beta)$ induced by a price p and a belief β. Suppose that the elasticity parameter is e, and that the monopolist chooses the price p. Then, the demand for the monopolist will be $a - e \cdot p$, and hence the total revenue, which is equal to the price times the demand, will be $p \cdot (a - e \cdot p)$. Since there are no fixed costs, and the marginal costs are constant and equal to c, the total costs will be the marginal cost times the demand, which is $c \cdot (a - e \cdot p)$. The total profit, given by the total revenue minus the total costs, will therefore be given by

$$\pi(p, e) = p \cdot (a - e \cdot p) - c \cdot (a - e \cdot p) = (p - c) \cdot (a - e \cdot p). \tag{2.11.2}$$

Assume now that the monopolist has a belief β about the elasticity parameter. To keep things easy, suppose that the belief β only assigns positive probability to finitely many states. By $supp(\beta)$ we denote the finite set of elasticity parameters that receive positive probability by β. This set is called the *support* of the belief β. The *expected* profit induced by the price p and the belief β is then given by

$$\pi(p,\beta) = \sum_{e \in supp(\beta)} \beta(e) \cdot \pi(p,e).$$

Hence, the monopolist's conditional preference relation $\succsim$ is such that for every belief β,

$$p_1 \succsim_\beta p_2 \text{ precisely when } \pi(p_1,\beta) \geq \pi(p_2,\beta).$$

This means that $\succsim$ has an expected utility representation u, given by

$$u(p,e) := \pi(p,e)$$

for every choice $p \geq 0$ and every state $e \in [e_1, e_2]$.

We will now compute, for every belief β, the optimal price for the monopolist. This will be precisely the price that maximizes the expected profit under β. In view of (2.11.2) this expected profit, for every price p, is given by

$$\begin{aligned} \pi(p,\beta) &= \sum_{e \in supp(\beta)} \beta(e) \cdot \pi(p,e) = \sum_{e \in supp(\beta)} \beta(e) \cdot [(p-c) \cdot (a - e \cdot p)] \\ &= (p-c) \cdot (a - [\sum_{e \in supp(\beta)} \beta(e) \cdot e] \cdot p). \end{aligned}$$

The expression $\sum_{e \in supp(\beta)} \beta(e) \cdot e$ has a clear interpretation: It is the *expected* elasticity parameter under the belief β. If we denote it by $E_\beta(e)$, then we conclude from above that

$$\pi(p,\beta) = (p-c) \cdot (a - E_\beta(e) \cdot p).$$

Thus, the expected profit is obtained if in the profit function $\pi(p,e)$ we replace the elasticity parameter e by the *expected* elastiscity parameter $E_\beta(e)$.

For a fixed belief β, the expected profit $\pi(p,\beta)$ is a second degree polynomial in p that becomes zero for $p = c$ and $p = a/E_\beta(e)$, and that obtains a maximum precisely in the middle between c and $a/E_\beta(e)$. Thus, the expected profit is maximized for the price

$$p^*(\beta) = \frac{1}{2} \cdot c + \frac{1}{2} \cdot \frac{a}{E_\beta(e)}. \tag{2.11.3}$$

In other words, for every belief β about the elasticity parameter, the optimal price for the monopolist is given by (2.11.3).

We check that for this optimal price the demand will always be positive, as it should be, irrespective of the value of e. By definition, the demand at the optimal price $p^*(\beta)$ for a given elasticity parameter e is

$$a - e \cdot p^*(\beta) = a - e \cdot [\frac{1}{2} \cdot c + \frac{1}{2} \cdot \frac{a}{E_\beta(e)}].$$

Recall that e lies in the interval $[e_1, e_2]$, and hence $e \leq e_2$ and $E_\beta(e) \geq e_1$. In view of the above, the demand at $p^*(\beta)$ and e is then

$$a - e \cdot [\frac{1}{2} \cdot c + \frac{1}{2} \cdot \frac{a}{E_\beta(e)}] \geq a - e_2 \cdot [\frac{1}{2} \cdot c + \frac{1}{2} \cdot \frac{a}{e_1}].$$

By the assumption (2.11.1) above, this demand will then be at least

$$a - e_2 \cdot [\frac{1}{2} \cdot c + \frac{1}{2} \cdot \frac{a}{e_1}] > a - \frac{2a}{c + a/e_1} \cdot [\frac{1}{2} \cdot c + \frac{1}{2} \cdot \frac{a}{e_1}] = 0.$$

Hence, the demand will always be greater than zero, no matter which value e takes. As such, the optimal price $p^*(\beta)$ in (2.11.3) is justified.

Note that the optimal price in (2.11.3) is *decreasing* in the expected value of the elasticity parameter. This makes intuitive sense: If the expected elasticity parameter rises, then the monopolist believes that, in expectation, a rise in the price will lead to a larger drop in demand. To compensate for this, the monopolist will end up charging a lower price than before.

2.12 Proofs

2.12.1 Proof for Section 2.8

In this subsection we will prove Theorem 2.8.1 for two choices. Before doing so, we first derive three preparatory results. The first characterizes the span of the set of beliefs where the DM is indifferent between a and b. Let $P_{a\sim b}$ be the set of beliefs p where the DM is indifferent between a and b.

Lemma 2.12.1 (Span of an indifference set) *Consider a conditional preference relation $\succsim$ that satisfies preservation of indifference, and two choices a and b. Then,*

$$span(P_{a\sim b}) = \{\lambda_1 \cdot p_1 + \lambda_2 \cdot p_2 \mid p_1, p_2 \in P_{a\sim b} \text{ and } \lambda_1, \lambda_2 \in \mathbf{R}\}.$$

Proof. Let

$$A := \{\lambda_1 \cdot p_1 + \lambda_2 \cdot p_2 \mid p_1, p_2 \in P_{a\sim b} \text{ and } \lambda_1, \lambda_2 \in \mathbf{R}\}.$$

We will show that $span(P_{a\sim b}) = A$. Clearly, $A \subseteq span(P_{a\sim b})$. Hence, it remains to show that $span(P_{a\sim b}) \subseteq A$.

Take some $p \in span(P_{a\sim b})$. Then, there are some beliefs $p_1, ..., p_k, p_{k+1}, ..., p_{k+m} \in P_{a\sim b}$ and numbers $\lambda_1, ..., \lambda_k, \lambda_{k+1}, ..., \lambda_{k+m} > 0$ such that

$$p = \lambda_1 p_1 + ... + \lambda_k p_k - \lambda_{k+1} p_{k+1} - ... - \lambda_{k+m} p_{k+m}. \tag{2.12.1}$$

Let $\alpha_1 := \lambda_1 + ... + \lambda_k$ and $\alpha_2 := \lambda_{k+1} + ... + \lambda_{k+m}$. If $\alpha_1 > 0$ and $\alpha_2 > 0$, then define the vectors

$$q_1 := \frac{\lambda_1}{\alpha_1} p_1 + ... + \frac{\lambda_k}{\alpha_1} p_k \text{ and } q_2 := \frac{\lambda_{k+1}}{\alpha_2} p_{k+1} + ... + \frac{\lambda_{k+m}}{\alpha_2} p_{k+m}.$$

It may be verified that q_1 and q_2 are convex combinations of beliefs in $P_{a\sim b}$. Hence, by repeatedly using preservation of indifference, it follows that $q_1, q_2 \in P_{a\sim b}$. By (2.12.1) we have that

$$p = \alpha_1 q_1 - \alpha_2 q_2,$$

and thus $p \in A$.

If $\alpha_1 > 0$ and $\alpha_2 = 0$, then we must have that $\lambda_{k+1} = ... = \lambda_{k+m} = 0$. We can define $q_1 \in P_{a\sim b}$ as above, and get $p = \alpha_1 q_1$. Thus, $p = \alpha_1 q_1 + 0 \cdot q_1$, which is in A. The case when $\alpha_1 = 0$ and $\alpha_2 > 0$ is similar. Finally, when $\alpha_1 = 0$ and $\alpha_2 = 0$, then $\lambda_1 = ... = \lambda_{k+m} = 0$, which means that $p = \underline{0}$. Thus, $p = 0 \cdot p_1 + 0 \cdot p_2$ for two arbitrary beliefs $p_1, p_2 \in P_{a\sim b}$, and hence $p \in A$.

In general, we thus see that every $p \in span(P_{a\sim b})$ is also in A, and thus $span(P_{a\sim b}) \subseteq A$. Together with the observation above that $A \subseteq span(P_{a\sim b})$, we conclude that $span(P_{a\sim b}) = A$. This completes the proof. ■

The second preparatory result contains some further properties of the set of beliefs where the DM is indifferent between a and b, gathered in Lemma 2.12.2. In this lemma, we denote by $S_{a\sim b}$ the set of states s where $a \sim_{[s]} b$. From now on, we often write $a \sim_s b$ instead of $a \sim_{[s]} b$. Recall, by Definition 2.2.1, that $\Delta(S)$ is the set of probability distributions on S, the set of states. That is, $\Delta(S)$ contains all possible beliefs.

Lemma 2.12.2 (Linear structure of indifference sets) *Suppose there are two choices, a and b, and n states. Consider a conditional preference relation $\succsim$ that satisfies the regularity axioms. Then, the following properties hold:*

(a) $P_{a\sim b} = span(P_{a\sim b}) \cap \Delta(S)$;

(b) if $\succsim$ has preference reversals between a and b, then $span(P_{a\sim b})$ is a hyperplane with dimension $n-1$, the beliefs $p_2, ..., p_n \in P_{a\sim b}$ selected by the utility design procedure in Section 2.4.1 of the book are linearly independent, and there is a full support belief $p \in P_{a\sim b}$ with $p(s) > 0$ for all $s \in S$;

(c) if a weakly dominates b under $\succsim$ then $P_{a\sim b} = \{p \in \Delta(S) \mid \sum_{s\in S_{a\sim b}} p(s) = 1\}$.

Proof. **(a)** Clearly, $P_{a\sim b} \subseteq span(P_{a\sim b}) \cap \Delta(S)$. It remains to show that $span(P_{a\sim b}) \cap \Delta(S) \subseteq P_{a\sim b}$.

Take some $p \in span(P_{a\sim b}) \cap \Delta(S)$. Then, by Lemma 2.12.1, there are beliefs $p_1, p_2 \in P_{a\sim b}$ and numbers λ_1, λ_2 such that

$$p = \lambda_1 p_1 + \lambda_2 p_2. \tag{2.12.2}$$

Since $p \in \Delta(S)$, we must have that $\sum_{s\in S} p(s) = 1$. Moreover, as p_1, p_2 are beliefs, it holds that $\sum_{s\in S} p_1(s) = \sum_{s\in S} p_2(s) = 1$. In view of (2.12.2),

$$1 = \sum_{s\in S} p(s) = \sum_{s\in S} (\lambda_1 p_1(s) + \lambda_2 p_2(s)) = \lambda_1 \left(\sum_{s\in S} p_1(s)\right) + \lambda_2 \left(\sum_{s\in S} p_2(s)\right) = \lambda_1 + \lambda_2.$$

Suppose first that $\lambda_1 = 0$. Then, $\lambda_2 = 1$, and hence $p = \lambda_2 p_2 = p_2$, which is in $P_{a\sim b}$. The case where $\lambda_2 = 0$ is similar.

Assume next that $\lambda_1, \lambda_2 > 0$. As $\lambda_1 + \lambda_2 = 1$, it follows from (2.12.2) that p is a convex combination of p_1 and p_2, which are both in $P_{a\sim b}$. By preservation of indifference, it follows that $p \in P_{a\sim b}$.

Suppose now that $\lambda_1 > 0$ and $\lambda_2 < 0$. Since $\lambda_1 + \lambda_2 = 1$, it must be that $\lambda_1 > 1$. Then, it follows from (2.12.2) that

$$p_1 = \frac{1}{\lambda_1} p - \frac{\lambda_2}{\lambda_1} p_2 = \frac{1}{\lambda_1} p + (1 - \frac{1}{\lambda_1}) p_2 \tag{2.12.3}$$

since $\lambda_2 = 1 - \lambda_1$. As $\lambda_1 > 1$, it follows that p_1 is a convex combination of p and p_2, where p_1 and p_2 are both in $P_{a\sim b}$.

We will show that p must be in $P_{a\sim b}$. Suppose, on the contrary, that $p \notin P_{a\sim b}$. Assume, without loss of generality, that $p \in P_{a\succ b}$, where $P_{a\succ b}$ is the set of beliefs q where $a \succ_q b$. Then, it follows from (2.12.3) and preservation of strict preference that $p_1 \in P_{a\succ b}$, which is a contradiction. Hence, $p \in P_{a\sim b}$.

The case where $\lambda_1 < 0$ and $\lambda_2 > 0$ is similar. In general, we conclude that every $p \in span(P_{a\sim b}) \cap \Delta(S)$ is also in $P_{a\sim b}$. Hence, $span(P_{a\sim b}) \cap \Delta(S) \subseteq P_{a\sim b}$. As we have already seen that $P_{a\sim b} \subseteq span(P_{a\sim b}) \cap \Delta(S)$, we have that $P_{a\sim b} = span(P_{a\sim b}) \cap \Delta(S)$.

(b) Suppose that $\succsim$ has preference reversals on $\{a, b\}$. Then, there must be a state x where $a \succ_x b$, and another state y where $b \succ_y a$. Indeed, assume that this would not be the case. Then, either $a \succsim_x b$ for all states x, or $b \succsim_x a$ for all states x. Assume, without loss of generality, that $a \succsim_x b$ for all states x. Then, it follows by preservation of indifference and preservation of strict preference that $a \succsim_p b$ for all beliefs p. This would contradict the assumption that there are preference reversals between a and b. Hence, we conclude that there are states x, y with $a \succ_x b$ and $b \succ_y a$.

By continuity, there must then be a belief $p_2 = (1-\lambda_2)[x] + \lambda_2[y]$ on the line segment between $[x]$ and $[y]$ where $a \sim_{p_2} b$. By preservation of strict preference, the DM will prefer a to b at every belief strictly between $[x]$ and p_2 on this line, and will prefer b to a at every belief strictly between $[y]$ and p_2 on this line. Hence, p_2 is the *unique* belief on the line between $[x]$ and $[y]$ where $a \sim_{p_2} b$.

Now, let the remaining states be numbered $s_3, ..., s_n$ such that

$$\begin{gathered} a \succ_{s_k} b \text{ for all } k \in \{3, ..., m\}, \\ b \succ_{s_k} a \text{ for all } k \in \{m+1, ..., m+l\}, \text{ and} \\ a \sim_{s_k} b \text{ for all } k \in \{m+l+1, ..., n\}. \end{gathered}$$

Following the utility design procedure of Section 2.4.1 in the book, we choose (i) for every $k \in \{3, ..., m\}$ the unique belief $p_k = (1-\lambda_k)[s_k] + \lambda_k[y]$ on the line segment between $[s_k]$ and $[y]$ with $a \sim_{p_k} b$, (ii) for every $k \in \{m+1, ..., m+l\}$ the unique belief $p_k = (1-\lambda_k)[s_k] + \lambda_k[x]$ on the line segment between $[s_k]$ and $[x]$ with $a \sim_{p_k} b$, and (iii) for every $k \in \{m+l+1, ..., n\}$ the belief $p_k = [s_k\}$ with $a \sim_{p_k} b$.

We will now show that $p_2, ..., p_n$ are linearly independent. Take some numbers $\alpha_2, ..., \alpha_n$ such that

$$\sum_{k=2}^{n} \alpha_k \cdot p_k = \underline{0}.$$

By construction, this sum is equal to

$$\begin{aligned} &\alpha_2((1-\lambda_2)[x] + \lambda_2[y]) + \sum_{k=3}^{m} \alpha_k((1-\lambda_k)[s_k] + \lambda_k[y]) + \\ &+ \sum_{k=m+1}^{m+l} \alpha_k((1-\lambda_k)[s_k] + \lambda_k[x]) + \sum_{k=m+l+1}^{n} \alpha_k[s_k] \\ &= \left(\alpha_2(1-\lambda_2) + \sum_{k=m+1}^{m+l} \alpha_k\lambda_k\right)[x] + \left(\alpha_2\lambda_2 + \sum_{k=3}^{m} \alpha_k\lambda_k\right)[y] \\ &+ \sum_{k=3}^{m+l} \alpha_k(1-\lambda_k)[s_k] + \sum_{k=m+l+1}^{n} \alpha_k[s_k] = \underline{0}. \end{aligned}$$

As the vectors $[x], [y], [s_3], ..., [s_n]$ are linearly independent, and $0 < \lambda_k < 1$ for all $k \in \{2, ..., n\}$, it follows that $\alpha_k = 0$ for all $k \in \{3, ..., n\}$. This, in turn, implies that also $\alpha_2 = 0$. Hence, the indifference beliefs $p_2, ..., p_n \in P_{a\sim b}$ are linearly independent.

As a consequence, the dimension of $span(P_{a\sim b})$ is at least $n-1$. The dimension of $span(P_{a\sim b})$ cannot be n, since otherwise we would have that $span(P_{a\sim b}) = \mathbf{R}^S$, and hence, by (a), $P_{a\sim b} = \mathbf{R}^S \cap \Delta(S) = \Delta(S)$. This would contradict the assumption that there are preference reversals between a and b. We thus conclude that the dimension of $span(P_{a\sim b})$ must be $n-1$, and therefore $span(P_{a\sim b})$ is a hyperplane.

To show that $P_{a\sim b}$ contains a belief p with $p(s) > 0$ for every state s, consider the vector

$$p := \tfrac{1}{n-1}p_2 + ... + \tfrac{1}{n-1}p_n.$$

It may be verified that p is a belief. Moreover, by construction of the beliefs $p_2, ..., p_n$, we have that $p(s) > 0$ for all states s.

(c) Let $A = \{p \in \Delta(S) \mid \sum_{s \in S_{a\sim b}} p(s) = 1\}$. To show that $P_{a\sim b} \subseteq A$, take some $p \in P_{a\sim b}$. Assume, contrary to what we want to show, that $p \notin A$. Then, $p(s) > 0$ for some $s \in S_{a\succ b}$, where $S_{a\succ b}$ is the set of states t with $a \succ_t b$. As $p = \sum_{s \in S_{a \sim b}} p(s) \cdot [s] + \sum_{s \in S_{a \succ b}} p(s) \cdot [s]$, it follows by preservation of indifference and preservation of strict preference that $p \in P_{a \succ b}$. This is a contradiction to the assumption that $p \in P_{a \sim b}$. We thus conclude that $p \in A$. Hence, $P_{a \sim b} \subseteq A$. The inclusion $A \subseteq P_{a \sim b}$ follows directly by preservation of indifference. We thus see that $P_{a \sim b} = A$. This completes the proof.■

The third preparatory result provides sufficient conditions for an expected utility representation between two choices.

Lemma 2.12.3 (Sufficient conditions for expected utility representation) *Consider a conditional preference relation $\succsim$ that satisfies the regularity axioms, two choices a and b, and a utility function u. Suppose that $\succsim$ has preference reversals between a and b, and that there are n states. If there is a belief p^* with $a \succ_{p^*} b$ and $u(a, p^*) > u(b, p^*)$, and $n - 1$ linearly independent beliefs $p_1, ..., p_{n-1}$ with $a \sim_{p_k} b$ and $u(a, p_k) = u(b, p_k)$ for all $k \in \{1, ..., n-1\}$, then u represents $\succsim$ on $\{a, b\}$.*

Proof. Let $P_{u(a)=u(b)}$ be the set of beliefs p with $u(a,p) = u(b,p)$. Moreover, let $P_{a \succ b}$ be the set of beliefs p with $a \succ_p b$, and $P_{u(a)>u(b)}$ the set of beliefs p where $u(a,p) > u(b,p)$. To show that u represents $\succsim$ on $\{a, b\}$, it is thus sufficient to show that $P_{a \sim b} = P_{u(a)=u(b)}$ and $P_{a \succ b} = P_{u(a)>u(b)}$.

We start by showing that $P_{a \sim b} = P_{u(a)=u(b)}$. For every vector $v \in \mathbf{R}^S$, define the "expected utility"

$$u(a, v) := \sum_{s \in S} v(s) \cdot u(a, s),$$

and similarly for $u(b, v)$. Consider the set $V_{u(a)=u(b)} := \{v \in \mathbf{R}^S \mid u(a, v) = u(b, v)\}$. It may be verified that $V_{u(a)=u(b)}$ is a linear space. Moreover, $P_{u(a)=u(b)} = V_{u(a)=u(b)} \cap \Delta(S)$.

We now show that $span(P_{a \sim b}) = V_{u(a)=u(b)}$. We first prove that $span(P_{a \sim b}) \subseteq V_{u(a)=u(b)}$. In Lemma 2.12.2 (b) we have seen that $span(P_{a \sim b})$ has dimension $n - 1$. Since the beliefs $p_1, ..., p_{n-1}$ in $span(P_{a \sim b})$ are linearly independent, we conclude that $\{p_1, ..., p_{n-1}\}$ is a basis of $span(P_{a \sim b})$. Take some $v \in span(P_{a \sim b})$. Then, we can write

$$v = \lambda_2 p_2 + ... + \lambda_n p_n$$

for some numbers $\lambda_2, ..., \lambda_n$. Since $u(a, p_k) = u(b, p_k)$ for all $k \in \{1, ...n - 1\}$, it follows that

$$u(a, v) - u(b, v) = \sum_{k=2}^{n} \lambda_k \cdot (u(a, p_k) - u(b, p_k)) = 0,$$

and hence $v \in V_{u(a)=u(b)}$. Thus, $span(P_{a \sim b}) \subseteq V_{u(a)=u(b)}$.

We next show that $V_{u(a)=u(b)} \subseteq span(P_{a \sim b})$. Since $V_{u(a)=u(b)}$ is a linear subspace of $\mathbf{R}^S$, its dimension can be at most n. Moreover, as $span(P_{a \sim b}) \subseteq V_{u(a)=u(b)}$ and $span(P_{a \sim b})$ has dimension $n - 1$, the dimension of $V_{u(a)=u(b)}$ is at least $n - 1$. Suppose, contrary to what we want to prove, that $V_{u(a)=u(b)}$ is not a subset of $span(P_{a \sim b})$. Then, the dimension of $V_{u(a)=u(b)}$ must be n, and hence $V_{u(a)=u(b)} = \mathbf{R}^S$. However, this is a contradiction since $u(a, p^*) > u(b, p^*)$, and hence $p^* \notin V_{u(a)=u(b)}$. We thus conclude that $V_{u(a)=u(b)} \subseteq span(P_{a \sim b})$. Since we have already seen that $span(P_{a \sim b}) \subseteq V_{u(a)=u(b)}$, it follows that $span(P_{a \sim b}) = V_{u(a)=u(b)}$.

Since $P_{u(a)=u(b)} = V_{u(a)=u(b)} \cap \Delta(S)$ and, by Lemma 2.12.2 (a), $P_{a \sim b} = span(P_{a \sim b}) \cap \Delta(S)$, we conclude that $P_{a \sim b} = P_{u(a)=u(b)}$.

We next prove that $P_{a \succ b} = P_{u(a)>u(b)}$. Let p^* be the belief where $a \succ_{p^*} b$ and $u(a, p^*) > u(b, p^*)$. Consider the set

$$A := \{p \in \Delta(S) \mid \text{there is no } \lambda \in [0, 1] \text{ with } (1 - \lambda)p + \lambda p^* \in P_{a \sim b}\}.$$

We show that $P_{a\succ b} = A$. To prove that $P_{a\succ b} \subseteq A$, take some $p \in P_{a\succ b}$. Since $p^* \in P_{a\succ b}$ it follows by preservation of strict preference that $(1-\lambda)p + \lambda p^* \in P_{a\succ b}$ for every $\lambda \in [0,1]$, and hence $p \in A$. Thus, $P_{a\succ b} \subseteq A$.

To show that $A \subseteq P_{a\succ b}$, take some $p \in A$. Suppose that $p \notin P_{a\succ b}$. Since $p \in A$, we must have that $p \notin P_{a\sim b}$, and hence $p \in P_{b\succ a}$. By continuity, there must then be some $\lambda \in (0,1)$ with $(1-\lambda)p + \lambda p^* \in P_{a\sim b}$. This, however, contradicts the assumption that $p \in A$. Hence, $p \in P_{a\succ b}$, which yields $A \subseteq P_{a\succ b}$. Altogether, we conclude that $P_{a\succ b} = A$.

We next show that $P_{u(a)>u(b)} = A$. Since $P_{a\sim b} = P_{u(a)=u(b)}$, it follows that

$$A = \{p \in \Delta(S) \mid \text{there is no } \lambda \in [0,1] \text{ with } (1-\lambda)p + \lambda p^* \in P_{u(a)=u(b)}\}.$$

As $p^* \in P_{u(a)>u(b)}$ by construction, it can be shown in a similar same way as above that $P_{u(a)>u(b)} = A$. As such, $P_{a\succ b} = A = P_{u(a)>u(b)}$.

Since $P_{a\sim b} = P_{u(a)=u(b)}$ and $P_{a\succ b} = P_{u(a)>u(b)}$, we conclude that $u(a,p) \geq u(b,p)$ if and only if $a \succsim_p b$. Hence, the utility function u represents $\succsim$ on $\{a,b\}$. This completes the proof. ■

We are now ready to prove Theorem 2.8.1.

Proof of Theorem 2.8.1. Suppose first that $\succsim$ has an expected utility representation. Then, it has been shown in Section 2.8.1 that $\succsim$ satisfies continuity, preservation of indifference and preservation of strict preference.

Assume next that $\succsim$ satisfies continuity, preservation of indifference and preservation of strict preference. We will show that $\succsim$ has an expected utility representation. We distinguish four cases: (a) there are preference reversals between a and b, (b) a weakly dominates b, (c) b weakly dominates a, and (d) a and b are equivalent, meaning that $a \sim_s b$ for all states s. For the remainder of this proof, we assume that the number of states is n.

(a) Suppose that there are preference reversals between a and b. Let the states x and y be such that $a \succ_x b$ and $b \succ_y a$, and use the utility design procedure from Section 2.4.1 in the book to generate utilities $u(a,s)$ and $u(b,s)$ for every state s. Recall that the procedure is based on the selection of specific beliefs $p_2, ..., p_n \in P_{a\sim b}$. Then, by construction of the procedure, we have that $u(b,y) > u(a,y)$ and $u(b,p_k) = u(a,p_k)$ for all $k \in \{2,...,n\}$. Moreover, we know from Lemma 2.12.2 (b) that these $n-1$ beliefs $p_2, ..., p_n$ are linearly independent. By Lemma 2.12.3 it thus follows that the utility function u generated by the utility design procedure represents $\succsim$.

(b) Suppose that a weakly dominates b. Choose a utility function u such that, for every state s, we have $u(a,s) > u(b,s)$ when $[s] \in P_{a\succ b}$, and $u(a,s) = u(b,s)$ when $[s] \in P_{a\sim b}$. As, by Lemma 2.12.2 (c),

$$P_{a\sim b} = \{p \in \Delta(S) \mid \sum_{s \in S_{a\sim b}} p(s) = 1\}$$

it follows that $P_{a\sim b} = P_{u(a)=u(b)}$. Since every belief p is either in $P_{a\sim b}$ or $P_{a\succ b}$, it follows that $P_{a\succ b} = P_{u(a)>u(b)}$. We thus conclude that the utility function u represents $\succsim$.

(c) This proof is similar to that for (b).

(d) Suppose that a and b are equivalent. Then, any utility function u with $u(a,s) = u(b,s)$ for every state s will represent $\succsim$. This completes the proof. ■

2.12.2 Proofs for Section 2.9

In this subsection we will prove Proposition 2.9.1 and Theorem 2.9.1. Before we can prove Proposition 2.9.1 we need the result below. In the statement, a *full support* belief is a belief p with $p(s) > 0$ for all states s.

Lemma 2.12.4 (Line containing three indifference beliefs) *Consider a conditional preference relation $\succsim$ that has preference reversals for all pairs of choices, and satisfies the regularity axioms and transitivity. Then, for every three choices a, b, c, there is a line of beliefs that contains full support beliefs p_{ab}, p_{ac}, p_{bc} where the DM is indifferent between the respective choices.*

Proof. Suppose first that there is a full support belief $p \in P_{a\sim b} \cap P_{b\sim c}$. Then, by transitivity, $p \in P_{a\sim c}$. We can then choose a line of beliefs through p. Such a line will satisfy the statement in the lemma.

Assume next that there is no full support belief in $P_{a\sim b} \cap P_{b\sim c}$. By transitivity, there will be no full support belief in $P_{a\sim b} \cap P_{a\sim c}$ or $P_{b\sim c} \cap P_{a\sim c}$ either. Let $\Delta^+(S)$ be the set of full support beliefs. Then, the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ will be pairwise disjoint on $\Delta^+(S)$. As, by Lemma 2.12.2 (a), these indifference sets are the intersections of hyperplanes with $\Delta(S)$, it must be that one of these indifference sets is "in between" the other two within $\Delta^+(S)$. Suppose, without loss of generality, that $P_{b\sim c}$ is in between $P_{a\sim b}$ and $P_{a\sim c}$. By Lemma 2.12.2 (b), there is a full support belief $p_{ab} \in P_{a\sim b}$ and a full support belief $p_{ac} \in P_{a\sim c}$. Let l be the line of beliefs that goes through p_{ab} and p_{ac}. As the set $P_{b\sim c}$ is in between $P_{a\sim b}$ and $P_{a\sim c}$, there must be a belief $p_{bc} \in P_{b\sim c}$ on the line l between p_{ab} and p_{ac}. Moreover, p_{bc} is a full support belief, since p_{ab} and p_{ac} are full support beliefs. The line l thus satisfies the requirements of the lemma. This completes the proof. ■

We are now ready to prove Proposition 2.9.1.

Proof of Proposition 2.9.1. Consider a conditional preference relation $\succsim$ that has preference reversals on every pair of choices, and satisfies the regularity axioms and transitivity.

(a) Assume first that $\succsim$ satisfies three choice linear preference intensity. Consider three choices a, b and c. We must show that $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle \subseteq \langle P_{a\sim c}\rangle$. Take some $q \in \langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$. By Lemma 2.12.4 there is a line l containing full support beliefs $p_{ab} \in P_{a\sim b}, p_{bc} \in P_{b\sim c}$ and $p_{ac} \in P_{a\sim c}$. Then, there is some $\varepsilon \in (0, 1)$ small enough such that (i) the vectors $p'_{ab} := (1-\varepsilon)p_{ab} + \varepsilon q$, $p'_{bc} := (1-\varepsilon)p_{bc} + \varepsilon q$ and $p' := (1-\varepsilon)p_{ac} + \varepsilon q$ are all in $\Delta(S)$, and (ii) the line l' through p'_{ab} and p'_{bc} contains a belief $p'_{ac} \in P_{a\sim c}$. Since $p'_{ab} - p'_{bc} = (1-\varepsilon)\cdot(p_{ab} - p_{bc})$, we conclude that the lines l and l' are parallel.

Moreover, the lines l and l' can be chosen such that they contain beliefs where the DM is not indifferent between any of the three choices. Hence, by preservation of strict preference, p'_{ac} is the unique belief in $P_{a\sim c}$ on the line l'. Also, the lines l and l' can be chosen such that the probability of no state is constant on l or l'.

As $q \in \langle P_{a\sim b}\rangle$ we know, in particular, that $q \in span(P_{a\sim b})$. Thus, we conclude that $p'_{ab} \in span(P_{a\sim b}) \cap \Delta(S)$. Since Lemma 2.12.2 (a) guarantees that $span(P_{a\sim b}) \cap \Delta(S) = P_{a\sim b}$, it follows that $p'_{ab} \in P_{a\sim b}$. As $q \in \langle P_{b\sim c}\rangle$ it can be shown, in a similar way, that $p'_{bc} \in P_{b\sim c}$.

We will now show that $p'_{ac} = p'$. Suppose first that $p_{ab} = p_{bc}$. Then, by transitivity, $p_{ac} = p_{ab} = p_{bc}$. Moreover, by definition of p'_{ab} and p'_{bc} it follows that $p'_{ab} = p'_{bc}$, and hence by transitivity we must have that $p'_{ac} = p'_{ab} = p'_{bc}$. Since

$$p' = (1-\varepsilon)p_{ac} + \varepsilon q = (1-\varepsilon)p_{ab} + \varepsilon q = p'_{ab},$$

we conclude that $p'_{ac} = p'$.

Suppose now that $p_{ab} \neq p_{bc}$. Then, by transitivity, the beliefs p_{ab}, p_{bc} and p_{ac} are pairwise different. By definition of p'_{ab} and p'_{bc}, we then have that $p'_{ab} \neq p'_{bc}$. Hence, by transitivity, the beliefs p'_{ab}, p'_{bc} and p'_{ac} are pairwise different.

By three choice linear preference intensity, we have for every state s that

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)). \tag{2.12.4}$$

Note that $p'_{ab}(s) = (1-\varepsilon)p_{ab}(s) + \varepsilon q(s)$ and $p'_{bc}(s) = (1-\varepsilon)p_{bc}(s) + \varepsilon q(s)$, which implies that

$$(p'_{ab}(s) - p'_{bc}(s)) = (1-\varepsilon)(p_{ab}(s) - p_{bc}(s)). \tag{2.12.5}$$

Recall that the beliefs p_{ab}, p_{bc} and p_{ac} are pairwise different, the beliefs p'_{ab}, p'_{bc} and p'_{ac} are pairwise different, and no state has constant probability on the lines l and l'. Hence, it follows from (2.12.4) and (2.12.5) that

$$(p'_{ac}(s) - p'_{bc}(s)) = (1-\varepsilon)(p_{ac}(s) - p_{bc}(s)),$$

and thus

$$\begin{aligned} p'_{ac}(s) &= (1-\varepsilon)(p_{ac}(s) - p_{bc}(s)) + p'_{bc}(s) \\ &= (1-\varepsilon)p_{ac}(s) + \varepsilon q(s) = p'(s). \end{aligned}$$

As this holds for every state s, we conclude that $p'_{ac} = p'$. Thus, the belief $p' = (1-\varepsilon)p_{ac} + \varepsilon q$ is in $P_{a\sim c}$. As such,

$$q = \frac{1}{\varepsilon}p' + (1 - \frac{1}{\varepsilon})p_{ac} \in \langle P_{a\sim c}\rangle .$$

As this holds for every $q \in \langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$, it follows that $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle \subseteq \langle P_{a\sim c}\rangle$.

(b) Suppose now that $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle \subseteq \langle P_{a\sim c}\rangle$ for all three choices a, b, c. We must show that $\succsim$ satisfies three choice linear preference intensity. Consider two parallel lines of beliefs l, l' that (i) contain beliefs where the DM is not indifferent between any two choices from $\{a, b, c\}$, (ii) where l contains indifference beliefs $p_{ab} \in P_{a\sim b}, p_{bc} \in P_{b\sim c}$ and $p_{ac} \in P_{a\sim c}$, and (iii) l' contains indifference beliefs $p'_{ab} \in P_{a\sim b}, p'_{bc} \in P_{b\sim c}$ and $p'_{ac} \in P_{a\sim c}$.

Let l_{ab} be the line through p_{ab} and p'_{ab}, let l_{bc} be the line through p_{bc} and p'_{bc}, and l_{ac} the line through p_{ac} and p'_{ac}. Note that all these lines belong to the same two-dimensional plane: the plane that goes through l and l'.

Assume first that the lines l_{ab}, l_{bc} and l_{ac} are all parallel. Then, there is a vector q such that

$$p'_{ab} = p_{ab} + q,\ p'_{bc} = p_{bc} + q \text{ and } p'_{ac} = p_{ac} + q.$$

As a consequence, for every state s,

$$\begin{aligned} (p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) &= (p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) \\ &= (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)). \end{aligned}$$

Hence, the formula for three choice linear preference intensity is satisfied.

Assume next that the lines l_{ab}, l_{bc} and l_{ac} are not all parallel. Without loss of generality, we suppose that l_{ab} and l_{bc} are not parallel. Since these two lines lie in the same two-dimensional plane, they must intersect at a unique vector q. Since q lies on l_{ab}, which goes through p_{ab} and p'_{ab} in $P_{a\sim b}$, we conclude that $q \in \langle P_{a\sim b}\rangle$. Similarly, as q lies on l_{bc}, which goes through p_{bc} and p'_{bc} in $P_{b\sim c}$, it follows that $q \in \langle P_{b\sim c}\rangle$. Since we assume that $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle \subseteq \langle P_{a\sim c}\rangle$, we conclude that $q \in \langle P_{a\sim c}\rangle$ too.

Let V be the two-dimensional plane that goes through the lines l and l'. Since, by condition (i) above, l and l' contain beliefs where the DM is not indifferent between a and c, it follows that $\langle P_{a\sim c}\rangle \cap V = l_{ac}$. As $q \in \langle P_{a\sim c}\rangle \cap V$, we conclude that q lies on the line l_{ac}.

As q lies on l_{ab}, l_{bc} and l_{ac}, the beliefs p_{ab}, p_{bc}, p_{ac} lie on l, the beliefs p'_{ab}, p'_{bc} and p'_{ac} lie on l', and the lines l and l' are parallel, there is a unique number λ such that

$$p'_{ab} = (1-\lambda)q + \lambda p_{ab},\ p'_{bc} = (1-\lambda)q + \lambda p_{bc} \text{ and } p'_{ac} = (1-\lambda)q + \lambda p_{ac}.$$

Hence, for every state s we have that

$$\begin{aligned}(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) &= \lambda \cdot (p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) \\ &= (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)).\end{aligned}$$

Thus, the formula for three choice linear preference intensity is satisfied.

We therefore conclude that $\succsim$ satisfies three choice linear preference intensity. This completes the proof. ■

Before we can prove Theorem 2.9.1, we need some preparatory results. The following lemma provides a connection between the linear extension and the span of an indifference set $P_{a\sim b}$.

Lemma 2.12.5 (Linear extension and span) *Consider a conditional preference relation $\succsim$ that satisfies preservation of indifference. Then, for every two choices a, b,*

$$\langle P_{a\sim b}\rangle = \{v \in span(P_{a\sim b}) \mid \sum_{s\in S} v(s) = 1\}.$$

Proof. Let $A := \{v \in span(P_{a\sim b}) \mid \sum_{s\in S} v(s) = 1\}$. We will prove that $\langle P_{a\sim b}\rangle = A$. To show that $\langle P_{a\sim b}\rangle \subseteq A$, take some $q \in \langle P_{a\sim b}\rangle$. Then, there are $p_1, p_2 \in P_{a\sim b}$ and some number λ such that $q = (1-\lambda)p_1 + \lambda p_2$. Clearly, $q \in span(P_{a\sim b})$. Moreover, since $\sum_{s\in S} p_1(s) = \sum_{s\in S} p_2(s) = 1$, it follows that $\sum_{s\in S} q(s) = 1$ also. Hence, $q \in A$.

To show that $A \subseteq \langle P_{a\sim b}\rangle$, take some $q \in A$. By Lemma 2.12.1, there are $p_1, p_1 \in P_{a\sim b}$ and numbers λ_1, λ_2 such that $q = \lambda_1 p_1 + \lambda_2 p_2$. As $\sum_{s\in S} p_1(s) = \sum_{s\in S} p_2(s) = 1$, it follows that $\sum_{s\in S} q(s) = \lambda_1 + \lambda_2$. Since $q \in A$, it must be that $\sum_{s\in S} q(s) = 1$, and hence $\lambda_1 + \lambda_2 = 1$. Thus, $\lambda_2 = 1 - \lambda_1$. But then, by definition, $q \in \langle P_{a\sim b}\rangle$.

We thus conclude that $\langle P_{a\sim b}\rangle \subseteq A$ and $A \subseteq \langle P_{a\sim b}\rangle$, which implies that $\langle P_{a\sim b}\rangle = A$. This completes the proof. ■

On the basis of Proposition 2.9.1 and Lemma 2.12.5 we can show the following characterization of three choice linear preference intensity, wich will be useful for proving Theorem 2.9.1.

Lemma 2.12.6 (Three choice linear preference intensity) *Suppose that the conditional preference relation $\succsim$ has preference reversals on every pair of choices, and satisfies the regularity axioms and transitivity. Then, $\succsim$ satisfies three choice linear preference intensity, if and only if, for every three choices a, b, c we have that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$.*

Proof. **(a)** Suppose first that $\succsim$ satisfies three choice linear preference intensity. We will show that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$. We distinguish two cases: (1) $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$ is not empty, and (2) $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$ is empty.

Case 1. Suppose that $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$ is not empty. Since $\succsim$ satisfies three choice linear preference intensity, we know by Proposition 2.9.1 that

$$\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle \subseteq \langle P_{a\sim c}\rangle . \tag{2.12.6}$$

To show that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$, take some $v \in span(P_{a\sim b}) \cap span(P_{b\sim c})$. Then, by Lemma 2.12.1, we have that

$$v = \lambda_1 p_1 + \lambda_2 p_2 = \mu_1 q_1 + \mu_2 q_2$$

for some $p_1, p_2 \in P_{a\sim b}$, $q_1, q_2 \in P_{b\sim c}$ and some numbers $\lambda_1, \lambda_2, \mu_1, \mu_2$. Note that

$$\sum_{s\in S} v(s) = \lambda_1 \left(\sum_{s\in S} p_1(s)\right) + \lambda_2 \left(\sum_{s\in S} p_2(s)\right) = \lambda_1 + \lambda_2,$$

since $\sum_{s\in S} p_1(s) = \sum_{s\in S} p_2(s) = 1$. Similarly, $\sum_{s\in S} v(s) = \mu_1 + \mu_2$, which yields

$$\lambda_1 + \lambda_2 = \mu_1 + \mu_2.$$

We distinguish the following cases: (1.1) $\sum_{s\in S} v(s) \neq 0$, and (1.2) $\sum_{s\in S} v(s) = 0$.

Case 1.1. Suppose first that $\sum_{s\in S} v(s) \neq 0$, which implies that $\lambda_1 + \lambda_2 \neq 0$. Then,

$$\frac{1}{\lambda_1 + \lambda_2} v = \frac{\lambda_1}{\lambda_1 + \lambda_2} p_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} p_2,$$

which means that $\frac{1}{\lambda_1+\lambda_2} v \in \langle P_{a\sim b}\rangle$. Since $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$, it follows in a similar way that $\frac{1}{\lambda_1+\lambda_2} v = \frac{1}{\mu_1+\mu_2} v \in \langle P_{b\sim c}\rangle$. Thus, $\frac{1}{\lambda_1+\lambda_2} v \in \langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$. By (2.12.6) we then conclude that $\frac{1}{\lambda_1+\lambda_2} v \in \langle P_{a\sim c}\rangle$, and hence $\frac{1}{\lambda_1+\lambda_2} v \in span(P_{a\sim c})$. This implies that $v \in span(P_{a\sim c})$ also.

Case 1.2. Suppose next that $\sum_{s\in S} v(s) = 0$. Since $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$ is not empty, we can take some $q \in \langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$. By (2.12.6) we then know that $q \in \langle P_{a\sim c}\rangle$, and hence, in particular, $q \in span(P_{a\sim c})$.

Choose some $\alpha \in (0,1)$, and let $q' := (1-\alpha)\cdot v + \alpha \cdot q$. As $v, q \in span(P_{a\sim b}) \cap span(P_{b\sim c})$, it follows that $q' \in span(P_{a\sim b}) \cap span(P_{b\sim c})$ also. Moreover, as $\sum_{s\in S} v(s) = 0$ and $\sum_{s\in S} q(s) = 1$, it follows that $\sum_{s\in S} q'(s) = \alpha > 0$. By Case 1.1, it thus follows that $q' \in span(P_{a\sim c})$. But then,

$$v = \frac{1}{1-\alpha} q' - \frac{\alpha}{1-\alpha} q \in span(P_{a\sim c})$$

since both q and q' are in $span(P_{a\sim c})$.

Case 2. Suppose that $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$ is empty. As, by Proposition 2.9.1, $\langle P_{a\sim b}\rangle \cap \langle P_{a\sim c}\rangle \subseteq \langle P_{b\sim c}\rangle$ and $\langle P_{a\sim c}\rangle \cap \langle P_{b\sim c}\rangle \subseteq \langle P_{a\sim b}\rangle$, it follows that $\langle P_{a\sim b}\rangle \cap \langle P_{a\sim c}\rangle$ and $\langle P_{a\sim c}\rangle \cap \langle P_{b\sim c}\rangle$ are empty as well.

Let $V_0 := \{v \in \mathbf{R}^X \mid \sum_{s\in S} v(s) = 0\}$ and $V_1 := \{v \in \mathbf{R}^X \mid \sum_{s\in S} v(s) = 1\}$. By Lemma 2.12.5 we know that $\langle P_{a\sim b}\rangle = span(P_{a\sim b}) \cap V_1$ and $\langle P_{b\sim c}\rangle = span(P_{b\sim c}) \cap V_1$, and hence $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle = span(P_{a\sim b}) \cap span(P_{b\sim c}) \cap V_1$. As $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$ is empty, we conclude that $span(P_{a\sim b}) \cap span(P_{b\sim c})$ has an empty intersection with V_1.

This implies, in turn that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq V_0$. To see this, assume to the contrary that there would be a vector $v \in span(P_{a\sim b}) \cap span(P_{b\sim c})$ with $\sum_{s\in S} v(s) = \alpha \neq 0$. Then, the vector $\frac{1}{\alpha} v$ would still be in $span(P_{a\sim b}) \cap span(P_{b\sim c})$, and $\sum_{s\in S} \frac{1}{\alpha} v(s) = 1$, so $v \in V_1$. But then,

$\frac{1}{\alpha}v \in span(P_{a\sim b}) \cap span(P_{b\sim c}) \cap V_1$, which is a contradiction since $span(P_{a\sim b}) \cap span(P_{b\sim c}) \cap V_1$ is empty. Thus, we know that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq V_0$.

On the basis of this fact, it can now be shown that $span(P_{a\sim b}) \cap span(P_{b\sim c}) = span(P_{a\sim b}) \cap V_0$. To see this, note first that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim b}) \cap V_0$, since $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq V_0$. Moreover, we also know that $span(P_{a\sim b}) \neq span(P_{b\sim c})$, since otherwise, by Lemma 2.12.5, $\langle P_{a\sim b}\rangle = \langle P_{b\sim c}\rangle$ and hence $\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle$ would not be empty, which would be a contradiction. Since, by Lemma 2.12.2 (b), $span(P_{a\sim b})$ and $span(P_{b\sim c})$ are linear subspaces of dimension $n-1$, it follows that $span(P_{a\sim b}) \cap span(P_{b\sim c})$ is a linear subspace of dimension $n-2$. Now, consider the linear subspace $span(P_{a\sim b}) \cap V_0$. Clearly, $span(P_{a\sim b}) \neq V_0$, since $span(P_{a\sim b})$ contains beliefs in $P_{a\sim b}$ which are not in V_0. Since $span(P_{a\sim b})$ and V_0 are linear subspaces of dimension $n-1$, it follows that $span(P_{a\sim b}) \cap V_0$ is a linear subspace of dimension $n-2$. Since $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim b}) \cap V_0$ and both linear subspaces have the same dimension, $n-2$, both spaces must be equal. Hence, $span(P_{a\sim b}) \cap span(P_{b\sim c}) = span(P_{a\sim b}) \cap V_0$.

Since we have seen above that also $\langle P_{a\sim b}\rangle \cap \langle P_{a\sim c}\rangle$ is empty, it can be shown in a similar way that $span(P_{a\sim b}) \cap span(P_{a\sim c}) = span(P_{a\sim b}) \cap V_0$. By combining the latter two equalities, we get

$$span(P_{a\sim b}) \cap span(P_{b\sim c}) = span(P_{a\sim b}) \cap V_0 = span(P_{a\sim b}) \cap span(P_{a\sim c}),$$

which implies that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$.

Since all cases have been covered, this completes part (a).

(b) Suppose now that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$ for all three choices a, b, c. Since, by Lemma 2.12.5, $\langle P_{a\sim b}\rangle = span(P_{a\sim b}) \cap V_1$, and similarly for $\langle P_{b\sim c}\rangle$ and $\langle P_{a\sim c}\rangle$, it follows that

$$\langle P_{a\sim b}\rangle \cap \langle P_{b\sim c}\rangle = span(P_{a\sim b}) \cap span(P_{b\sim c}) \cap V_1 \subseteq span(P_{a\sim c}) \cap V_1 = \langle P_{a\sim c}\rangle .$$

By Proposition 2.9.1 it follows that $\succsim$ satisfies three choice linear preference intensity. The proof is hereby complete. ■

In our last preparatory result, we characterize the span of an indifference set $P_{a\sim b}$ in case of an expected utility representation. We use the following notation: For a given utility function u, choice a and vector $q \in \mathbf{R}^S$, we denote by $u(a,q) := \sum_{s\in S} q(s) \cdot u(a,s)$ the "expected utility" induced by a at the vector q.

Lemma 2.12.7 (Span of indifference set under utility representation) *Consider a conditional preference relation $\succsim$ with an expected utility representation u. Suppose there are preferene reversals between choices a and b. Then,*

$$span(P_{a\sim b}) = \{q \in \mathbf{R}^S \mid u(a,q) = u(b,q)\}.$$

Proof. Let $A := \{q \in \mathbf{R}^S \mid u(a,q) = u(b,q)\}$. We first show that $span(P_{a\sim b}) \subseteq A$. Take some $q \in span(P_{a\sim b})$. Then, by Lemma 2.12.1, there are $p_1, p_2 \in P_{a\sim b}$ and numbers λ_1, λ_2 such that $q = \lambda_1 p_1 + \lambda_2 p_2$. As $u(a,p_1) = u(b,p_1)$ and $u(a,p_2) = u(b,p_2)$, it follows that

$$u(a,q) = \lambda_1 u(a,p_1) + \lambda_2 u(a,p_2) = \lambda_1 u(b,p_1) + \lambda_2 u(b,p_2) = u(b,q),$$

and hence $q \in A$. Thus, $span(P_{a\sim b}) \subseteq A$.

By Lemma 2.12.2 (b) we know that $span(P_{a\sim b})$ has dimension $n-1$. Since A is a linear subspace with dimension $n-1$ also, and $span(P_{a\sim b}) \subseteq A$, it must be that $span(P_{a\sim b}) = A$. This completes the proof. ■

We are now ready to prove Theorem 2.9.1.

Proof of Theorem 2.9.1. **(a)** Suppose first that $\succsim$ has an expected utility representation u. From Section 2.8.1 we know that $\succsim$ satisfies continuity, preservation of indifference and preservation of strict preference. To show transitivity, assume that $a \succsim_p b \succsim_p c$ for some belief p, and for some choices a, b and c. Then, we must have that $u(a,p) \geq u(b,p) \geq u(c,p)$, which implies that $u(a,p) \geq u(c,p)$, and hence $a \succsim_p c$.

To show three choice linear preference intensity it suffices, in view of Lemma 2.12.6, to show that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$ for all three choices a, b, c. Take some $q \in span(P_{a\sim b}) \cap span(P_{b\sim c})$. Then, by Lemma 2.12.7, we have that $u(a,q) = u(b,q)$ and $u(b,q) = u(c,q)$, and hence $u(a,q) = u(c,q)$. By the same Lemma 2.12.7 it thus follows that $q \in span(P_{a\sim c})$. Hence, $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$, which implies by Lemma 2.12.6 that $\succsim$ satisfies three choice linear preference intensity.

We finally show four choice linear preference intensity. Consider a line of beliefs l, and four choices a, b, c, d such that there is a belief on the line where the DM is not indifferent between any pair of choices in $\{a, b, c, d\}$. Moreover, let $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}$ and p_{cd} be corresponding indifference beliefs on this line. Consider some state s. If the probability of s is constant on the line l, then the formula for four choice linear preference intensity holds trivially.

We therefore assume from now on that the probability of s is not constant on l, so that every belief on l is uniquely given by the probability it assigns to s. Suppose that $p_{ab} = p_{ac}$. Then, by transitivity, it must be that $p_{ab} = p_{ac} = p_{bc}$, and the formula for four choice linear preference intensity would hold trivially. Similarly, the formula would trivially hold if $p_{ab} = p_{ad}$ or $p_{ac} = p_{ad}$.

We now assume that p_{ab}, p_{ac}, p_{ad} are pairwise different. Then, by transitivity, p_{bc} is different from p_{ab} and p_{ac}, the belief p_{bd} is different from p_{ab} and p_{ad}, and the belief p_{cd} is different from p_{ac} and p_{ad}.

Consider two arbitrary, but different, beliefs p_1, p_2 on l, and define

$$\Delta(u(a) - u(b)) := (u(a,p_1) - u(b,p_1)) - (u(a,p_2) - u(b,p_2)).$$

As there is a belief on the line where the DM is indifferent between a and b, and another belief on the line where the DM is not, we must have that $\Delta(u(a) - u(b)) \neq 0$. In a similar way, we define $\Delta(u(a) - u(c))$ and $\Delta(u(a) - u(d))$.

In (2.9.4) of Section 2.9.1 we have seen that

$$\frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(c))} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)}. \tag{2.12.7}$$

Recall that also $\Delta(u(a) - u(c)) \neq 0$. Moreover, since $p_{ab} \neq p_{bc}$ and the belief on the line is uniquely given by its probability on s, we have that $p_{ab}(s) \neq p_{bc}(s)$. Thus, the two ratios above are well-defined. In a similar fashion, it follows that

$$\frac{\Delta(u(a) - u(c))}{\Delta(u(a) - u(d))} = \frac{p_{ad}(s) - p_{cd}(s)}{p_{ac}(s) - p_{cd}(s)} \tag{2.12.8}$$

and

$$\frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(d))} = \frac{p_{ad}(s) - p_{bd}(s)}{p_{ab}(s) - p_{bd}(s)} \tag{2.12.9}$$

As, by definition,

$$\frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(d))} = \frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(c))} \cdot \frac{\Delta(u(a) - u(c))}{\Delta(u(a) - u(d))},$$

it follows by (2.12.7), (2.12.8) and (2.12.9) that

$$\frac{p_{ad}(s)-p_{bd}(s)}{p_{ab}(s)-p_{bd}(s)}=\frac{p_{ac}(s)-p_{bc}(s)}{p_{ab}(s)-p_{bc}(s)}\cdot\frac{p_{ad}(s)-p_{cd}(s)}{p_{ac}(s)-p_{cd}(s)}.$$

By cross-multiplication, this yields the formula for four choice linear preference intensity. Thus, $\succsim$ satisfies four choice linear preference intensity.

(b) Suppose that $\succsim$ satisfies continuity, preservation of indifference, preservation of strict preference, transitivity, three choice linear preference intensity and four choice linear preference intensity. If there are only two choices, then we know from Theorem 2.8.1 that there is an expected utility representation. We therefore assume, from now on, that there are at least three choices.

Suppose first that no two choices are equivalent under $\succsim$. To show that $\succsim$ has an expected utility representation, we distinguish two cases: (1) $P_{a\sim b}=P_{c\sim d}$ for every two pairs of choices $\{a,b\}$ and $\{c,d\}$, and (2) $P_{a\sim b}\neq P_{c\sim d}$ for some pairs of choices $\{a,b\}$ and $\{c,d\}$.

Case 1. Suppose that $P_{a\sim b}=P_{c\sim d}$ for every two pairs of choices $\{a,b\}$ and $\{c,d\}$. Let $A:=P_{a\sim b}$ for some pair of choices $\{a,b\}$. If $A=\Delta(S)$, then the DM is always indifferent between any pair of choices. This would be a contradiction, as we assume that no two choices are equivalent under $\succsim$.

Hence, it must be that $A\neq\Delta(S)$. By preservation of indifference, there must be a state x with $[x]\notin A$. Thus, $[x]\notin P_{a\sim b}$ for every two choices a and b. By transitivity, we can order the choices $c_1,c_2,...,c_K$ such that

$$c_1\succ_{[x]}c_2\succ_{[x]}c_3\succ_{[x]}...\succ_{[x]}c_K.$$

Choose numbers $v_1,...,v_K$ with $v_1>v_2>...>v_K$.

For choice c_1, set $u(c_1,x)=v_1$, and set the utilities $u(c_1,s)$ for states $s\neq x$ arbitrarily.

By Lemma 2.12.2 (b) we know that $span(A)$ has dimension $n-1$, where n is the number of states. Let $\{p_1,...,p_{n-1}\}$ be a basis for $span(A)$. As $[x]\notin span(A)$, we know that $\{p_1,...,p_{n-1},[x]\}$ is a basis for $\mathbf{R}^S$. For every choice c_k with $k\geq 2$ find the unique utilities $u(c_k,s)$ such that

$$u(c_k,p_1)=u(c_1,p_1),...,u(c_k,p_{n-1})=u(c_1,p_{n-1})\text{ and }u(c_k,x)=v_k. \quad (2.12.10)$$

We will show that the utility function u represents $\succsim$.

Take two choices a,b with $a\succ_{[x]}b$. Then, by construction of the utility function, we have that $u(a,p_k)=u(b,p_k)$ for all $k\in\{1,...,n-1\}$, and $u(a,x)>u(b,x)$. As $\{p_1,...,p_{n-1}\}$ is a basis for $span(P_{a\sim b})$, we know that $p_1,...,p_{n-1}$ are linearly independent. It thus follows by Lemma 2.12.3 that u represents $\succsim$ on the pair of choices $\{a,b\}$. As this holds for every pair of choices $\{a,b\}$, we conclude that u represents $\succsim$.

Case 2. Suppose that $P_{a\sim b}\neq P_{c\sim d}$ for some pairs of choices $\{a,b\}$ and $\{c,d\}$. Then, there must be some choices a,b,c such that $P_{a\sim c}\neq P_{b\sim c}$. To see this, suppose on the contrary that $P_{a\sim c}=P_{b\sim c}$ for all three choices a,b,c. Then, take two arbitrary pairs of choices $\{a,b\}$ and $\{c,d\}$ where $\{a,b\}\cap\{c,d\}=\emptyset$. By assumption we would then have that

$$P_{a\sim b}=P_{b\sim c}=P_{c\sim d},$$

and hence $P_{a\sim b}=P_{c\sim d}$ for all pairs $\{a,b\}$ and $\{c,d\}$. This would be a contradiction. Hence, $P_{a\sim c}\neq P_{b\sim c}$ for some choices a,b,c.

Now take some choice d different from a,b and c, if it exists. Then, either $P_{a\sim d}\neq P_{b\sim d}$ or $P_{a\sim d}\neq P_{c\sim d}$. To see this, suppose on the contrary that $P_{a\sim d}=P_{b\sim d}=P_{c\sim d}$. Define $A:=P_{a\sim d}=$

$P_{b\sim d} = P_{c\sim d}$. Since, by transitivity, $P_{a\sim d} \cap P_{b\sim d} \subseteq P_{a\sim b}$ and $P_{b\sim d} \cap P_{c\sim d} \subseteq P_{b\sim c}$, it follows that $A \subseteq P_{a\sim b}$ and $A \subseteq P_{b\sim c}$. Since, by Lemma 2.12.2 (b), $span(A)$, $span(P_{a\sim b})$ and $span(P_{b\sim c})$ all have dimension $n-1$, it follows that $A = span(P_{a\sim b}) = span(P_{b\sim c})$.

Hence, by Lemma 2.12.2 (a),

$$A = \Delta(S) \cap span(A) = \Delta(S) \cap span(P_{a\sim b}) = P_{a\sim b}.$$

In a similar way, it can be shown that also $A = P_{b\sim c}$. This would imply that $A = P_{a\sim b} = P_{b\sim c}$.

By transitivity, $A = P_{a\sim b} \cap P_{b\sim c} \subseteq P_{a\sim c}$. In a similar way as above, it can be shown that, in fact, $A = P_{a\sim c}$. We thus conclude that $P_{a\sim b} = P_{b\sim c} = P_{a\sim c}$. This is a contradiction to our assumption that $P_{a\sim c} \neq P_{b\sim c}$. Hence, either $P_{a\sim d} \neq P_{b\sim d}$ or $P_{a\sim d} \neq P_{c\sim d}$.

We can thus apply the utility design procedure for more than two choices, from Section 2.4.2. We will show that the utility function u so obtained represents $\succsim$. We distinguish the following cases: (2.1) there are three choices, (2.2) there are four choices, and (2.3) there are more than four choices.

Case 2.1. Suppose there are three choices. Let these choices be a, b, c with $P_{a\sim c} \neq P_{b\sim c}$. In the procedure, we first derive the utilities $u(a, s)$ and $u(b, s)$ for the choices a and b, using the utility design procedure for two choices. By the proof of Theorem 2.8.1 we know that these utilities represent $\succsim$ on $\{a, b\}$.

To derive the utilities for c, we first fix the beliefs $p_1, p_2, ..., p_n$ as described in the utility design procedure for more than two choices, where n is the number of states. In particular, $p_1 \in P_{a\sim c} \backslash P_{b\sim c}$, and $p_2, ..., p_n \in P_{b\sim c}$. Note that such a belief $p_1 \in P_{a\sim c} \backslash P_{b\sim c}$ can be found, since $P_{a\sim c} \neq P_{b\sim c}$ and, by Lemma 2.12.2 (a) and (b), $P_{a\sim c} = span(P_{a\sim c}) \cap \Delta(S)$, $P_{b\sim c} = span(P_{b\sim c}) \cap \Delta(S)$ where $span(P_{a\sim c})$ and $span(P_{b\sim c})$ both have dimension $n-1$.

Since $p_1 \notin P_{b\sim c}$, we must have that $p_1 \in P_{b\succ c}$ or $p_1 \in P_{c\succ b}$. Let us assume, without loss of generality, that $p_1 \in P_{c\succ b}$. As $p_1 \in P_{a\sim c}$, it follows by transitivity that $p_1 \in P_{a\succ b}$. Above we have seen that u represents $\succsim$ on $\{a, b\}$, and thus we know that $u(a, p_1) > u(b, p_1)$.

We now show that u represents $\succsim$ on $\{b, c\}$. By construction of the utility design procedure, we have that $u(c, p_k) = u(b, p_k)$ for all $k \in \{2, ..., n\}$. Moreover, we know by the proof of Lemma 2.12.2 (b) that $\{p_2, ..., p_n\}$ is a basis for $span(P_{b\sim c})$, and hence $p_2, ..., p_n$ are linearly independent. Consider the belief p_1 above, with $p_1 \in P_{a\succ b}$. Since we know that $u(a, p_1) > u(b, p_1)$ and, by construction of the utility design procedure, $u(c, p_1) = u(a, p_1)$, it follows that $u(c, p_1) > u(b, p_1)$. But then, it follows by Lemma 2.12.3 that u represents $\succsim$ on $\{b, c\}$.

We finally show that u represents $\succsim$ on $\{a, c\}$. Since $P_{a\sim c} \neq P_{b\sim c}$, we know by Lemma 2.12.2 (a) that $span(P_{a\sim c}) \neq span(P_{b\sim c})$. As both linear subspaces have dimension $n-1$, it follows that $span(P_{a\sim c}) \cap span(P_{b\sim c})$ is a linear subspace with dimension $n-2$. Choose a basis $\{q_2, ..., q_{n-1}\}$ for $span(P_{a\sim c}) \cap span(P_{b\sim c})$. As $p_1 \notin P_{b\sim c}$, we know by Lemma 2.12.2 (a) that $p_1 \notin span(P_{b\sim c})$, and hence $\{p_1, q_2, ..., q_{n-1}\}$ are linearly independent. Since all these vectors are in $span(P_{a\sim c})$, and $span(P_{a\sim c})$ has dimension $n-1$, we conclude that $\{p_1, q_2, ..., q_{n-1}\}$ is a basis for $P_{a\sim c}$.

By Lemma 2.12.6 we know that $span(P_{a\sim c}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim b})$. As such, we conclude that $q_2, ..., q_{n-1} \in span(P_{a\sim b}) \cap span(P_{b\sim c})$. Since u represents $\succsim$ on $\{a, b\}$ and $\{b, c\}$, it follows from Lemma 2.12.7 that

$$u(a, q_k) = u(b, q_k) \text{ and } u(b, q_k) = u(c, q_k) \text{ for all } k \in \{2, ..., n-1\}$$

which implies that

$$u(a, q_k) = u(c, q_k) \text{ for all } k \in \{2, ..., n-1\}. \tag{2.12.11}$$

Moreover, by construction of the utility design procedure,

$$u(a, p_1) = u(c, p_1). \tag{2.12.12}$$

Since $P_{a\sim c} \neq P_{b\sim c}$ we can choose, by a similar argument as above, a belief $p \in P_{b\sim c} \backslash P_{a\sim c}$. Assume, without loss of generality, that $p \in P_{c\succ a}$. Then, by transitivity, $p \in P_{b\succ a}$. As u represents $\succsim$ on $\{a, b\}$ and $\{b, c\}$, we know that $u(c, p) = u(b, p)$ and $u(b, p) > u(a, p)$, which implies that

$$u(c, p) > u(a, p) \text{ for some } p \in P_{c\succ a}. \tag{2.12.13}$$

In view of (2.12.11), (2.12.12) and (2.12.13), it follows by Lemma 2.12.3 that u represents the restriction of $\succsim$ to $\{a, c\}$. As such, u represents $\succsim$.

Case 2.2. Suppose there are four choices, a, b, c, d, where $P_{a\sim c} \neq P_{b\sim c}$. By construction, the utility design procedure first computes the utilities for a, b and c, and afterwards computes the utilities for d. We show that u represents $\succsim$.

By Case 2.1, we know that u represents $\succsim$ on $\{a, b, c\}$. That is, it is left to show that u represents $\succsim$ on $\{d, a\}, \{d, b\}$ and $\{d, c\}$.

We have seen above that either $P_{d\sim a} \neq P_{d\sim b}$ or $P_{d\sim a} \neq P_{d\sim c}$. Suppose, without loss of generality, that $P_{d\sim a} \neq P_{d\sim b}$. Then, by construction of the utility design procedure, we find the utilities for d in a similar way as for $\{a, b, c\}$, but now applied to the choices $\{a, b, d\}$ instead of $\{a, b, c\}$. In the same way as above, it then follows that u represents $\succsim$ on $\{d, a\}$ and $\{d, b\}$.

It remains to show that u represents $\succsim$ on $\{c, d\}$. We distinguish three cases: (2.2.1) $P_{d\sim a} = P_{a\sim c}$ or $P_{d\sim b} = P_{b\sim c}$, (2.2.2) $P_{d\sim a} \neq P_{a\sim c}$, $P_{d\sim b} \neq P_{b\sim c}$ and $span(P_{d\sim a}) \cap span(P_{a\sim c}) \neq span(P_{d\sim b}) \cap span(P_{b\sim c})$, and (2.2.3) $P_{d\sim a} \neq P_{a\sim c}$, $P_{d\sim b} \neq P_{b\sim c}$ and $span(P_{d\sim a}) \cap span(P_{a\sim c}) = span(P_{d\sim b}) \cap span(P_{b\sim c})$.

Case 2.2.1: Suppose that $P_{d\sim a} = P_{a\sim c}$ or $P_{d\sim b} = P_{b\sim c}$. Assume, without loss of generality, that $P_{d\sim a} = P_{a\sim c}$. By transitivity, $P_{d\sim a} = P_{d\sim a} \cap P_{a\sim c} \subseteq P_{d\sim c}$. As, by Lemma 2.12.2 (a) and (b), $P_{d\sim a} = span(P_{d\sim a}) \cap \Delta(S)$, where $span(P_{d\sim a})$ has dimension $n-1$, and a similar property holds for $P_{d\sim c}$, it follows in the same way as at the beginning of Case 2 that $P_{d\sim a} = P_{a\sim c} = P_{d\sim c}$.

Let $\{p_2, ..., p_n\}$ be a basis for $P_{d\sim c}$. As $P_{d\sim a} = P_{a\sim c} = P_{d\sim c}$, and u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, it follows that $u(d, p_m) = u(a, p_m)$ and $u(a, p_m) = u(c, p_m)$ for all $m \in \{2, ..., n\}$. Hence,

$$u(d, p_m) = u(c, p_m) \text{ for all } m \in \{2, ..., n\}. \tag{2.12.14}$$

We now show that $P_{d\sim b} \neq P_{d\sim c}$. To see this, suppose on the contrary that $P_{d\sim b} = P_{d\sim c}$. Then, by transitivity, it would follow in the same was as above that $P_{d\sim b} = P_{d\sim c} = P_{b\sim c}$. As we have seen above that $P_{d\sim c} = P_{a\sim c}$, it would follow that $P_{b\sim c} = P_{a\sim c}$, which is a contradiction. Hence, $P_{d\sim b} \neq P_{d\sim c}$.

We can thus choose some $p \in P_{d\sim b} \backslash P_{d\sim c}$. Assume, without loss of generality, that $p \in P_{d\succ c}$. Then, by transitivity, $p \in P_{b\succ c}$. As u represents $\succsim$ on $\{d, b\}$ and $\{b, c\}$, we know that $u(d, p) = u(b, p)$ and $u(b, p) > u(c, p)$. Thus,

$$u(d, p) > u(c, p) \text{ for some } p \in P_{d\succ c}. \tag{2.12.15}$$

In view of (2.12.14) and (2.12.15), it follows by Lemma 2.12.3 that u represents $\succsim$ on $\{d, c\}$.

Case 2.2.2: Suppose that $P_{d\sim a} \neq P_{a\sim c}$, $P_{d\sim b} \neq P_{b\sim c}$ and $span(P_{d\sim a}) \cap span(P_{a\sim c}) \neq span(P_{d\sim b}) \cap span(P_{b\sim c})$. By Lemma 2.12.2 (a) we know that $span(P_{d\sim a}) \neq span(P_{a\sim c})$ and $span(P_{d\sim b}) \neq span(P_{b\sim c})$. As, by Lemma 2.12.2 (b), each of these four linear subspaces has dimension $n-1$, we conclude that $span(P_{d\sim a}) \cap span(P_{a\sim c})$ and $span(P_{d\sim b}) \cap span(P_{b\sim c})$ are linear subspaces with dimension

$n-2$. Since $span(P_{d\sim a}) \cap span(P_{a\sim c}) \neq span(P_{d\sim b}) \cap span(P_{b\sim c})$, there is some $q_1 \in (span(P_{d\sim a}) \cap span(P_{a\sim c})) \backslash (span(P_{d\sim b}) \cap span(P_{b\sim c}))$. Moreover, let $\{q_2, ..., q_{n-1}\}$ be a basis for $span(P_{d\sim b}) \cap span(P_{b\sim c})$. Then $\{q_1, q_2, ..., q_{n-1}\}$ are linearly independent.

As, by Lemma 2.12.6,

$$span(P_{d\sim a}) \cap span(P_{a\sim c}) \subseteq span(P_{d\sim c}) \text{ and } span(P_{d\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{d\sim c})$$

we conclude that $q_1, q_2, ..., q_{n-1} \in span(P_{d\sim c})$. Since $\{q_1, q_2, ..., q_{n-1}\}$ are linearly independent, and $span(P_{d\sim c})$ has dimension $n-1$, we know that $\{q_1, q_2, ..., q_{n-1}\}$ is a basis for $span(P_{d\sim c})$.

By construction, $q_1 \in span(P_{d\sim a}) \cap span(P_{a\sim c})$. As u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, it follows from Lemma 2.12.7 that $u(d, q_1) = u(a, q_1)$ and $u(a, q_1) = u(c, q_1)$, and thus

$$u(d, q_1) = u(c, q_1). \tag{2.12.16}$$

As $q_2, ..., q_{n-1} \in span(P_{d\sim b}) \cap span(P_{b\sim c})$, and u represents $\succsim$ on $\{d, b\}$ and $\{b, c\}$, it follows in a similar fashion that

$$u(d, q_m) = u(c, q_m) \text{ for all } m \in \{2, ..., n-1\}. \tag{2.12.17}$$

Since $P_{d\sim a} \neq P_{a\sim c}$, it follows by transitivity that $P_{d\sim a} \neq P_{d\sim c}$. Thus, there is some $p \in P_{d\sim a} \backslash P_{d\sim c}$. Suppose, without loss of generality, that $p \in P_{d\succ c}$. By transitivity, we then have that $p \in P_{a\succ c}$. As u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, it follows that $u(d, p) = u(a, p)$ and $u(a, p) > u(c, p)$, and thus

$$u(d, p) > u(c, p) \text{ for some } p \in P_{d\succ c}. \tag{2.12.18}$$

In view of (2.12.16), (2.12.17) and (2.12.18), it follows by Lemma 2.12.3 that u represents $\succsim$ on $\{d, c\}$.

Case 2.2.3: Suppose that $P_{d\sim a} \neq P_{a\sim c}$, $P_{d\sim b} \neq P_{b\sim c}$ and $span(P_{d\sim a}) \cap span(P_{a\sim c}) = span(P_{d\sim b}) \cap span(P_{b\sim c})$. Let

$$A := span(P_{d\sim a}) \cap span(P_{a\sim c}).$$

Recall that, by Lemma 2.12.6, $span(P_{d\sim a}) \cap span(P_{a\sim c}) \subseteq span(P_{d\sim c})$ and $span(P_{d\sim a}) \cap span(P_{d\sim b}) \subseteq span(P_{a\sim b})$. Hence,

$$\begin{aligned} & span(P_{d\sim a}) \cap span(P_{a\sim c}) \cap span(P_{d\sim b}) \cap span(P_{b\sim c}) \cap span(P_{d\sim c}) \cap span(P_{a\sim b}) \\ = \; & span(P_{d\sim a}) \cap span(P_{a\sim c}) \cap span(P_{d\sim b}) \cap span(P_{b\sim c}) \\ = \; & span(P_{d\sim a}) \cap span(P_{a\sim c}) = A, \end{aligned} \tag{2.12.19}$$

where the second equality follows from the fact that $span(P_{d\sim a}) \cap span(P_{a\sim c}) = span(P_{d\sim b}) \cap span(P_{b\sim c})$.

As $P_{d\sim a} \neq P_{a\sim c}$ we know, by Lemma 2.12.2 (a), that $span(P_{d\sim a}) \neq span(P_{a\sim c})$. Since, by Lemma 2.12.2 (b), $span(P_{d\sim a})$ and $span(P_{a\sim c})$ have dimension $n-1$, we know that A has dimension $n-2$. Moreover, by (2.12.19) we know, for every $e, f \in \{a, b, c, d\}$, that $A \subseteq span(P_{e\sim f})$.

Let $\Delta^+(S) := \{p \in \Delta(S) \mid p(s) > 0 \text{ for all } s \in S\}$ be the set of full support beliefs. We distinguish two cases: (2.2.3.1) $A \cap \Delta^+(S)$ is empty, and (2.2.3.2) $A \cap \Delta^+(S)$ is non-empty.

Case 2.2.3.1. Suppose that $A \cap \Delta^+(S)$ is empty. Recall from Lemma 2.12.2 (b) that each of the indifference sets $P_{e\sim f}$, where $e, f \in \{a, b, c, d\}$, has a full support belief in $\Delta^+(S)$, and thus $P_{e\sim f} \cap \Delta^+(S)$ is non-empty. Moreover, by (2.12.19) we have that $P_{e\sim f} \cap P_{g\sim h} = A$ whenever $P_{e\sim f} \neq P_{g\sim h}$. As $A \cap \Delta^+(S)$ is empty, it follows that $P_{e\sim f} \cap P_{g\sim h} \cap \Delta^+(S)$ is empty whenever $P_{e\sim f} \neq P_{g\sim h}$.

Recall from the assumption in Case 2.2.3 that $P_{a\sim c} \neq P_{b\sim c}$, $P_{d\sim a} \neq P_{a\sim c}$ and $P_{d\sim b} \neq P_{b\sim c}$. Moreover, we have seen at the beginning of Case 2.2 that $P_{d\sim a} \neq P_{d\sim b}$. By transitivity, it follows that the sets $P_{a\sim b}, P_{a\sim c}, P_{b\sim c}$ are pairwise different, that the sets $P_{a\sim c}, P_{a\sim d}$ and $P_{c\sim d}$ are pairwise different, that the sets $P_{a\sim b}, P_{a\sim d}$ and $P_{b\sim d}$ are different, and that the sets $P_{b\sim c}, P_{b\sim d}$ and $P_{c\sim d}$ are pairwise different. Let $P_1, ..., P_k$ be the pairwise different sets from $P_{a\sim b}, ..., P_{c\sim d}$. From the above, it follows that $k \geq 3$.

As $P_{e\sim f} \cap P_{g\sim h} \cap \Delta^+(S)$ is empty whenever $P_{e\sim f} \neq P_{g\sim h}$, it follows that the sets $P_1 \cap \Delta^+(S), ..., P_k \cap \Delta^+(S)$ are pairwise disjoint. Moreover, we have seen that each of the latter sets are non-empty. Since $span(P_1), ..., span(P_k)$ are hyperplanes of dimension $n-1$, we can order the sets $P_1, ..., P_k$ such that $P_2 \cap \Delta^+(S), ..., P_{k-1} \cap \Delta^+(S)$ are in between $P_1 \cap \Delta^+(S)$ and $P_k \cap \Delta^+(S)$. Take some $p_1 \in P_1 \cap \Delta^+(S)$ and $p_k \in P_k \cap \Delta^+(S)$, and let l be the line through p_1 and p_k. Then, the corresponding line segment from p_1 to p_k is included in $\Delta^+(S)$. As $P_2 \cap \Delta^+(S), ..., P_{k-1} \cap \Delta^+(S)$ are in between $P_1 \cap \Delta^+(S)$ and $P_k \cap \Delta^+(S)$, the line l contains for every $m \in \{2, ..., k-1\}$ a unique belief p_m in P_m.

In particular, for every pair of choices e, f in $\{a, b, c, d\}$, there is a unique belief $p_{ef} \in P_{e\sim f}$ on the line l, and the line l contains a belief where the DM is not indifferent between any of the choices in $\{a, b, c, d\}$.

Recall, from above, that the sets $P_{a\sim b}, P_{a\sim c}, P_{b\sim c}$ are pairwise different, that the sets $P_{a\sim b}, P_{a\sim d}$ and $P_{b\sim d}$ are pairwise different, that the sets $P_{a\sim c}, P_{a\sim d}$ and $P_{c\sim d}$ are pairwise different, and that the sets $P_{b\sim c}, P_{b\sim d}$ and $P_{c\sim d}$ are pairwise different. Moreover, we have seen that $P_1 \cap \Delta^+(S), ..., P_k \cap \Delta^+(S)$ are pairwise disjoint. Hence, by construction, p_{ab}, p_{ac}, p_{bc} are pairwise different, p_{ab}, p_{ad}, p_{bd} are pairwise different, and p_{ac}, p_{ad}, p_{cd} are pairwise different. Let s be a state such that the probability of s is not constant on the line l. By four choice linear preference intensity, we have that

$$\frac{p_{ac}(s) - p_{cd}(s)}{p_{ad}(s) - p_{cd}(s)} = \frac{(p_{ab}(s) - p_{bd}(s))(p_{ac}(s) - p_{bc}(s))}{(p_{ab}(s) - p_{bc}(s))(p_{ad}(s) - p_{bd}(s))}. \tag{2.12.20}$$

Note that both fractions are well-defined since $p_{ad} \neq p_{cd}$, $p_{ab} \neq p_{bc}$ and $p_{ad} \neq p_{bd}$. Moreover, as p_{ac}, p_{ad}, p_{cd} are pairwise different, we have that $p_{ac}(s) - p_{cd}(s) \neq p_{ad}(s) - p_{cd}(s)$, and hence the fraction on the lefthand side is not equal to 1. As such, the fraction on the righthand side is not equal to 1 either. Let this fraction on the righthand side be called F. Then, by (2.12.20), p_{cd} is the unique belief on l where

$$p_{cd}(s) = \frac{F \cdot p_{ad}(s) - p_{ac}(s)}{F - 1}. \tag{2.12.21}$$

Remember that $A \subseteq span(P_{c\sim d})$, that A has dimension $n-2$, and that $span(P_{c\sim d})$ has dimension $n-1$. Let $\{q_2, ..., q_{n-1}\}$ be a basis for A. Since $p_{cd} \in \Delta^+(S)$ and $A \cap \Delta^+(S)$ is empty, we conclude that $p_{cd} \notin A$. Hence, $\{p_{cd}, q_2, ..., q_{n-1}\}$ is a basis for $span(P_{c\sim d})$.

Now, let $\succsim^u$ be the conditional preference relation generated by the utility function u. We have already seen that u represents $\succsim$ on all pairs of choices in $\{a, b, c, d\}$, except $\{c, d\}$. In particular, we thus know that

$$u(a, p_{ab}) = u(b, p_{ab}),\ u(a, p_{ac}) = u(c, p_{ac}),\ u(a, p_{ad}) = u(d, p_{ad}),$$
$$u(b, p_{bc}) = u(c, p_{bc}) \text{ and } u(b, p_{bd}) = u(d, p_{bd}).$$

As we have seen in part (a) of the proof that $\succsim^u$ satisfies four choice linear preference intensity, the unique belief on the line l where the DM is indifferent between c and d under $\succsim^u$ is given by (2.12.21). Therefore,

$$u(c, p_{cd}) = u(d, p_{cd}). \tag{2.12.22}$$

Recall that $A = span(P_{d\sim a}) \cap span(P_{a\sim c})$. As u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, it follows that $u(d, v) = u(a, v)$ and $u(a, v) = u(c, v)$ for every $v \in span(P_{d\sim a}) \cap span(P_{a\sim c})$. Therefore, $u(c, v) =$

$u(d, v)$ for every $v \in A$. In particular,

$$u(c, q_k) = u(d, q_k) \text{ for every } k \in \{2, ..., n-1\}, \tag{2.12.23}$$

where $\{q_2, ..., q_{n-1}\}$ is a basis for A. Moreover, we have seen that $\{p_{cd}, q_2, ..., q_{n-1}\}$ is a basis for $span(P_{c\sim d})$.

As $P_{d\sim a} \neq P_{a\sim c}$, we can choose some $p \in P_{d\sim a} \backslash P_{a\sim c}$. Assume, without loss of generality, that $p \in P_{a\succ c}$. By transitivity, we then have that $p \in P_{d\succ c}$. Since u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, we know that $u(d, p) = u(a, p)$ and $u(a, p) > u(c, p)$, and hence

$$u(d, p) > u(c, p) \text{ for some } p \in P_{d\succ c}. \tag{2.12.24}$$

In view of (2.12.22), (2.12.23) and (2.12.24), it follows by Lemma 2.12.3 that u represents $\succsim$ on $\{c, d\}$. Thus, u represents $\succsim$ on $\{a, b, c, d\}$, which completes the proof for Case 2.2.3.1.

Case 2.2.3.2. Suppose that $A \cap \Delta^+(S)$ is non-empty. Then, there is some full support belief p^* in A, with $p^*(s) > 0$ for all states s. As we have seen that $A \subseteq span(P_{e\sim f})$ for all $e, f \in \{a, b, c, d\}$, it follows that $p^* \in P_{e\sim f}$ for all pairs $e, f \in \{a, b, c, d\}$.

Since we have seen that A has dimension $n - 2$, the linear subspace A is contained in some hyperplane containing the zero vector. Hence, there is some vector $n^A \in \mathbf{R}^S$ such that

$$n^A \cdot v = 0 \text{ for all } v \in A. \tag{2.12.25}$$

Moreover, we can choose the vector n^A such that for every pair $e, f \in \{a, b, c, d\}$ there is some $p \in P_{e\sim f}$ with $n^A \cdot p \neq 0$.

In that case, there is for every pair $e, f \in \{a, b, c, d\}$ some $p \in P_{e\sim f}$ with $n^A \cdot p > 0$. To see this, suppose that e, f are such that $n^A \cdot p \leq 0$ for every $p \in P_{e\sim f}$. As there is some $p \in P_{e\sim f}$ with $n^A \cdot p \neq 0$, there must be some $p \in P_{e\sim f}$ with $n^A \cdot p < 0$. Since $p^* \in \Delta^+(S)$, there is some $\lambda > 1$ close enough to 1 such that $q := (1 - \lambda)p + \lambda p^* \in \Delta(S)$. Note that $p^* \in A \subseteq span(P_{e\sim f})$ and $p \in P_{e\sim f}$, which implies that $q \in span(P_{e\sim f}) \cap \Delta(S) = P_{e\sim f}$. At the same time we know, by (2.12.25) and the fact that $p^* \in A$, that $n^A \cdot p^* = 0$. Since $n^A \cdot p < 0$ and $\lambda > 1$, it follows that

$$n^A \cdot q = (1 - \lambda) \cdot (n^A \cdot p) + \lambda \cdot (n^A \cdot p^*) > 0.$$

Thus,

$$\text{for every } e, f \in \{a, b, c, d\} \text{ there is some } p \in P_{e\sim f} \text{ with } n^A \cdot p > 0. \tag{2.12.26}$$

Let

$$P^+ := \{p \in \Delta(S) \mid n^A \cdot p > 0\}.$$

Then, in view of (2.12.26),

$$P_{e\sim f} \cap P^+ \text{ is non-empty for all } e, f \in \{a, b, c, d\}. \tag{2.12.27}$$

Recall that $P_{e\sim f} \cap P_{g\sim h} = A$ for every two pairs $\{e, f\}, \{g, h\}$ in $\{a, b, c, d\}$ with $P_{e\sim f} \neq P_{g\sim h}$. In view of (2.12.25) and (2.12.27) we conclude that $P_{e\sim f} \cap P_{g\sim h} \cap P^+$ is empty whenever $P_{e\sim f} \neq P_{g\sim h}$. Hence, $(P_{e\sim f} \cap P^+)$ and $(P_{g\sim h} \cap P^+)$ are disjoint whenever $P_{e\sim f} \neq P_{g\sim h}$. But then, the different sets in $P_{a\sim b}, ..., P_{c\sim d}$ can be numbered $P_1, ..., P_k$, with $k \geq 3$, such that $P_2 \cap P^+, ..., P_{k-1} \cap P^+$ are in between $P_1 \cap P^+$ and $P_k \cap P^+$. In a similar way as in Case 2.2.3.1, it can then be shown that u represents $\succsim$ on $\{c, d\}$. This would complete the proof for Case 2.2.3.2.

Hence, u represents $\succsim$ on $\{a, b, c, d\}$. This completes the proof for Case 2.2.

Case 2.3. Suppose there are more than four choices. Label these choices $a, b, c, d_1, d_2, ..., d_K$, where $P_{c \sim a} \neq P_{c \sim b}$, and the choices are ordered according to the order in which their respective utilities are computed in the utility design procedure.

We will prove, by induction on k, that u represents $\succsim$ on $\{a, b, c, d_1, ..., d_k\}$. In Case 2.2 we have already shown that u represents $\succsim$ on $\{a, b, c, d_1\}$, which yields the induction start.

Now, let $k \geq 2$, and suppose that u represents $\succsim$ on $\{a, b, c, d_1, ..., d_{k-1}\}$. We will show that u represents $\succsim$ on $\{a, b, c, d_1, ..., d_{k-1}, d_k\}$, by showing that it does so on $\{d_k, a\}, \{d_k, b\}$ and $\{d_k, e\}$ for every $e \in \{c, d_1, ..., d_{k-1}\}$.

In a similar way as in Case 2.2, it can be shown that either $P_{d_k \sim a} \neq P_{d_k \sim b}$, or $P_{d_k \sim a} \neq P_{d_k \sim c}$. Assume, without loss of generality, that $P_{d_k \sim a} \neq P_{d_k \sim b}$. Then, it follows from the proof of Case 2.1 that u represents $\succsim$ on $\{d_k, a\}$ and $\{d_k, b\}$.

Now, choose some $e \in \{c, d_1, ..., d_{k-1}\}$. By mimicking the proof of Case 2.2, it can then be shown u represents $\succsim$ on $\{d_k, e\}$. Indeed, instead of applying the proof to $\{a, b, c, d\}$, we can now apply it to $\{a, b, e, d_k\}$. As such, u represents $\succsim$ on $\{a, b, c, d_1, ..., d_{k-1}, d_k\}$. By induction, the proof for Case 2.3 is complete.

By combining Cases 1, 2.1, 2.2 and 2.3, we have shown that u represents $\succsim$ whenever no two choices are equivalent.

Suppose now that two, or more, choices are equivalent. In this case, we can select a subset C^* of choices such that (i) no two choices in C^* are equivalent, and (ii) every choice outside C^* is equivalent to a choice inside C^*. By the proof above, we then know that there is a utility function u^* on C^* that represents $\succsim$ on C^*. This utility function can be extended to a utility function u on C, by setting, for every choice $c \notin C^*$,

$$u(c, s) := u(c^*, s)$$

for all states s, where c^* is the unique choice in C^* that is equivalent to c. Then, the utility function u will represent $\succsim$ on the whole choice set C. This completes the proof. ■

2.12.3 Proof for Section 2.10

Before we can prove Theorem 2.10.1 we need a preparatory result. It describes, for a given signed conditional preference relation meeting the axioms, the structure of the set of signed beliefs for which the DM is "indifferent" between two choices. To formally state the preparatory result, we must introduce some new notions and notation. For a signed conditional preference relation $\succsim^*$ and two choices a and b, we denote by $Q_{a \sim^* b}$ the set of signed beliefs q for which $a \sim_q^* b$. By

$$\Delta^*(S) := \{q \in \mathbf{R}^S \mid \sum_{s \in S} q(s) = 1\}$$

we denote the set of all signed beliefs. Two subsets $Q, Q' \subseteq \Delta^*(S)$ are called *parallel* if there is some vector $v \in \mathbf{R}^S$ such that

$$Q' = \{q + v \mid q \in Q\}.$$

In particular, two parallel sets Q, Q' with $Q \neq Q'$ are always disjoint, that is, $Q \cap Q'$ is empty.

Lemma 2.12.8 (Signed indifference sets) *Let $\succsim^*$ be a signed conditional preference relation without equivalent choices which satisfies continuity, preservation of indifference and preservation of strict preference.*

(a) Consider two choices a, b such that there is no constant preference intensity between a and b. Then,

$span(Q_{a\sim^* b})$ has dimension $|S|-1$, and $Q_{a\sim^* b} = \Delta^*(S) \cap span(Q_{a\sim^* b})$;

(b) Consider three choices a, b, c such that there is constant preference intensity between a and b, but not between a and c, and not between b and c. Then, the sets $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$ are parallel.

Proof. **(a)** In a similar way as in the proof of Lemma 2.12.2 (a), it can be shown that $Q_{a\sim^* b} = \Delta^*(S) \cap span(Q_{a\sim^* b})$. We therefore omit this proof here.

As there is no constant preference intensity between a and b, it is not the case that $a \succ^*_q b$ for all signed beliefs q, and it is not the case that $b \succ^*_q a$ for all signed beliefs q. In that case, there must be signed beliefs q_1 and q_2 such that $a \succ^*_{q_1} b$ and $b \succ^*_{q_2} a$. To see this, suppose such signed beliefs q_1 and q_2 would not exist. Then, either $a \succsim^*_q b$ for all signed beliefs q, or $b \succsim^*_q a$ for all signed beliefs q. Assume, without loss of generality, that $a \succsim^*_q b$ for all signed beliefs q. As there is no constant preference intensity between a and b, and a and b are not equivalent, there must be signed beliefs q_1 and q_2 with $a \succ^*_{q_1} b$ and $a \sim^*_{q_2} b$. Let the signed belief q_3 be such that $q_2 = (1/2)q_1 + (1/2)q_3$. We will see that $b \succ^*_{q_3} a$. Suppose, on the contrary, that $a \succsim^*_{q_3} b$. Since $q_2 = (1/2)q_1 + (1/2)q_3$ and $a \succ^*_{q_1} b$, it would follow by preservation of strict preference that $a \succ^*_{q_2} b$, which is a contradiction to the fact that $a \sim^*_{q_2} b$. Hence, we see that $b \succ^*_{q_3} a$. This is a contradiction to the assumption above that $a \succsim^*_q b$ for all signed beliefs q. Thus, we conclude that there must be signed beliefs q_1, q_2 with $a \succ^*_{q_1} b$ and $b \succ^*_{q_2} a$.

Choose some full support belief p^* with $p^*(s) > 0$ for all states s such that $p^* \notin Q_{a\sim^* b}$. For every number λ, consider the conditional preference relation $\succsim^\lambda$ where for every two choices c and d, and every belief p,

$$c \succsim^\lambda_p d \text{ if and only if } c \succsim^*_{(1-\lambda)p^* + \lambda p} d. \tag{2.12.28}$$

We now show that λ can be chosen large enough such that $\succsim^\lambda$ has preference reversals for $\{a, b\}$. Recall from above that there are signed beliefs q_1, q_2 with $a \succ^*_{q_1} b$ and $b \succ^*_{q_2} a$. As p^* is a full support belief with $p^*(s) > 0$ for all states s, we can choose $\varepsilon > 0$ small enough such that both $p_1 := (1-\varepsilon)p^* + \varepsilon q_1$ and $p_2 := (1-\varepsilon)p^* + \varepsilon q_2$ are beliefs. By setting $\lambda := 1/\varepsilon$, we have that $q_1 = (1-\lambda)p^* + \lambda p_1$ and $q_2 = (1-\lambda)p^* + \lambda p_2$. Since $a \succ^*_{q_1} b$ and $b \succ^*_{q_2} a$ it follows, by definition of $\succsim^\lambda$, that $a \succ^\lambda_{p_1} b$ and $b \succ^\lambda_{p_2} a$. As we have chosen ε small enough, and $\lambda = 1/\varepsilon$, we can choose λ large enough such that there are beliefs p_1 and p_2 with $a \succ^\lambda_{p_1} b$ and $b \succ^\lambda_{p_2} a$. That is, we can choose λ large enough such that there are preference reversals between a and b. In particular, we can choose $\lambda > 1$.

It can also be shown that the conditional preference relation $\succsim^\lambda$ satisfies continuity, preservation of indifference and preservation of strict preference. We start with continuity. Take two choices c and d and two beliefs p_1, p_2 with $c \succ^\lambda_{p_1} d$ and $d \succ^\lambda_{p_2} c$. Then, by definition, $c \succ^*_{(1-\lambda)p^* + \lambda p_1} d$ and $d \succ^*_{(1-\lambda)p^* + \lambda p_2} c$. Define the signed beliefs $q_1 := (1-\lambda)p^* + \lambda p_1$ and $q_2 := (1-\lambda)p^* + \lambda p_2$. Since $\succsim^*$ satisfies continuity, there is some $\mu \in (0,1)$ such that $c \sim^*_{(1-\mu)q_1 + \mu q_2} d$. Since

$$\begin{aligned}(1-\mu)q_1 + \mu q_2 &= (1-\mu)((1-\lambda)p^* + \lambda p_1) + \mu((1-\lambda)p^* + \lambda p_2) \\ &= (1-\lambda)p^* + \lambda((1-\mu)p_1 + \mu p_2)\end{aligned}$$

it follows by definition of $\succsim^\lambda$ that $c \sim^\lambda_{(1-\mu)p_1 + \mu p_2} d$. Thus, $\succsim^\lambda$ satisfies continuity. In a similar fashion, it can be shown that $\succsim^\lambda$ satisfies preservation of indifference and preservation of strict preference.

Summarizing, we see that $\succsim^\lambda$ satisfies continuity, preservation of indifference and preservation of strict preference, and there are preference reversals between a and b. By Lemma 2.12.2 (a) and (b) we can thus conclude that $P_{a\sim^\lambda b} = span(P_{a\sim^\lambda b}) \cap \Delta(S)$, and $span(P_{a\sim^\lambda b})$ has dimension $n-1$, where n is the number of states.

Select beliefs $p_1, ..., p_{n-1}$ such that $\{p_1, ..., p_{n-1}\}$ is a basis for $span(P_{a\sim^\lambda b})$. Define the signed beliefs $q_1, ..., q_{n-1}$ by

$$q_k := (1-\lambda)p^* + \lambda p_k \tag{2.12.29}$$

for all $k \in \{1, ..., n-1\}$. We show that $\{q_1, ..., q_{n-1}\}$ is a basis for $span(Q_{a\sim^* b})$.

We start by verifying that $q_k \in Q_{a\sim^* b}$ for every k. Since $a \sim^\lambda_{p_k} b$, it follows by (2.12.28) and (2.12.29) that $a \sim^*_{q_k} b$, and hence $q_k \in Q_{a\sim^* b}$, for all $k \in \{1, ..., n-1\}$.

We next show that $q_1, ..., q_{n-1}$ are linearly independent. Recall that $p^* \notin Q_{a\sim^* b}$. By (2.12.28) it then follows that $p^* \notin P_{a\sim^\lambda b}$. Since we know that $P_{a\sim^\lambda b} = span(P_{a\sim^\lambda b}) \cap \Delta(S)$, it follows that $p^* \notin span(P_{a\sim^\lambda b})$.

Now, suppose that $\mu_1 q_1 + ... + \mu_{n-1} q_{n-1} = \underline{0}$ for some numbers $\mu_1, ..., \mu_{n-1}$. Then, by (2.12.29),

$$(1-\lambda)(\mu_1 + ... + \mu_{n-1})p^* + \lambda\mu_1 p_1 + ... + \lambda\mu_{n-1} p_{n-1} = \underline{0}. \tag{2.12.30}$$

As $p^* \notin span(P_{a\sim^\lambda b})$, and $\{p_1, ..., p_{n-1}\}$ is a basis for $span(P_{a\sim^\lambda b})$, we conclude that $p^*, p_1, ..., p_{n-1}$ are linearly independent. Hence, (2.12.30) implies that $(1-\lambda)(\mu_1 + ... + \mu_{n-1}) = 0$ and $\lambda\mu_k = 0$ for all $k \in \{1, ..., n-1\}$. As $\lambda > 1$, this means that $\mu_k = 0$ for all $k \in \{1, ..., n-1\}$. Thus, the signed beliefs $q_1, ..., q_{n-1} \in Q_{a\sim^* b}$ are linearly independent.

This means, in turn, that $span(Q_{a\sim^* b})$ has dimension at least $n-1$. Recall that $Q_{a\sim^* b} = \Delta^*(S) \cap span(Q_{a\sim^* b})$. If $span(Q_{a\sim^* b})$ would have dimension n, then $span(Q_{a\sim^* b}) = \mathbf{R}^S$, which would imply that $Q_{a\sim^* b} = \Delta^*(S)$. This would be a contradiction, since a and b are not equivalent. We thus conclude that $span(Q_{a\sim^* b})$ has dimension $n-1$.

(b) Suppose that there is constant preference intensity between a and b, but not between a and c, and not between b and c. Then, we know from (a) that $Q_{a\sim^* c} = span(Q_{a\sim^* c}) \cap \Delta^*(S)$ and $Q_{b\sim^* c} = span(Q_{b\sim^* c}) \cap \Delta^*(S)$ where $span(Q_{a\sim^* c})$ and $span(Q_{b\sim^* c})$ both have dimension $|S|-1$. Suppose, contrary to what we want to show, that $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$ are not parallel. Then, it must be that $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$ intersect, and hence there is some signed belief q which is both in $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$. By transitivity, it would then follow that $q \in Q_{a\sim^* b}$. This, however, is a contradiction, since there is constant preference intensity between a and b, and the choices a and b are not equivalent. We thus conclude that $Q_{a\sim^* c}$ and $Q_{b\sim^* c}$ are parallel. This completes the proof. ■

We are now ready to prove Theorem 2.10.1.

Proof of Theorem 2.10.1. (a) Suppose first that $\succsim$ has an expected utility representation u. Let $\succsim^*$ be the signed conditional preference relation where for every signed belief q, and every two choices a and b, we have that

$$a \succsim^*_q b \text{ if and only if } u(a,q) \geq u(b,q).$$

Then, $\succsim^*$ extends $\succsim$. Similarly to the arguments in Section 2.8 and the proof of Theorem 2.9.1, it can then be shown that $\succsim^*$ satisfies continuity, preservation of indifference, preservation of strict preference, transitivity, three choice preference intensity and four choice preference intensity.

We now show that $\succsim^*$ satisfies transitive constant preference intensity. Suppose that there is constant preference intensity between a and b and between b and c. Then, there must be numbers α_1, α_2 such that $u(a,q) - u(b,q) = \alpha_1$ for every signed belief q, and $u(b,q) - u(c,q) = \alpha_2$ for every signed belief q. Then, $u(a,q) - u(c,q) = \alpha_1 + \alpha_2$ for every signed belief q, and therefore there is constant preference intensity between a and c.

We next show that $\succsim^*$ satisfies four choice linear preference intensity with constant preference intensity. Consider a line l of signed beliefs, and four choices a, b, c and d, such that there is a signed belief on the line where the DM is not "indifferent" between any of the four choices.

To prove part (a) of this axiom, assume that there is constant preference intensity between c and d, but not between the other five pairs of choices. Let $q_{ab}, q_{ac}, q_{ad}, q_{bc}$ and q_{bd} be signed beliefs on the line where the DM is "indifferent" between the respective choices. Consider two arbitrary, but different, signed beliefs q_1, q_2 on l, and define, for every two choices e, f in $\{a, b, c, d\}$,

$$\Delta(u(e) - u(f)) := (u(e, q_1) - u(f, q_1)) - (u(e, q_2) - u(f, q_2)).$$

Now, consider a state s such that the probability of s is not constant on the line l. In a similar way as in the proof of Theorem 2.9.1 it can be shown, for every three choices e, f, g in $\{a, b, c, d\}$, that

$$\frac{\Delta(u(e) - u(f))}{\Delta(u(e) - u(g))} = \frac{q_{eg}(s) - q_{fg}(s)}{q_{ef}(s) - q_{fg}(s)}.$$

In particular, we have that

$$\frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(c))} = \frac{q_{ac}(s) - q_{bc}(s)}{q_{ab}(s) - q_{bc}(s)} \text{ and } \frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(d))} = \frac{q_{ad}(s) - q_{bd}(s)}{q_{ab}(s) - q_{bd}(s)}. \tag{2.12.31}$$

Recall that the preference intensity between c and d is constant. This means that there is a number α such that $u(d, q) = u(c, q) + \alpha$ for every signed belief q. But then,

$$\begin{aligned}
\Delta(u(a) - u(d)) &= (u(a, q_1) - u(d, q_1)) - (u(a, q_2) - u(d, q_2)) \\
&= (u(a, q_1) - u(c, q_1) - \alpha) - (u(a, q_2) - u(c, q_2) - \alpha) \\
&= (u(a, q_1) - u(c, q_1)) - (u(a, q_2) - u(c, q_2)) = \Delta(u(a) - u(c)).
\end{aligned}$$

As such,

$$\frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(c))} = \frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(d))}.$$

Together with (2.12.31), this yields

$$\frac{q_{ac}(s) - q_{bc}(s)}{q_{ab}(s) - q_{bc}(s)} = \frac{q_{ad}(s) - q_{bd}(s)}{q_{ab}(s) - q_{bd}(s)},$$

and hence

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q_{ad}(s) - q_{bd}(s)) = (q_{ab}(s) - q_{bd}(s)) \cdot (q_{ac}(s) - q_{bc}(s)).$$

Thus, part (a) of four choice linear preference intensity with constant preference intensity holds.

To prove part (b) of the axiom, assume that the preference intensities between a and b, and between c and d, are constant, but not between the other four pairs of choices. Then, there are numbers α, β such that $u(b, q) = u(a, q) + \alpha$ and $u(d, q) = u(c, q) + \beta$ for every signed belief q. Let q_{ac}, q_{bc}, q_{ad} and q_{bd} be signed beliefs on the line l where the DM is "indifferent" between the respective choices. Define the signed belief $q := q_{ad} - q_{bd} + q_{bc}$, which is again on the line l. Then,

$$\begin{aligned}
u(a, q) - u(c, q) &= (u(a, q_{ad}) - u(c, q_{ad})) - (u(a, q_{bd}) - u(c, q_{bd})) + (u(a, q_{bc}) - u(c, q_{bc})) \\
&= (u(a, q_{ad}) - u(d, q_{ad}) + \beta) - (u(b, q_{bd}) - u(d, q_{bd}) - \alpha + \beta) + (u(b, q_{bc}) - u(c, q_{bc}) - \alpha) = 0,
\end{aligned}$$

since $u(a, q_{ad}) = u(d, q_{ad})$, $u(b, q_{bd}) = u(d, q_{bd})$ and $u(b, q_{bc}) = u(c, q_{bc})$.

Recall that there is a signed belief on the line where the DM is not "indifferent" between any of the choices in $\{a, b, c, d\}$. But then, by preservation of indifference and preservation of strict preference, there is only one signed belief on the line l where the DM is "indifferent" between a and c, which is q_{ac}. Thus, we conclude that $q = q_{ac}$, and therefore

$$q_{ac} - q_{bc} = q_{ad} - q_{bd},$$

which yields part (b) of four choice linear preference intensity with constant preference intensity.

Thus, $\succsim$ can be extended to a signed conditional preference relation that satisfies all of the axioms above.

(b) Suppose now that $\succsim$ can be extended to a signed conditional preference relation $\succsim^*$ that satisfies all of the axioms above. We will show that there is a utility function u that represents $\succsim^*$, and thereby represents $\succsim$ as well. To start, we assume that no two choices are equivalent under $\succsim$. At the end of the proof, we show how to deal with the case where some choices are equivalent. We distinguish two cases: (1) for every two choices a, b there is no constant preference intensity between a and b, and (2) there are at least two choices a and b with a constant preference intensity between them.

Case 1. Suppose that, for every two choices a and b, there is no constant preference intensity between a and b. By the proof of Lemma 2.12.8 (a) we then know that for every two choices a and b, there must be signed beliefs q_1 and q_2 such that $a \succ^*_{q_1} b$ and $b \succ^*_{q_2} a$.

Choose some full support belief p^* with $p^*(s) > 0$ for all states s. For every number λ, consider the conditional preference relation $\succsim^\lambda$ where for every two choices a and b, and every belief p,

$$a \succsim^\lambda_p b \text{ if and only if } a \succsim^*_{(1-\lambda)p^*+\lambda p} b.$$

We have seen in the proof of Lemma 2.12.8 (a) that for every two choices a and b there is some large enough λ such that $\succsim^\lambda$ has preference reversals between a and b. But then, we can choose λ large enough such that $\succsim^\lambda$ has preference reversals for all pairs of choices.

We will now show that $\succsim^\lambda$ satisfies the regularity axioms, transitivity, three choice linear preference intensity and four choice linear preference intensity. In the proof of Lemma 2.12.8 (a) we saw that $\succsim^\lambda$ satisfies the regularity axioms. Transitivity of $\succsim^\lambda$ follows immediately from the assumption that $\succsim^*$ satisfies transitivity.

We now show three choice linear preference intensity. Consider three choices a, b, c, two parallel lines of beliefs l and l' that contain beliefs where the DM is not indifferent under $\succsim^\lambda$ between any of the three choices, and beliefs p_{ab}, p_{ac}, p_{bc} on l and beliefs $p'_{ab}, p'_{ac}, p'_{bc}$ on l' where the DM is indifferent under $\succsim^\lambda$ between the respective choices.

Define the lines L and L' of signed beliefs where

$$L := \{(1-\lambda)p^* + \lambda p \mid p \text{ on } l\} \text{ and } L' := \{(1-\lambda)p^* + \lambda p' \mid p' \text{ on } l'\}.$$

Then, it may be verified that the lines L and L' are parallel as well.

Recall that l and l' contain beliefs where the DM is not indifferent under $\succsim^\lambda$ between any of the three choices. By definition of $\succsim^\lambda$ it follows that L and L' contain signed beliefs where the DM is not "indifferent" under $\succsim^*$ between any of the three choices.

Moreover, define the signed belief $q_{ab} := (1-\lambda)p^* + \lambda p_{ab}$, and similarly for $q_{ac}, q_{bc}, q'_{ab}, q'_{ac}$ and q'_{bc}. Then, q_{ab}, q_{ac}, q_{bc} are on L and $q'_{ab}, q'_{ac}, q'_{bc}$ are on L'. Also, by definition of $\succsim^\lambda$ we can conclude that at the signed beliefs $q_{ab}, ..., q'_{bc}$ the DM is "indifferent" under $\succsim^*$ between the corresponding pair of choices. Since $\succsim^*$ satisfies three choice linear preference intensity, we know for every state s that

$$(q_{ab}(s) - q_{bc}(s)) \cdot (q'_{ac}(s) - q'_{bc}(s)) = (q'_{ab}(s) - q'_{bc}(s)) \cdot (q_{ac}(s) - q_{bc}(s)). \quad (2.12.32)$$

As $q_{ab} = (1-\lambda)p^* + \lambda p_{ab}$, it follows that $p_{ab} = (1-1/\lambda)p^* + (1/\lambda)q_{ab}$. Similarly for the other five beliefs. Together with (2.12.32) we conclude that

$$\begin{aligned}(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) &= \frac{1}{\lambda^2}(q_{ab}(s) - q_{bc}(s)) \cdot (q'_{ac}(s) - q'_{bc}(s)) \\ &= \frac{1}{\lambda^2}(q'_{ab}(s) - q'_{bc}(s)) \cdot (q_{ac}(s) - q_{bc}(s)) \\ &= (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)).\end{aligned}$$

Thus, $\succsim^\lambda$ satisfies three choice linear preference intensity. In a similar fashion, it can be shown that $\succsim^\lambda$ satisfies four choice linear preference intensity.

Summarizing, we see that the conditional preference relation $\succsim^\lambda$ has preference reversals for all pairs of choices, and satisfies each of the axioms from Theorem 2.9.1. By the same theorem, we then conclude that $\succsim^\lambda$ has an expected utility representation u^λ.

Define the utility function u by

$$u(c,s) := (1-1/\lambda) \cdot u^\lambda(c,p^*) + (1/\lambda) \cdot u^\lambda(c,s)$$

for every choice c and state s. We will show that u represents $\succsim$.

Take some arbitrary belief p. Then, $p = (1-\lambda)p^* + \lambda p'$ for the belief $p' := (1-1/\lambda)p^* + (1/\lambda)p$. We conclude, for two arbitrary choices a and b, that

$$\begin{gathered} a \succsim_p b \text{ if and only if } a \succsim^*_p b \\ \text{if and only if } a \succsim^*_{(1-\lambda)p^* + \lambda p'} b \text{ if and only if } a \succsim^\lambda_{p'} b \\ \text{if and only if } u^\lambda(a,p') \geq u^\lambda(b,p') \\ \text{if and only if } u^\lambda(a,(1-1/\lambda)p^* + (1/\lambda)p) \geq u^\lambda(b,(1-1/\lambda)p^* + (1/\lambda)p) \\ \text{if and only if } (1-1/\lambda)u^\lambda(a,p^*) + (1/\lambda)u^\lambda(a,p) \geq (1-1/\lambda)u^\lambda(b,p^*) + (1/\lambda)u^\lambda(b,p) \\ \text{if and only if } u(a,p) \geq u(b,p). \end{gathered}$$

Here, the first equivalence follows from the assumption that $\succsim^*$ extends $\succsim$, the second equivalence from the definition of p', the third equivalence from the definition of $\succsim^\lambda$, the fourth equivalence from the fact that u^λ represents $\succsim^\lambda$, the fifth equivalence from the definition of p', the sixth equivalence from the fact that expected utility is linear in the belief, and the last equivalence from the definition of the utility function u.

Thus, we see that the utility function u represents $\succsim$, which completes the proof of Case 1.

Case 2. Suppose now that there are at least two choices a and b such that $\succsim^*$ exhibits a constant preference intensity between a and b. We start by constructing a set of choices D, as follows. Take an arbitrary choice $d_1 \in C$. If there is a choice $d_2 \neq d_1$ such that there is no constant preference intensity between d_2 and d_1, then select such a choice d_2. In the next step, if there is a choice $d_3 \neq d_1, d_2$ such that there is no constant preference intensity between d_3 and d_1 and between d_3 and d_2 then select such a choice d_3. Continue in this way until no further choice can be selected in this way. Let $D = \{d_1, ..., d_K\}$ be the resulting set. Then, by construction, there is no constant preference intensity between any two choices in D, and for every choice $c \notin D$ there is a choice $d \in D$ such that there is constant preference intensity between c and d. But we can show even more, as the following claim shows.

Claim. For every choice $c \notin D$ there is exactly one choice $d(c) \in D$ such that there is constant preference intensity between c and $d(c)$.

Proof of claim. Suppose there are two choices $d_1, d_2 \in D$ such that there is a constant preference intensity between c and d_1 and between c and d_2. By transitivity of constant preference intensity, it would then follow that there is a constant preference intensity between d_1 and d_2, which is a contradiction. This completes the proof of the claim.

We distinguish two cases: (2.1) the set D only contains one choice, and (2.2) the set D contains more than one choice.

Case 2.1. Suppose that D only contains one choice, say d. Then, for every choice $c \neq d$, there is constant preference intensity between c and d. By transitivity of constant preference intensity, it would follow that for every two choices $a, b \in C$ we have constant preference intensity between a and b. Consider an arbitrary signed belief q, with the induced ranking $c_1 \succ_q^* c_2 \succ_q^* ... \succ_q^* c_M$. Since there is constant preference intensity between any two choices, this same ranking is induced at *every* signed belief. Take some numbers $\alpha_1 > \alpha_2 > ... > \alpha_M$. Then, the utility function u with $u(c_m, s) := \alpha_m$ for every choice c_m and every state s represents $\succsim^*$, and thereby $\succsim$.

Case 2.2. Suppose that D contains at least two choices. By the claim, there are for every choice $a \notin D$ two choices $d(a), e(a) \in D$ such that there is constant preference intensity between a and $d(a)$, but not between a and $e(a)$. We define the utility function u as follows.

Since there is no constant preference intensity between any two choices in D, we know from Case 1 that there is a utility function v that represents $\succsim^*$ on D. We set $u(d, s) := v(d, s)$ for every choice $d \in D$ and state $s \in S$.

Now take some choice $a \notin D$. As there is no constant preference intensity between a and $e(a)$, there is a signed belief $q_{ae(a)}$ where the DM is "indifferent" between a and $e(a)$. Recall that there is constant preference intensity between a and $d(a) \in D$. We define, for every state s,

$$u(a, s) := u(d(a), s) + u(e(a), q_{ae(a)}) - u(d(a), q_{ae(a)}). \tag{2.12.33}$$

We show that this utility function u represents $\succsim^*$, by proving that u represents $\succsim^*$ on $\{a, b\}$ for every two choices $a, b \in C$. We distinguish the following cases: (2.2.1) $a, b \in D$, (2.2.2) $a \notin D$ and $b = d(a)$, (2.2.3) $a \notin D$ and $b = e(a)$, (2.2.4) $a \notin D$ and $b \in D \backslash \{d(a), e(a)\}$, and (2.2.5) $a, b \notin D$.

Case 2.2.1. Suppose that $a, b \in D$. Then, u represents $\succsim^*$ on $\{a, b\}$ since v represents $\succsim^*$ on D.

Case 2.2.2. Suppose that $a \notin D$ and $b = d(a)$. Since there is constant preference intensity between a and $d(a)$, it must be that either $a \succ_q^* d(a)$ for all signed beliefs q, or $d(a) \succ_q^* a$ for all signed beliefs q. Assume, without loss of generality, that $a \succ_q^* d(a)$ for all signed beliefs q. Since $e(a) \sim_{q_{ae(a)}}^* a$, it follows that $e(a) \succ_{q_{ae(a)}}^* d(a)$. As u represents $\succsim^*$ on D, we have that $u(e(a), q_{ae(a)}) > u(d(a), q_{ae(a)})$. By (2.12.33) we conclude that $u(a, q) > u(d(a), q)$ for all signed beliefs q, and hence u represents $\succsim^*$ on $\{a, d(a)\}$.

Case 2.2.3. Assume that $a \notin D$ and $b = e(a)$. Recall from above that $e(a) \sim_{q_{ae(a)}}^* a$. Moreover, by (2.12.33), we know that $u(a, q_{ae(a)}) = u(e(a), q_{ae(a)})$, and thus $q_{ae(a)} \in Q_{u(a)=u(e(a))}$. Here, we denote by $Q_{u(a)=u(e(a))}$ the set of signed beliefs q where $u(a, q) = u(e(a), q)$. As there is constant preference intensity between a and $d(a)$, but not between a and $e(a)$ and not between $d(a)$ and $e(a)$, we know from Lemma 2.12.8 (b) that the sets $Q_{a \sim^* e(a)}$ and $Q_{d(a) \sim^* e(a)}$ are parallel. Since, by (2.12.33), the expected utility difference between a and $d(a)$ is constant across all signed beliefs, we know that also the sets $Q_{u(a)=u(e(a))}$ and $Q_{u(d(a))=u(e(a))}$ are parallel. As u represents $\succsim^*$ on D, we must have that $Q_{d(a) \sim^* e(a)} = Q_{u(d(a))=u(e(a))}$.

Summarizing, we thus see that (i) $Q_{u(a)=u(e(a))}$ and $Q_{u(d(a))=u(e(a))}$ are parallel, (ii) $Q_{u(d(a))=u(e(a))} = Q_{d(a)\sim^* e(a)}$, and (iii) $Q_{d(a)\sim^* e(a)}$ and $Q_{a\sim^* e(a)}$ are parallel. Thus, $Q_{u(a)=u(e(a))}$ and $Q_{a\sim^* e(a)}$ are parallel. Since $q_{ae(a)}$ is in both $Q_{a\sim^* e(a)}$ and $Q_{u(a)=u(e(a))}$, it follows that $Q_{u(a)=u(e(a))} = Q_{a\sim^* e(a)}$.

Since there is no constant preference intensity between $d(a)$ and $e(a)$, there must be some $q_{d(a)e(a)}$ with $d(a) \sim^*_{q_{d(a)e(a)}} e(a)$. Recall from above that $a \succ^*_q d(a)$ for all signed beliefs q, and thus $a \succ^*_{q_{d(a)e(a)}} d(a)$. Hence, $a \succ^*_{q_{d(a)e(a)}} e(a)$. As u represents $\succsim^*$ on $\{a, d(a)\}$ and $\{d(a), e(a)\}$, we have that $u(a, q_{d(a)e(a)}) > u(d(a), q_{d(a)e(a)})$ and $u(d(a), q_{d(a)e(a)}) = u(e(a), q_{d(a)e(a)})$. This implies $u(a, q_{d(a)e(a)}) > u(e(a), q_{d(a)e(a)})$. We have thus found a belief $q_{d(a)e(a)}$ with $a \succ^*_{q_{d(a)e(a)}} e(a)$ and $u(a, q_{d(a)e(a)}) > u(e(a), q_{d(a)e(a)})$.

As $Q_{u(a)=u(e(a))} = Q_{a\sim^* e(a)}$ it can be shown, in a similar way as in the proof of Lemma 2.12.3, that u represents $\succsim^*$ on $\{a, e(a)\}$.

Case 2.2.4. Assume that $a \notin D$ and $b \in D\backslash\{d(a), e(a)\}$. We distinguish three cases: (2.2.4.1) $Q_{a\sim^* e(a)}$ is not parallel to $Q_{b\sim^* e(a)}$, (2.2.4.2) $Q_{a\sim^* e(a)}$ is parallel to $Q_{b\sim^* e(a)}$ but $Q_{a\sim^* e(a)} \neq Q_{b\sim^* e(a)}$, and (2.2.4.3) $Q_{a\sim^* e(a)} = Q_{b\sim^* e(a)}$.

Case 2.2.4.1. Suppose that $Q_{a\sim^* e(a)}$ is not parallel to $Q_{b\sim^* e(a)}$. Then, there is some signed belief $q_{ab} \in Q_{a\sim^* e(a)} \cap Q_{b\sim^* e(a)}$. As $\succsim^*$ is transitive, it follows that $q_{ab} \in Q_{a\sim^* b}$. Since u represents $\succsim^*$ on $\{a, e(a)\}$ and $\{b, e(a)\}$, we know that $u(a, q_{ab}) = u(e(a), q_{ab}) = u(b, q_{ab})$. We have thus found a signed belief $q_{ab} \in Q_{a\sim^* b}$ with $q_{ab} \in Q_{u(a)=u(b)}$.

As there is constant preference intensity between a and $d(a)$, but not between a and b and not between b and $d(a)$, we know by Lemma 2.12.8 (b) that $Q_{a\sim^* b}$ is parallel to $Q_{b\sim^* d(a)}$. Moreover, as u represents $\succsim^*$ on $\{b, d(a)\}$, we know that $Q_{b\sim^* d(a)} = Q_{u(b)=u(d(a))}$. Since, by (2.12.33), the expected utility between a and $d(a)$ is constant across all signed beliefs, we have that $Q_{u(a)=u(b)}$ is parallel to $Q_{u(b)=u(d(a))}$. Summarizing, we see that (i) $Q_{u(a)=u(b)}$ is parallel to $Q_{u(b)=u(d(a))}$, (ii) $Q_{u(b)=u(d(a))} = Q_{b\sim^* d(a)}$, and (iii) $Q_{b\sim^* d(a)}$ is parallel to $Q_{a\sim^* b}$. Thus, $Q_{u(a)=u(b)}$ is parallel to $Q_{a\sim^* b}$. Since q_{ab} is both in $Q_{a\sim^* b}$ and $Q_{u(a)=u(b)}$, we conclude that $Q_{u(a)=u(b)} = Q_{a\sim^* b}$.

Take some signed belief $q_{d(a)b}$ in $Q_{d(a)\sim^* b}$. Since we assume that $a \succ^*_q d(a)$ for all signed beliefs q, we have that $a \succ^*_{q_{d(a)b}} d(a) \sim^*_{q_{d(a)b}} b$, and thus $a \succ^*_{q_{d(a)b}} b$. As u represents $\succsim^*$ on $\{a, d(a)\}$ and $\{d(a), b\}$, we have that $u(a, q_{d(a)b}) > u(d(a), q_{d(a)b}) = u(b, q_{d(a)b})$. Hence, we have a found a belief $q_{d(a)b}$ with $a \succ^*_{q_{d(a)b}} b$ and $u(a, q_{d(a)b}) > u(b, q_{d(a)b})$. Since $Q_{u(a)=u(b)} = Q_{a\sim b}$, we can use a similar argument as in the proof of Lemma 2.12.3 to show that u represents $\succsim^*$ on $\{a, b\}$.

Case 2.2.4.2. Suppose that $Q_{a\sim^* e(a)}$ is parallel to $Q_{b\sim^* e(a)}$ but $Q_{a\sim^* e(a)} \neq Q_{b\sim^* e(a)}$. We show that the sets $Q_{a\sim^* e(a)}, Q_{b\sim^* e(a)}, Q_{a\sim^* b}, Q_{d(a)\sim^* e(a)}$ and $Q_{d(a)\sim^* b}$ must all be parallel. As there is constant preference intensity between a and $d(a)$, but not between a and $e(a)$ and not between $e(a)$ and $d(a)$, it follows by Lemma 2.12.8 (b) that $Q_{a\sim^* e(a)}$ and $Q_{d(a)\sim^* e(a)}$ are parallel. Similarly, since there is constant preference intensity between a and $d(a)$, but not between a and b and not between b and $d(a)$, it follows by Lemma 2.12.8 (b) that $Q_{a\sim^* b}$ and $Q_{d(a)\sim^* b}$ are parallel. Moreover, by assumption, $Q_{a\sim^* e(a)}$ is parallel to $Q_{b\sim^* e(a)}$. Now suppose, contrary to what we want to show, that $Q_{a\sim^* b}$ is not parallel to $Q_{a\sim^* e(a)}$. Then, there is some $q \in Q_{a\sim^* b} \cap Q_{a\sim^* e(a)}$ and hence, by transitivity of $\succsim^*$, we have that $q \in Q_{b\sim^* e(a)}$. But then, q is in both $Q_{a\sim^* e(a)}$ and $Q_{b\sim^* e(a)}$, which is impossible since both sets are parallel but not equal. Hence, we must conclude that $Q_{a\sim^* b}$ is parallel to $Q_{a\sim^* e(a)}$. But then, all five sets $Q_{a\sim^* e(a)}, Q_{b\sim^* e(a)}, Q_{a\sim^* b}, Q_{d(a)\sim^* e(a)}$ and $Q_{d(a)\sim^* b}$ are parallel.

Take a line l of signed beliefs that crosses each of these five sets once, and let $q_{ae(a)}, q_{be(a)}, q_{ab}, q_{d(a)e(a)}$ and $q_{d(a)b}$ be the signed beliefs on the line where the DM is "indifferent" between the respective choices.

As u represents $\succsim^*$ on $\{a, e(a)\}, \{b, e(a)\}, \{d(a), e(a)\}$ and $\{d(a), b\}$, we conclude that

$$u(a, q_{ae(a)}) = u(e(a), q_{ae(a)}),\ u(b, q_{be(a)}) = u(e(a), q_{be(a)}),$$
$$u(d(a), q_{d(a)e(a)}) = u(e(a), q_{d(a)e(a)}) \text{ and } u(d(a), q_{d(a)b}) = u(b, q_{d(a)b}).$$

Recall that there is constant preference intensity between a and $d(a)$. Since $\succsim^*$ satisfies part (a) of four choice linear preference intensity with constant preference intensity, we know that q_{ab} is uniquely given by the other four signed indifference beliefs. Moreover, as the signed conditional preference relation $\succsim^{*u}$ induced by u also satisfies part (a) of four choice linear preference intensity with constant preference intensity, and coincides with $\succsim^*$ on $\{a, e(a)\}, \{b, e(a)\}, \{d(a), e(a)\}$ and $\{d(a), b\}$, we conclude that $q_{ab} \in Q_{a\sim^{*u}b}$ and hence $u(a, q_{ab}) = u(b, q_{ab})$. Thus, we have found a signed belief $q_{ab} \in Q_{a\sim^*b}$ with $q_{ab} \in Q_{u(a)=u(b)}$.

Since the expected utility difference between a and $d(a)$ is constant across all signed beliefs, we know that (i) $Q_{u(a)=u(b)}$ is parallel to $Q_{u(d(a))=u(b)}$. Moreover, as u represents $\succsim^*$ on $\{d(a), b\}$, we have that (ii) $Q_{u(d(a))=u(b)} = Q_{d(a)\sim^*b}$. Finally, we know that (iii) $Q_{d(a)\sim^*b}$ is parallel to $Q_{a\sim^*b}$. By combining (i), (ii) and (iii), we conclude that $Q_{u(a)=u(b)}$ is parallel to $Q_{a\sim^*b}$. But since we have found a signed belief $q_{ab} \in Q_{a\sim^*b}$ with $q_{ab} \in Q_{u(a)=u(b)}$, it must be that $Q_{u(a)=u(b)} = Q_{a\sim^*b}$.

Now, take some signed belief q with $d(a) \sim_q^* b$. As $a \succ_{q'}^* d(a)$ for all signed beliefs q', we conclude that $a \succ_q^* b$. Since u represents $\succsim^*$ on $\{d(a), b\}$ and $\{a, d(a)\}$, we know that $u(a, q) > u(d(a), q) = u(b, q)$. Hence, we have found some signed belief q with $a \succ_q^* b$ and $u(a, q) > u(b, q)$. Since $Q_{u(a)=u(b)} = Q_{a\sim^*b}$, we can show in a similar way as in the proof of Lemma 2.12.3 that u represents $\succsim^*$ on $\{a, b\}$.

Case 2.2.4.3. Assume that $Q_{a\sim^*e(a)} = Q_{b\sim^*e(a)}$. As a and b are not equivalent, it follows by transitivity of $\succsim^*$ that $Q_{a\sim^*b} = Q_{a\sim^*e(a)} = Q_{b\sim^*e(a)}$. Take an arbitrary $q_{ab} \in Q_{a\sim^*b}$. As q_{ab} is in both $Q_{a\sim^*e(a)}$ and $Q_{b\sim^*e(a)}$, and u represents $\succsim^*$ on $\{a, e(a)\}$ and $\{b, e(a)\}$, it follows that $u(a, q) = u(e(a), q) = u(b, q)$. Thus, $Q_{a\sim^*b} \subseteq Q_{u(a)=u(b)}$.

Take some signed belief q with $d(a) \sim_q^* b$. Since $a \succ_{q'}^* d(a)$ for all signed beliefs q', we know that $a \succ_q^* b$. As u represents $\succsim^*$ on $\{d(a), b\}$ and $\{a, d(a)\}$, it follows that $u(a, q) > u(d(a), q) = u(b, q)$. Thus, we have found some signed belief q with $a \succ_q^* b$ and $u(a, q) > u(b, q)$.

We now show that $Q_{a\sim^*b} = Q_{u(a)=u(b)}$. To see this, recall from above that $Q_{a\sim^*b} \subseteq Q_{u(a)=u(b)}$, which implies that $span(Q_{a\sim^*b}) \subseteq span(Q_{u(a)=u(b)})$. Recall also that $span(Q_{a\sim^*b})$ has dimension $n-1$, which means that $span(Q_{u(a)=u(b)})$ has dimension $n-1$ or n. Suppose, contrary to what we want to show, that $span(Q_{u(a)=u(b)})$ has dimension n. Then, $span(Q_{u(a)=u(b)}) = \mathbf{R}^S$, and thus, by Lemma 2.12.8 (a), $Q_{u(a)=u(b)} = \Delta^*(S)$. However, we have found above a signed belief q with $u(a, q) > u(b, q)$, and thus $q \notin Q_{u(a)=u(b)}$. This is a contradiction. We thus conclude that $span(Q_{u(a)=u(b)})$ has dimension $n-1$. Since $span(Q_{a\sim^*b}) \subseteq span(Q_{u(a)=u(b)})$ and $span(Q_{a\sim^*b})$ has dimension $n-1$, it follows that $span(Q_{a\sim^*b}) = span(Q_{u(a)=u(b)})$. By Lemma 2.12.8 (a), it then follows that $Q_{a\sim^*b} = Q_{u(a)=u(b)}$.

Summarizing, we see that $Q_{u(a)=u(b)} = Q_{a\sim^*b}$, and there is a signed belief q where $a \succ_q^* b$ and $u(a, q) > u(b, q)$. We can then show in a similar way as in the proof of Lemma 2.12.3 that u represents $\succsim^*$ on $\{a, b\}$.

Case 2.2.5. Suppose finally that $a, b \notin D$. We distinguish two cases: (2.2.5.1) $d(a) = d(b)$, and (2.2.5.2) $d(a) \neq d(b)$.

Case 2.2.5.1. Assume that $d(a) = d(b)$. Then, there is constant preference intensity between a and $d(a)$ and between b and $d(a)$. By transitivity of constant preference intensity, there is also constant preference intensity between a and b. That is, either $a \succ_q^* b$ for all signed beliefs q, or $b \succ_q^* a$ for all signed beliefs q. Assume, without loss of generality, that $a \succ_q^* b$ for all signed beliefs q.

Take some choice $c \in D \backslash \{d(a)\}$. Then, we know by the claim that there is no constant preference intensity between a and c, and hence there is a signed belief q with $a \sim_q^* c$. As $a \succ_q^* b$, we know by transitivity of $\succsim^*$ that $c \succ_q^* b$. Since, by the previous cases, u represents $\succsim^*$ on $\{a, c\}$ and $\{b, c\}$, it follows that $u(a,q) = u(c,q) > u(b,q)$. We have thus found a signed belief q with $u(a,q) > u(b,q)$.

Since $d(a) = d(b)$ we know, by construction of the utility function u in (2.12.33), that the expected utility difference between a and b is constant across all signed beliefs. As we have found a signed belief q with $u(a,q) > u(b,q)$, we conclude that $u(a,q') > u(b,q')$ for all signed beliefs q'. Since $a \succ_{q'}^* b$ for all signed beliefs q', we conclude that u represents $\succsim^*$ on $\{a, b\}$.

Case 2.2.5.2. Suppose that $d(a) \neq d(b)$. Then, we know by the claim that there is no constant preference intensity between a and $d(b)$, and also not between b and $d(a)$. Since there is constant preference intensity between a and $d(a)$, but not between a and $d(b)$ and not between $d(a)$ and $d(b)$, it follows by Lemma 2.12.8 (b) that (i) $Q_{d(a) \sim^* d(b)}$ is parallel to $Q_{a \sim^* d(b)}$. In a similar fashion, it follows that (ii) $Q_{d(a) \sim^* d(b)}$ is also parallel to $Q_{b \sim^* d(a)}$.

Moreover, since there is constant preference intensity between a and $d(a)$, but not between b and $d(a)$, it must be that there is also no constant preference intensity between a and b. Otherwise, it would follow by transitivity of constant preference intensity that there would also be constant preference intensity between b and $d(a)$, which would be a contradiction. But then, since there is constant preference intensity between a and $d(a)$ but not between b and $d(a)$, and not between a and b, it follows by Lemma 2.12.8 (b) that (iii) $Q_{b \sim^* d(a)}$ is parallel to $Q_{a \sim^* b}$. By combining (i), (ii) and (iii) we conclude that $Q_{a \sim^* b}, Q_{b \sim^* d(a)}, Q_{d(a) \sim^* d(b)}$ and $Q_{a \sim^* d(b)}$ are all parallel.

Take a line l of signed beliefs that cross each of these four parallel sets exactly once, and let $q_{ab}, q_{bd(a)}, q_{d(a)d(b)}$ and $q_{ad(b)}$ be the signed beliefs on this line where the DM is "indifferent" between the respective choices. As there is constant preference intensity between a and $d(a)$, and between b and $d(b)$, and since $\succsim^*$ satisfies part (b) of four choice linear preference intensity with constant preference intensity, we know that q_{ab} is uniquely given by the other three signed "indifference" beliefs.

Now, consider the conditional preference relation $\succsim^{*u}$ induced by the utility function u. Since also $\succsim^{*u}$ satisfies part (b) of four choice linear preference intensity with constant preference intensity, and since, by the previous cases, u represents $\succsim^*$ on $\{b, d(a)\}, \{d(a), d(b)\}$ and $\{a, d(b)\}$, we know that $q_{ab} \in Q_{a \sim^{*u} b}$, and hence $u(a, q_{ab}) = u(b, q_{ab})$. We have thus found a signed belief q_{ab} with $q_{ab} \in Q_{a \sim^* b}$ and $q_{ab} \in Q_{u(a)=u(b)}$.

Since, by (2.12.33), the expected utility difference between a and $d(a)$ is constant across all signed beliefs, we know that (i) $Q_{u(a)=u(b)}$ is parallel to $Q_{u(b)=u(d(a))}$. Since, by the previous cases, u represents $\succsim^*$ on $\{d(a), b\}$, it follows that (ii) $Q_{u(b)=u(d(a))} = Q_{b \sim^* d(a)}$. Moreover, we have seen above that (iii) $Q_{b \sim^* d(a)}$ is parallel to $Q_{a \sim^* b}$. By combining (i), (ii) and (iii) we conclude that $Q_{u(a)=u(b)}$ is parallel to $Q_{a \sim^* b}$. Since above we have found a signed belief q_{ab} with $q_{ab} \in Q_{a \sim^* b}$ and $q_{ab} \in Q_{u(a)=u(b)}$, it follows that $Q_{a \sim^* b} = Q_{u(a)=u(b)}$.

Now, take some signed belief q with $b \sim_q^* d(a)$. Since we are assuming that $a \succ_{q'}^* d(a)$ for all signed beliefs q', it follows by transitivity of $\succsim^*$ that $a \succ_q^* b$. As u represents $\succsim^*$ on $\{b, d(a)\}$ and $\{a, d(a)\}$, we know that $u(a,q) > u(d(a),q) = u(b,q)$. We have thus found a signed belief q with $a \succ_q^* b$ and $u(a,q) > u(b,q)$. Since $Q_{a \sim^* b} = Q_{u(a)=u(b)}$ we can show, in a similar way as in the proof of Lemma 2.12.3, that u represents $\succsim^*$ on $\{a, b\}$.

Since we have covered all the possible cases, we conclude that u represents $\succsim^*$ on every pair of choices $\{a, b\}$, and thus u represents $\succsim^*$. Since $\succsim^*$ extends $\succsim$, it follows that u represents $\succsim$.

Recall that so far we have been assuming that no two choices are equivalent. Now, suppose that two, or more, choices are equivalent. In this case, we can select a subset C^* of choices such that (i) no two choices in C^* are equivalent, and (ii) every choice outside C^* is equivalent to a choice inside C^*.

By the proof above, we then know that there is a utility function u^* on C^* that represents $\succsim$ on C^*. This utility function can be extended to a utility function u on C, by setting, for every choice $c \notin C^*$,

$$u(c, s) := u(c^*, s)$$

where c^* is the unique choice in C^* that is equivalent to c. Then, the utility function u will represent $\succsim$ on the whole choice set C. This completes the proof.

■

Chapter 3

Common Belief in Rationality in Standard Games

3.7 Economic Applications

In this section we discuss two economic applications of common belief in rationality in standard games: One where there is competition in prices between two firms, and one where there is competition in quantities between two firms.

3.7.1 Competition in Prices

Consider two firms, 1 and 2, that compete for the market of a certain good. The goods they offer are *differentiated*, which means that the two goods have different characteristics. Some consumers prefer the characteristics of the good of firm 1, whereas other consumers like the good of firm 2 more. That is, even if a firm charges a higher price than its competitor it will still attract some consumers, because they enjoy the characteristics of this firm's good sufficiently more than those of the competitor. Of course, if firm 1 raises its price, then the demand for firm 1 will drop, whereas the demand for firm 2 will rise, and similarly for firm 2. More precisely, if firms 1 and 2 charge prices p_1 and p_2, then the demands for both firms are given by

$$q_1 = a - d \cdot p_1 + e \cdot p_2 \text{ and } q_2 = a - d \cdot p_2 + e \cdot p_1. \qquad (3.7.1)$$

Hence, d is a measure for the elastisticy of a firm's demand with respect to its own price, whereas e reflects the elasticity of demand with respect to the opponent's price.

We assume, for simplicity, that both firms have a constant marginal cost equal to c. Suppose that both firms choose a price from the interval $[0, M]$, that $e < 2d$ and that

$$M \geq \frac{c+a/d}{2-e/d}.$$

The objective of each firm is to maximize its profit. That is, for every pair of prices p_1, p_2, firm i's utility is equal to the induced profit. This type of competition is called *Bertrand competition.* What price(s) can both firms rationally choose under common belief in rationality?

To answer this question we first model the situation above as a game. Obviously, the two players are the two firms 1 and 2. The set of choices C_i for firm i is the set of possible prices it can choose, and hence $C_i = [0, M]$. Note that C_i is an infinite set, whereas we have been assuming in Chapter 3 that the sets of choices are finite. We will see, however, that the idea of common belief in rationality and its associated recursive procedure can easily be extended to the case of infinite choice sets.

What about player i's utility function? We have indicated above that firm i's utility is equal to its profit. Assume that the two firms choose prices p_1 and p_2. Then, the revenue for firm 1 is its price times its demand, which is

$$p_1 \cdot q_1 = p_1 \cdot (a - d \cdot p_1 + e \cdot p_2),$$

whereas its costs are the marginal cost times the demand, resulting in

$$c \cdot q_1 = c \cdot (a - d \cdot p_1 + e \cdot p_2).$$

The profit for firm 1, which is the revenue minus the costs, is therefore given by

$$\pi_1(p_1, p_2) = (p_1 - c) \cdot (a - d \cdot p_1 + e \cdot p_2). \tag{3.7.2}$$

Similarly, firm 2's profit is given by

$$\pi_2(p_1, p_2) = (p_2 - c) \cdot (a - d \cdot p_2 + e \cdot p_1). \tag{3.7.3}$$

Although the choice sets for both firms are infinite, we can still define probabilistic beliefs in the same way as in the book. Indeed, a belief for firm i would be a probability distribution β_i over the set $C_j = [0, M]$ of opponent's prices. To keep things simple, let us concentrate on beliefs β_i that only assign positive probability to finitely many choices of the opponent. Like in Section 2.11 of this online appendix, we denote by $supp(\beta_i)$ the set of opponent's choices to which β_i assigns positive probability, and call it the *support* of belief β_i. But then, we can define belief hierarchies, epistemic models and the condition of common belief in rationality in essentially the same way as in Chapter 3 of the book.

What about the recursive procedure for common belief in rationality in this setting? In Theorem 3.4.1 of the book we have seen that for the case of *finitely* many choices, the procedure of *iterated elimination of strictly dominated choices* selects precisely those choices that can rationally be made under common belief in rationality. Moreover, we know from Theorem 2.6.1 in the book that the choices that are strictly dominated are precisely the choices that are not optimal for a probabilistic belief. Hence, this procedure is equivalent to the following procedure, which we call *iterated elimination of suboptimal choices*: In the first round we eliminate all choices that are not optimal for any probabilistic belief. In the second round, we start by eliminating those states that involve an opponent's choice that has been eliminated in round 1, which leads to a reduced decision problem for every player. In every reduced decision problem, we then eliminate those choices that are not optimal for any probabilistic belief. And so on.

This procedure, which reveals the idea of common belief in rationality, can be applied to games with infinite choice sets as well. In particular, it can be applied to our setting here, to identify the prices that both firms can rationally choose under common belief in rationality. However, as we will see, the procedure will no longer terminate after finitely many rounds. Indeed, since we start with infinitely many choices, it is no longer guaranteed that the procedure will stop after finitely many rounds.

Let us now apply the *iterated elimination of suboptimal choices* to the price competition game above. At the end of every round k, let firm i's decision problem be given by (P_i^k, S_i^k), where P_i^k is the set of prices for firm i that survive round k, and S_i^k is the set of states that survive round k.

Clearly, before we start the procedure we have $P_i^0 = [0, M]$ and $S_i^0 = [0, M]$ for both firms i.

Round 1. Consider firm 1, whose utility function is given by (3.7.2). By definition, we allow for all possible states in round 1, and hence $S_1^1 = [0, M]$. Which prices are optimal for some probabilistic belief, and which are not? Suppose firm 1 holds the belief β_1 about firm 2's price. Then, the *expected* profit for firm 1 is

$$\begin{aligned}
\pi_1(p_1, \beta_1) &= \sum_{p_2 \in supp(\beta_1)} \beta_1(p_2) \cdot \pi_1(p_1, p_2) \\
&= \sum_{p_2 \in supp(\beta_1)} \beta_1(p_2) \cdot (p_1 - c) \cdot (a - d \cdot p_1 + e \cdot p_2) \\
&= (p_1 - c) \cdot (a - d \cdot p_1 + e \cdot [\sum_{p_2 \in supp(\beta_1)} \beta_1(p_2) \cdot p_2]) \\
&= (p_1 - c) \cdot (a - d \cdot p_1 + e \cdot E_{\beta_1}(p_2)),
\end{aligned} \tag{3.7.4}$$

where

$$E_{\beta_1}(p_2) := \sum_{p_2 \in supp(\beta_1)} \beta_1(p_2) \cdot p_2$$

denotes the *expected* price for firm 2 under the belief β_1.

From (3.7.4) we see that firm 1's expected profit, when viewed as a function of its price p_1, is a second-degree polynomial in p_1 that becomes zero for $p_1 = c$ and $p_1 = (a + e \cdot E_{\beta_1}(p_2))/d$, and that obtains a maximum exacty halfway these two points. That is, the unique optimal price for firm 1 under the belief β_1 is given by

$$\begin{aligned}
p_1 &= \tfrac{1}{2} \cdot c + \tfrac{1}{2} \cdot \tfrac{a + e \cdot E_{\beta_1}(p_2)}{d} \\
&= \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot E_{\beta_1}(p_2).
\end{aligned} \tag{3.7.5}$$

As $E_{\beta_1}(p_2)$ can only take values between 0 and M, and the optimal price is increasing in $E_{\beta_1}(p_2)$, the optimal price can only take values between $\frac{1}{2}(c + \frac{a}{d})$ and $\frac{1}{2}(c + \frac{a}{d}) + \frac{eM}{2d}$. Moreover, $\frac{1}{2}(c + \frac{a}{d}) + \frac{eM}{2d} \leq M$ since, by assumption, $e < 2d$ and $M \geq \frac{c + a/d}{2 - e/d}$. Hence the set of optimal prices in round 1 is

$$P_1^1 = [\tfrac{1}{2}(c + \tfrac{a}{d}),\ \tfrac{1}{2}(c + \tfrac{a}{d}) + \tfrac{eM}{2d}]. \tag{3.7.6}$$

Similarly for firm 2.

Round 2. Consider firm 1 first. By definition of the procedure, S_1^2 contains those opponent's prices that have survived round 1. Hence, $S_1^2 = P_2^1 = [\frac{1}{2}(c + \frac{a}{d}), \frac{1}{2}(c + \frac{a}{d}) + \frac{eM}{2d}]$, which leads to a reduced decision problem for firm 1.

Which prices are optimal for firm 1 if it forms a belief about the states in S_1^2. In other words, when firm 1 holds a belief β_1 that only assigns positive probability to opponent's prices in P_2^1. In that case, the optimal price for firm 1 is given by (3.7.5), where the expected price $E_{\beta_1}(p_2)$ for firm 2 under the belief β_1 must be in P_2^1. Therefore, the lowest price l for firm 1 that is optimal for such a belief β_1 is given by (3.7.5) if we substitute $E_{\beta_1}(p_2) = \frac{1}{2}(c + \frac{a}{d})$. Similarly, the highest price h for firm 1 that is optimal for such a belief β_1 is given by (3.7.5) if we substitute $E_{\beta_1}(p_2) = \frac{1}{2}(c + \frac{a}{d}) + \frac{eM}{2d}$. Thus,

$$l = \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot (\tfrac{1}{2}(c + \tfrac{a}{d})) = \tfrac{1}{2}(c + \tfrac{a}{d})(1 + \tfrac{e}{2d})$$

and

$$\begin{aligned} h &= \tfrac{1}{2}\cdot(c+\tfrac{a}{d}) + \tfrac{e}{2d}\cdot(\tfrac{1}{2}(c+\tfrac{a}{d}) + \tfrac{eM}{2d}) \\ &= \tfrac{1}{2}(c+\tfrac{a}{d})(1+\tfrac{e}{2d}) + (\tfrac{e}{2d})^2 M. \end{aligned}$$

Hence, the set of prices for firm 1 that are optimal for some belief in round 2 is given by

$$P_1^2 = [\tfrac{1}{2}(c+\tfrac{a}{d})(1+\tfrac{e}{2d}),\ \tfrac{1}{2}(c+\tfrac{a}{d})(1+\tfrac{e}{2d}) + (\tfrac{e}{2d})^2 M].$$

Similarly for firm 2.

If we continue like this, we can derive for every round k the set P_1^k of prices that survive for firm 1, and similarly for firm 2. In fact, it can be shown that

$$P_1^k = [\tfrac{c+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^k),\ \tfrac{c+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^k) + (\tfrac{e}{2d})^k M] \tag{3.7.7}$$

for every round k, and similarly for firm 2. We will now show, by induction on k, that (3.7.7) holds.

For $k = 0$ we have that $P_1^0 = [0, M]$ and hence (3.7.7) holds.

Take now some $k \geq 1$, and assume that (3.7.7) holds for $k-1$. Concentrate on firm 1. By definition, $S_1^k = P_2^{k-1}$, and P_1^k contains those prices that are optimal for some belief on S_1^k. Take a belief β_1 on S_1^k. Since $S_1^k = P_2^{k-1}$, it follows by the induction assumption that

$$S_1^k = [\tfrac{c+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k-1}),\ \tfrac{c+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k-1}) + (\tfrac{e}{2d})^{k-1} M],$$

and hence the expected price $E_{\beta_1}(p_2)$ must be in this interval as well. Recall that the optimal price under the belief β_1 is given by (3.7.5). As the lowest value for $E_{\beta_1}(p_2)$ is $\frac{c+a/d}{2-e/d}\cdot(1-(\frac{e}{2d})^{k-1}$, the lowest price that is optimal for such a belief β_1 is

$$\begin{aligned} l &= \tfrac{1}{2}(c+\tfrac{a}{d}) + \tfrac{e}{2d}\cdot(\tfrac{c+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k-1})) \\ &= \tfrac{c+a/d}{2-e/d}\cdot(\tfrac{1}{2}(2-\tfrac{e}{d}) + \tfrac{e}{2d} - (\tfrac{e}{2d})^k) \\ &= \tfrac{c+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^k). \end{aligned}$$

Similarly, as the highest value for $E_{\beta_1}(p_2)$ is $\frac{c+a/d}{2-e/d}\cdot(1-(\frac{e}{2d})^{k-1} + (\frac{e}{2d})^{k-1}M$, the highest price that is optimal for such a belief β_1 is

$$\begin{aligned} h &= \tfrac{1}{2}(c+\tfrac{a}{d}) + \tfrac{e}{2d}\cdot(\tfrac{c+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k-1}) + (\tfrac{e}{2d})^{k-1}M) \\ &= \tfrac{c+a/d}{2-e/d}\cdot(\tfrac{1}{2}(2-\tfrac{e}{d}) + \tfrac{e}{2d} - (\tfrac{e}{2d})^k) + (\tfrac{e}{2d})^k M \\ &= \tfrac{c+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^k) + (\tfrac{e}{2d})^k M. \end{aligned}$$

Thus, $P_1^k = [l, h]$, which is the interval given by (3.7.7). By induction on k, we conclude that (3.7.7) holds for every k. And similarly for firm 2.

Note that with every round k the interval of prices P_1^k becomes strictly smaller, and therefore the procedure does not terminate within finitely many rounds. In the limit, when k tends to infinity, the term $(\frac{e}{2d})^k$ goes to zero, since we assume that $e < 2d$, and therefore $\frac{e}{2d} < 1$. Hence, when k tends to infinity, the interval of prices P_1^k collapses to a single price, which is

$$p^* = \tfrac{c+a/d}{2-e/d}. \tag{3.7.8}$$

This is thus the only price that survives all rounds of the *iterated elimination of suboptimal choices,* and similarly for firm 2. We thus conclude that under common belief in rationality, both firms can only rationally choose the price p^* above.

But we can say a bit more: Similarly to Theorem 3.4.1 in the book, we can conclude that for every $k \geq 1$, the prices that firm 1 can rationally choose if it expresses up to k-fold belief in rationality are given by the set P_1^{k+1} in (3.7.7), and similarly for firm 2.

Let us finally investigate how the price p^* depends on the various parameters in the model, and why this makes intuitive sense. In view of (3.7.8) we see that the price p^* increases in its marginal cost c. This is natural, since an increase in the marginal cost leads the firm to choose a higher price to compensate for it. Moreover, the price p^* is increasing in a, which somehow measures the size of the market. Also this is intuitive, since a larger market allows the firms to choose higher prices and still obtain a "reasonable" demand. The price p^* is decreasing in the elasticity parameter d, which measures how quickly the demand for firm 1 drops if it increases its price. Indeed, if d becomes larger, then consumers react more fiercely to a price raise of firm 1, which forces firm 1 to choose a lower price in order to still obtain a "reasonable" demand. Finally, the price p^* is increasing in e, which measures how quickly the demand for firm 1 rises if the opponent raises its price. This makes sense, since a higher e means that consumers will switch more quickly to firm 1 if firm 2 raises its price, which allows firm 1 to choose a higher price.

3.7.2 Competition in Quantities

Consider two firms, 1 and 2, that produce *homogeneous* goods, meaning they produce goods that are either identical or very similar. Both firms compete for the market of that good, not by choosing prices but by choosing the *quantities* they wish to produce. This type of competition is called *Cournot competition.*

Assume that both firms i can choose a quantity q_i in the interval $[0, M]$. The resulting market price for the good is then given by

$$p = a - e \cdot (q_1 + q_2), \tag{3.7.9}$$

where e measures how quickly the market price drops if the total supply $q_1 + q_2$ of the good increases. We refer to e as the *elasticity* parameter. Like in the Bertrand competition described above, we suppose that both firms have a constant marginal cost equal to c. We assume that $c < a$, and that $M \in [\frac{a-c}{2e}, \frac{a-c}{e}]$. To objective of each firm is to maximize its profit. Hence, the firms' utility functions are equal to their profit functions.

If we model this situation as a game, then the players are the two firms, and player i's choice set is given by $C_i = [0, M]$ – the set of possible quantities that can be chosen. We will next derive player 1's utility function. If firm 1 chooses a quantity q_1 and firm 2 chooses a quantity q_2, then its revenue is equal to the quantity it sells times the market price, which is given by $q_1 \cdot (a - e \cdot (q_1 + q_2))$. The costs for firm 1 are given by the quantity it produces times the constant marginal cost, which is $q_1 \cdot c$. The profit for firm 1, which is the revenue minus the costs, is thus equal to

$$\begin{aligned} \pi_1(q_1, q_2) &= q_1 \cdot (a - e \cdot (q_1 + q_2)) - q_1 \cdot c \\ &= q_1 \cdot (a - c - e \cdot (q_1 + q_2)). \end{aligned} \tag{3.7.10}$$

This is the utility function of firm 1. Similarly for firm 2.

What quantity, or quantities, can both firms rationally choose under common belief in rationality? Note that both firms have an infinite choice set and an infinite set of states. We therefore use the *iterated elimination of suboptimal choices* explained in the previous subsection to find the quantities that are possible under common belief in rationality.

For every round k, let Q_i^k and S_i^k be the sets of quantities and states that survive for firm i in round k. Then, $Q_i^0 = S_i^0 = [0, M]$ for both firms i.

Round 1. Focus on firm 1. By definition, the set of states in round 1 is $S_1^1 = [0, M]$. Suppose that firm 1 holds a belief β_1 about the quantity of firm 2. As in the previous subsection, we assume that β_1 assigns positive probability only to a finite number of quantities. By $supp(\beta_1)$ we denote the set of quantities for firm 2 that β_1 assigns a positive probability to. Then, the *expected* profit for firm 1 of choosing a quantity q_1 under the belief β_1 is

$$\begin{aligned} \pi_1(q_1, \beta_1) &= \sum_{q_2 \in supp(\beta_1)} \beta_1(q_2) \cdot \pi_1(q_1, q_2) \\ &= \sum_{q_2 \in supp(\beta_1)} \beta_1(q_2) \cdot (q_1 \cdot (a - c - e \cdot (q_1 + q_2))) \\ &= q_1 \cdot (a - c - e \cdot (q_1 + [\sum_{q_2 \in supp(\beta_1)} \beta_1(q_2) \cdot q_2])) \\ &= q_1 \cdot (a - c - e \cdot (q_1 + E_{\beta_1}(q_2))). \end{aligned} \tag{3.7.11}$$

Here,

$$E_{\beta_1}(q_2) := \sum_{q_2 \in supp(\beta_1)} \beta_1(q_2) \cdot q_2$$

denotes the expected quantity of firm 2 under the belief β_1.

Note that $\pi_1(q_1, \beta_1)$, when viewed as a function of q_1, is a second-degree polynomial in q_1 that becomes zero at $q_1 = 0$ and $q_1 = \frac{a-c}{e} - E_{\beta_1}(q_2)$, and that has a maximum exactly halfway between these two points. Thus, the optimal quantity for firm 1 under the belief β_1 is given by

$$\begin{aligned} q_1 &= \tfrac{1}{2} \cdot 0 + \tfrac{1}{2} \cdot (\tfrac{a-c}{e} - E_{\beta_1}(q_2)) \\ &= \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot E_{\beta_1}(q_2). \end{aligned} \tag{3.7.12}$$

Note that this optimal quantity is at least zero, since $M \leq \frac{a-c}{e}$ and hence $E_{\beta_1}(q_2) \leq \frac{a-c}{e}$. Moreover, the optimal quantity is at most M, since $M \geq \frac{a-c}{2e}$.

As the optimal quantity is decreasing in $E_{\beta_1}(q_2)$ and the value of $E_{\beta_1}(q_2)$ is between 0 and M, the quantities q_1 that are optimal for some belief β_1 are between $\frac{a-c}{2e} - \frac{1}{2}M$ and $\frac{a-c}{2e}$. Thus, the set of quantities in round 1 that are optimal for some belief β_1 is

$$Q_1^1 = [\tfrac{a-c}{2e} - \tfrac{1}{2}M,\ \tfrac{a-c}{2e}].$$

Similarly for firm 2.

Round 2. Focus again on firm 1. By definition, the set of states in round 2 is $S_1^2 = Q_2^1 = [\frac{a-c}{2e} - \frac{1}{2}M, \frac{a-c}{2e}]$. Now, let firm 1 have a belief β_1 on the set of states S_2^1. Recall that the optimal quantity for firm 1 is given by (3.7.12) above. As the optimal quantity is decreasing in $E_{\beta_1}(q_2)$, and $E_{\beta_1}(q_2)$ is between $\frac{a-c}{2e} - \frac{1}{2}M$ and $\frac{a-c}{2e}$, the set of quantities in round 2 that are optimal for some belief β_1 on S_2^1 is

$$\begin{aligned} Q_1^2 &= [\tfrac{a-c}{2e} - \tfrac{1}{2} \cdot \tfrac{a-c}{2e},\ \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot (\tfrac{a-c}{2e} - \tfrac{1}{2}M)] \\ &= [\tfrac{a-c}{4e},\ \tfrac{a-c}{4e} + \tfrac{1}{4}M]. \end{aligned}$$

Similarly for firm 2.

If we continue in this fashion, we can compute the sets of quantities Q_1^k, Q_2^k for every round k. We will show, by induction on k, that

$$Q_1^k = \begin{cases} [(\frac{1}{3} - \frac{1}{3 \cdot 2^k}) \cdot \frac{a-c}{e},\ (\frac{1}{3} - \frac{1}{3 \cdot 2^k}) \cdot \frac{a-c}{e} + \frac{M}{2^k}], & \text{if } k = 0 \text{ or } k \text{ is even} \\ [(\frac{1}{3} + \frac{1}{3 \cdot 2^k}) \cdot \frac{a-c}{e} - \frac{M}{2^k},\ (\frac{1}{3} + \frac{1}{3 \cdot 2^k}) \cdot \frac{a-c}{e}], & \text{if } k \text{ is odd} \end{cases}, \tag{3.7.13}$$

and similarly for firm 2.

If $k = 0$ then $Q_1^0 = [0, M]$, which matches the equation (3.7.13).

Now suppose that $k \geq 1$, and that (3.7.13) holds for $k - 1$. We distinguish two cases: (1) k is odd, and (2) k is even.

Case 1. Assume that k is odd. Then, $k-1$ is either 0 or even. By the induction assumption, we know that (3.7.13) holds for Q_2^{k-1}. Take a belief β_1 on the set of states $S_1^k = Q_2^{k-1}$. Hence, the lowest value for $E_{\beta_1}(q_2)$ is $(\frac{1}{3} - \frac{1}{3 \cdot 2^{k-1}}) \cdot \frac{a-c}{e}$. As the optimal quantity q_1 is given by (3.7.12), which is decreasing in $E_{\beta_1}(q_2)$, the highest value of q_1 in Q_1^k is

$$h = \frac{a-c}{2e} - \frac{1}{2} \cdot (\frac{1}{3} - \frac{1}{3 \cdot 2^{k-1}}) \cdot \frac{a-c}{e} = (\frac{1}{3} + \frac{1}{3 \cdot 2^k}) \cdot \frac{a-c}{e},$$

which matches (3.7.13).

Moreover, the highest value for $E_{\beta_1}(q_2)$ is $(\frac{1}{3} - \frac{1}{3 \cdot 2^{k-1}}) \cdot \frac{a-c}{e} + \frac{M}{2^{k-1}}$, which implies that the lowest value for q_1 in Q_1^k is

$$l = \frac{a-c}{2e} - \frac{1}{2} \cdot [(\frac{1}{3} - \frac{1}{3 \cdot 2^{k-1}}) \cdot \frac{a-c}{e} + \frac{M}{2^{k-1}}] = (\frac{1}{3} + \frac{1}{3 \cdot 2^k}) \cdot \frac{a-c}{e} - \frac{M}{2^k}.$$

We thus conclude that $Q_1^k = [l, h]$, which matches (3.7.13).

Case 2. Assume that k is even. Then, $k - 1$ is odd. By the induction assumption, we know that (3.7.13) holds for Q_2^{k-1}. Take a belief β_1 on the set of states $S_1^k = Q_2^{k-1}$. Hence, the lowest value for $E_{\beta_1}(q_2)$ is $(\frac{1}{3} + \frac{1}{3 \cdot 2^{k-1}}) \cdot \frac{a-c}{e} - \frac{M}{2^{k-1}}$. As the optimal quantity q_1 is given by (3.7.12), which is decreasing in $E_{\beta_1}(q_2)$, the highest value of q_1 in Q_1^k is

$$h = \frac{a-c}{2e} - \frac{1}{2} \cdot [(\frac{1}{3} + \frac{1}{3 \cdot 2^{k-1}}) \cdot \frac{a-c}{e} - \frac{M}{2^{k-1}}] = (\frac{1}{3} - \frac{1}{3 \cdot 2^k}) \cdot \frac{a-c}{e} + \frac{M}{2^k},$$

which matches (3.7.13).

Moreover, the highest value for $E_{\beta_1}(q_2)$ is $(\frac{1}{3} + \frac{1}{3 \cdot 2^{k-1}}) \cdot \frac{a-c}{e}$, which implies that the lowest value for q_1 in Q_1^k is

$$l = \frac{a-c}{2e} - \frac{1}{2} \cdot (\frac{1}{3} + \frac{1}{3 \cdot 2^{k-1}}) \cdot \frac{a-c}{e} = (\frac{1}{3} - \frac{1}{3 \cdot 2^k}) \cdot \frac{a-c}{e}.$$

We thus conclude that $Q_1^k = [l, h]$, which matches (3.7.13).

By induction on k we see that (3.7.13) holds for every $k \geq 0$. Note that the sets Q_1^k, Q_2^k become strictly smaller with every round k, and therefore the procedure does not terminate within finitely many rounds, similarly to the case of Bertrand competition above. As k tends to infinity, the term 2^k tends to infinity as well, and therefore the sets Q_1^k and Q_2^k collapse to the quantity

$$q^* = \frac{a-c}{3e}. \tag{3.7.14}$$

This is the only quantity that survives all rounds of the iterated elimination of suboptimal choices for both firms. As such, we conclude that under common belief in rationality, both firms can only rationally choose the quantity q^* above.

Moreover, for every $k \geq 1$, the quantities that both firms can rationally choose if they express up to k-fold belief in rationality are given by Q_1^{k+1} and Q_2^{k+1} as specified by (3.7.13).

We finally investigate how the quantity q^* in (3.7.14) depends on the parameters in the model, and why this makes sense. First of all, the quantity q^* is increasing in a. If the parameter a increases, then this will lead to a larger market price for every combination of quantities chosen by the two firms. This, in turn, allows the firm to choose a larger quantity and still obtain a "reasonable" profit. Moreover, the quantity q^* is decreasing in the marginal cost c. Indeed, if c rises, then producing the same quantity becomes more costly than before, which forces the firm to reduce its production. Finally, the quantity q^* is decreasing in the elasticity parameter e. Also this is intuitive, because a larger e leads to a lower market price for every combination of quantities chosen. To compensate for this, the firm will reduce the quantity supplied.

Chapter 4

Correct and Symmetric Beliefs in Standard Games

4.6 Economic Applications

In this section we reconsider the models of competition in prices and competition in quantities introduced in Section 3.7 of this online appendix. For both models we will apply the concept of *Nash equilibrium* to explore what choice(s) both firms can rationally make under common belief in rationality with a simple belief hierarchy.

4.6.1 Competition in Prices

Recall the *Bertrand competition* model we introduced in Section 3.7.1 of this online appendix. We have argued that under common belief in rationality both firms can only rationally choose the price

$$p^* = \frac{c+a/d}{2-e/d}. \tag{4.6.1}$$

Hence, under common belief in rationality with a *simple belief hierarchy,* the only possible rational choice for both firms is to choose this price p^*. We will now verify that p^* can indeed rationally be chosen under common belief in rationality with a simple belief hierarchy.

Recall from Theorem 4.1.2 that in a game with *finitely* many choices, the choices that can rationally be made under common belief in rationality with a simple belief hierarchy are precisely the choices that are optimal in a *Nash equilibrium.* The same is true for games with *infinitely* many choices, if we extend the notion of Nash equilibrium to such games. In fact, the definition of a Nash equilibrium for a game with infinitely many choices is precisely the same as for finitely many choices.

Indeed, consider a game with infinitely many choices, where for every player i the choice set is the infinite set C_i, and the utility function is given by u_i. Then, a *Nash equilibrium* is a combination of beliefs $(\sigma_1, ..., \sigma_n)$ where, for every player i, the belief σ_i assigns positive probability to finitely many choices of player i, and $\sigma_i(c_i) > 0$ only if choice c_i is optimal for player i under the belief σ_{-i}.

Consider the price p^* in (4.6.1). Then, it can be shown that the belief combination ($\sigma_1 = p^*$, $\sigma_2 = p^*$), where σ_1 and σ_2 assign probability 1 to the price p^*, is a Nash equilibrium. Indeed, in this belief combination firm 1 believes that firm 2 chooses the price p^*. By (3.7.5), the optimal price for firm 1 is given by

$$\begin{aligned} p_1 &= \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot E_{\sigma_2}(p_2) = \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot p^* \\ &= \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \tfrac{c+a/d}{2-e/d} = \tfrac{c+a/d}{2-e/d} = p^*. \end{aligned}$$

Hence, σ_1 assigns probability 1 to the only price that is optimal for firm 1 under the belief σ_2. Similarly, σ_2 assigns probability 1 to the only price that is optimal for firm 2 under the belief σ_1. As such, ($\sigma_1 = p^*$, $\sigma_2 = p^*$) is a Nash equilibrium.

By Theorem 4.1.2 applied to games with infinitely many choices, we then know that both firms can rationally choose the price p^* under common belief in rationality with a simple belief hierarchy. Since we have seen in Section 3.7.1 that p^* is the *only* price that can rationally be chosen under common belief in rationality, we conclude that p^* is the *only* price that can rationally be chosen under common belief in rationality with a simple belief hierarchy. As every simple belief hierarchy is *symmetric* and uses *one theory per choice*, it immediately follows that p^* is also the only price that can rationally be chosen under common belief in rationality with a *symmetric* belief hierarchy, with or without insisting on *one theory per choice*.

Suppose now that we would not know which prices are possible under common belief in rationality. Is there then a quick way to find the prices that can rationally be chosen under common belief in rationality with a *simple* belief hierarchy? The answer is "yes", by directly computing the set of Nash equilibria in the game.

To see this, suppose that (σ_1, σ_2) is a Nash equilibrium in the Bertrand competition model. Then, $\sigma_1(p_1) > 0$ only if the price p_1 is optimal for firm 1 under the belief σ_2 about firm 2's price. By (3.7.5) we conclude that σ_1 must assign probability 1 to the unique price

$$p_1^* = \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot E_{\sigma_2}(p_2) \tag{4.6.2}$$

that is optimal for firm 1 under the belief σ_2. Similarly, σ_2 must assign probability 1 to the unique price

$$p_2^* = \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot E_{\sigma_1}(p_1) \tag{4.6.3}$$

that is optimal for firm 2 under the belief σ_1.

By (4.6.2) and (4.6.3) we know that $E_{\sigma_2}(p_2) = p_2^*$ and $E_{\sigma_1}(p_1) = p_1^*$. Thus,

$$p_1^* = \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot p_2^* \text{ and} \tag{4.6.4}$$

$$p_2^* = \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot p_1^*. \tag{4.6.5}$$

The equations (4.6.5) and (4.6.4) are often called *best response functions.* If we substitute (4.6.5) into (4.6.4) we obtain

$$\begin{aligned} p_1^* &= \tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot [\tfrac{1}{2} \cdot (c + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot p_1^*] \\ &= \tfrac{1}{2} \cdot (1 + \tfrac{e}{2d}) \cdot (c + \tfrac{a}{d}) + (\tfrac{e}{2d})^2 \cdot p_1^*. \end{aligned}$$

Hence,

$$p_1^* = \frac{1/2 \cdot (1+e/2d) \cdot (c+a/d)}{1-(e/2d)^2} = \frac{1/2 \cdot (1+e/2d) \cdot (c+a/d)}{(1+(e/2d)) \cdot (1-(e/2d))} = \frac{c+a/d}{2-e/d} = p^*.$$

If we substitute this into (4.6.5) we get that $p_2^* = p^*$ also. Hence, our conclusion is that there is only one Nash equilibrium, which is ($\sigma_1 = p^*$, $\sigma_2 = p^*$), assigning probability 1 to the price p^* for both firms. In the literature, this Nash equilibrium is known as the *Bertrand equilibrium.*

4.6.2 Competition in Quantities

Recall the *Cournot competition* model we discussed in Section 3.7.2. We saw that under common belief in rationality both firms can only rationally choose the quantity

$$q^* = \tfrac{a-c}{3e}. \tag{4.6.6}$$

Can this quantity also rationally be chosen under common belief in rationality with a simple belief hierarchy?

To answer that question we verify that the belief combination $(\sigma_1 = q^*, \sigma_2 = q^*)$, which assigns probability 1 to the quantity q^* for both firms, is a Nash equilibrium. Firm 1 believes that, with probability 1, firm 2 chooses the quantity q^*. By (3.7.12) we know that the unique optimal quantity for firm 1 under that belief is given by

$$q_1 = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot E_{\sigma_2}(q_2) = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot q^* = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot \tfrac{a-c}{3e} = \tfrac{a-c}{3e} = q^*.$$

Hence, σ_1 assigns probability 1 to the unique quantity that is optimal for firm 1 under the belief σ_2. Similarly, σ_2 assigns probability 1 to the unique quantity that is optimal for firm 2 under the belief σ_1. Therefore, $(\sigma_1 = q^*, \sigma_2 = q^*)$ is indeed a Nash equilibrium.

In view of Theorem 4.1.2 applied to games with infinitely many choices we conclude that both firms can rationally choose the quantity q^* under common belief in rationality with a simple belief hierarchy. As q^* is the *only* quantity that can rationally be chosen under common belief in rationality, it follows that q^* is the *only* quantity that can rationally be chosen under common belief in rationality with a *simple* belief hierarchy. As a consequence, q^* is the *only* quantity that can rationally be chosen under common belief in rationality with a *symmetric* belief hierarchy, with or without insisting on the *one theory per choice* condition.

We finally show how to *directly* find the Nash equilibria in this game, without relying on the sets of quantities that can rationally be chosen under common belief in rationality. Suppose that (σ_1, σ_2) is a Nash equilibrium in the Cournot competition model. By (3.7.12) we know that σ_1 must assign probability 1 to the unique quantity

$$q_1^* = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot E_{\sigma_2}(q_2) \tag{4.6.7}$$

that is optimal for firm 1 under the belief σ_2 about firm 2's quantity. Similarly, σ_2 must assign probability 1 to the unique quantity

$$q_2^* = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot E_{\sigma_1}(q_1) \tag{4.6.8}$$

that is optimal for firm 2 under the belief σ_1 about firm 1's quantity.

Since $E_{\sigma_1}(q_1) = q_1^*$ and $E_{\sigma_2}(q_2) = q_2^*$ it follows from (4.6.7) and (4.6.8) that

$$q_1^* = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot q_2^* \text{ and} \tag{4.6.9}$$
$$q_2^* = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot q_1^*. \tag{4.6.10}$$

These equations reflect the *best response functions* for the firms. If we substitute (4.6.10) into (4.6.9) we get

$$q_1^* = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot (\tfrac{a-c}{2e} - \tfrac{1}{2} \cdot q_1^*) = \tfrac{a-c}{4e} + \tfrac{1}{4} \cdot q_1^*$$

and hence

$$q_1^* = \tfrac{(a-c)/4e}{3/4} = \tfrac{a-c}{3e} = q^*.$$

By substituting this into (4.6.10) we obtain

$$q_2^* = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot q^* = \tfrac{a-c}{2e} - \tfrac{1}{2} \cdot \tfrac{a-c}{3e} = \tfrac{a-c}{3e} = q^*.$$

Thus, the combination of beliefs ($\sigma_1 = q^*$, $\sigma_2 = q^*$), where both firms are believed to choose q^* with probability 1, is the only Nash equilibrium in this model. This Nash equilibrium is called the *Cournot equilibrium.*

Chapter 5

Common Belief in Rationality with Incomplete Information

5.7 Economic Applications

In this section we reconsider the Bertrand competition model and Cournot competition model from Section 3.7 of this online appendix. This time we assume that both firms are uncertain about the opponent's cost function, and we show how this gives rise to a game with incomplete information. We extend the generalized iterated strict dominance procedure to this setting with infinitely many choices, states and utility functions, and use this procedure to find the prices and quantities that both firms can rationally choose under common belief in rationality.

5.7.1 Competition in Prices

Recall the Bertrand competition model from Section 3.7.1. We saw that the profit function for firm 1 was given by

$$\pi_1(p_1, p_2) = (p_1 - c_1) \cdot (a - d \cdot p_1 + e \cdot p_2), \tag{5.7.1}$$

where c_1 is the constant marginal cost for firm 1. Similarly for firm 2.

In Section 3.7.1 we assumed that $c_1 = c_2 = c$, and that this is transparent amongst the two firms. In this section we drop this assumption, and assume that both firms are *uncertain* about the marginal cost of the other firm. Of course, both firms know their own marginal cost. More precisely, it is commonly known that the marginal cost of both firms lies somewhere in the interval $[\underline{c}, \overline{c}]$, where $\underline{c}$ is the lowest possible marginal cost and $\overline{c}$ is the highest possible marginal cost. That is, firm 1 believes that firm 2's marginal cost lies in $[\underline{c}, \overline{c}]$ and firm 2 believes that firm 1's marginal cost lies in $[\underline{c}, \overline{c}]$. We assume that

$$e < 2d \text{ and } M \geq \frac{\overline{c} + a/d}{2 - e/d}.$$

In view of (5.7.1), the marginal cost c_i for firm i completely determines firm i's utility function. We thus obtain a scenario with incomplete information where there are infinitely many possible utility functions for the two players, since there are infinitely many values that c_i can take. Moreover, there are infinitely many possible choices for the two players, and therefore also infinitely many states. Nevertheless, the concept of *common belief in rationality* and the *generalized iterated strict dominance procedure* can naturally be extended to such scenarios.

Indeed, consider a game with incomplete information $(C_i, U_i)_{i \in I}$ where the set C_i of possible choices and the set U_i of possible utility functions may be infinite for every player i. In the generalized iterated strict dominance procedure for *finite* games we would in round 1 eliminate, for every possible utility function u_i of player i, those choices that are strictly dominated. By Theorem 2.6.1 in the book, these are precisely the choices that are not optimal for any belief at the utility function u_i. Thus, in the *infinite* case we can eliminate, in round 1, for every player i and for every utility function u_i those choices that are not optimal for any probabilistic belief. Then, we are left with the 1-fold reduced decision problem for every player i and utility function $u_i \in U_i$. Note that there may be *infinitely* many decision problems, as there are possibly infinitely many utility functions.

Like in the procedure for finite games, we would start in round 2 by eliminating, for every player i and every utility function u_i, those states that involve opponent's choices c_j that did not survive round 1 at *any* of player j's utility functions. We thus obtain a smaller decision problem at u_i. In the finite case we would then eliminate the choices for player i that are strictly dominated within this smaller decision problem. Again, by Theorem 2.6.1, this is equivalent to eliminating those choices that are not optimal for any belief within this smaller decision problem at u_i. In the infinite case, we can then eliminate those choices for player i that are not optimal for any belief within this smaller decision problem at u_i.

By continuing in this fashion, we can extend the generalized iterated strict dominance procedure to games with incomplete information that may contain infinitely many choices, states and utility functions. We will call this the generalized iterated strict dominance procedure for *infinite games.*

Let us now apply the generalized iterated strict dominance procedure for infinite games to the Bertrand competition model above, to find those prices that both firms can rationally choose under common belief in rationality.

Round 1. Consider firm 1. For every value of $c_1 \in [\underline{c}, \overline{c}]$ there is a new decision problem for firm 1, with the utility function as given in (5.7.1). Which prices are optimal for firm 1 for some belief with the marginal cost c_1? By (3.7.5) we know that for every belief β_1 about firm 2's price, the unique optimal price for firm 1 is given by

$$p_1 = \tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot E_{\beta_1}(p_2). \tag{5.7.2}$$

As $E_{\beta_1}(p_2)$ lies between 0 and M, and the optimal price is increasing in $E_{\beta_1}(p_2)$, the set of prices $P_1^1(c_1)$ that is optimal for firm 1 for some belief with the marginal cost c_1 is

$$P_1^1(c_1) = [\tfrac{1}{2}(c_1 + \tfrac{a}{d}),\ \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{eM}{2d}]. \tag{5.7.3}$$

This yields the 1-fold reduced decision problems for firm 1 – one for every value of c_1. Similarly for firm 2.

Round 2. Consider the decision problem for firm 1 at marginal cost c_1. By definition, the set of states $S_1^2(c_1)$ contains precisely those prices for firm 2 that have survived round 1 for *some* marginal cost $c_2 \in [\underline{c}, \overline{c}]$. In view of (5.7.3), the lowest price for firm 2 that has survived for some c_2 is $\frac{1}{2}(\underline{c} + \frac{a}{d})$

for marginal cost $\underline{c}$, whereas the highest price for firm 2 that has survived for some c_2 is $\frac{1}{2}(\overline{c}+\frac{a}{d})+\frac{eM}{2d}$ for marginal cost $\overline{c}$. Thus, the set of states is given by

$$S_1^2(c_1) = [\tfrac{1}{2}(\underline{c}+\tfrac{a}{d}),\ \tfrac{1}{2}(\overline{c}+\tfrac{a}{d})+\tfrac{eM}{2d}].$$

Hence, firm 1 must form a belief β_1 about the set of states $S_1^2(c_1)$ above, which implies that $E_{\beta_1}(p_2)$ must be between $\frac{1}{2}(\underline{c}+\frac{a}{d})$ and $\frac{1}{2}(\overline{c}+\frac{a}{d})+\frac{eM}{2d}$. But then, by (5.7.2), the lowest price for firm 1 that is optimal for such a belief at marginal cost c_1 is

$$\tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}\tfrac{1}{2}(\underline{c}+\tfrac{a}{d}),$$

whereas the highest price for firm 1 that is optimal for such a belief at marginal cost c_1 is

$$\tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}(\tfrac{1}{2}(\overline{c}+\tfrac{a}{d})+\tfrac{eM}{2d}).$$

The set of prices that survive round 2 for firm 1 at marginal cost c_1 is thus given by

$$P_1^2(c_1) = [\tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}\tfrac{1}{2}(\underline{c}+\tfrac{a}{d}),\ \tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}(\tfrac{1}{2}(\overline{c}+\tfrac{a}{d})+\tfrac{eM}{2d})].$$

This yields the 2-fold reduced decision problems for firm 1. Similarly for firm 2.

If we continue in this fashion we can derive, for every round $k \geq 1$, both players i and every marginal cost c_i, the set $P_i^k(c_i)$ of prices that survive. We will show, by induction on k, that

$$P_1^k(c_1) = [\tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}(\tfrac{\underline{c}+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k-1}),\ \tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}(\tfrac{\overline{c}+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k-1})+(\tfrac{e}{2d})^{k-1}M)], \quad (5.7.4)$$

and similarly for firm 2. Let the lower bound and upper bound in this interval be denoted by $l_1^k(c_1)$ and $h_1^k(c_1)$, and similarly for player 2. Then, it may be verified that

$$l_1^k(\underline{c}) = \tfrac{1}{2}(\underline{c}+\tfrac{a}{d})+\tfrac{e}{2d}(\tfrac{\underline{c}+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k-1})) = \tfrac{\underline{c}+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k}) \quad (5.7.5)$$

and

$$\begin{aligned} h_1^k(\overline{c}) &= \tfrac{1}{2}(\overline{c}+\tfrac{a}{d})+\tfrac{e}{2d}(\tfrac{\overline{c}+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k-1})+(\tfrac{e}{2d})^{k-1}M) \\ &= \tfrac{\overline{c}+a/d}{2-e/d}\cdot(1-(\tfrac{e}{2d})^{k})+(\tfrac{e}{2d})^{k-1}M. \end{aligned} \quad (5.7.6)$$

For $k=1$ we know that $P_1^1(c_1)$ is given by (5.7.3), which matches (5.7.4).

Consider now some $k \geq 2$, and assume that (5.7.4) holds for firm 2 and $k-1$. Let us focus on the decision problem of firm 1 at the marginal cost c_1. By definition, $S_1^k(c_1)$ contains all prices for firm 2 that have survived round $k-1$ for some marginal cost of firm 2. By the induction assumption, $S_1^k(c_1)$ thus contains all prices p_2 such that $p_2 \in [l_2^{k-1}(c_2),\ h_2^{k-1}(c_2)]$ for some $c_2 \in [\underline{c}, \overline{c}]$. Hence, $S_1^k(c_1) = [l_2^{k-1}(\underline{c}),\ h_2^{k-1}(\overline{c})]$.

Take some belief β_1 on $S_1^k(c_1)$. Then, $E_{\beta_1}(p_2) \in [l_2^{k-1}(\underline{c}),\ h_2^{k-1}(\overline{c})]$. By (5.7.2) it then follows that the optimal price for firm 1 under that belief β_1 is in the interval

$$[\tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}l_2^{k-1}(\underline{c}),\ \tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}h_2^{k-1}(\overline{c})].$$

Thus, the set of prices that are optimal for some belief on $S_1^k(c_1)$ for the marginal cost c_1 is

$$P_1^k(c_1) = [\tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}l_2^{k-1}(\underline{c}),\ \tfrac{1}{2}(c_1+\tfrac{a}{d})+\tfrac{e}{2d}h_2^{k-1}(\overline{c})].$$

Moreover, by (5.7.5) and (5.7.6), this interval is equal to

$$P_1^k(c_1) = [\tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d}(\tfrac{\underline{c}+a/d}{2-e/d} \cdot (1-(\tfrac{e}{2d})^{k-1}),\ \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d}(\tfrac{\overline{c}+a/d}{2-e/d} \cdot (1-(\tfrac{e}{2d})^{k-1}) + (\tfrac{e}{2d})^{k-1}M)],$$

which matches (5.7.4). Similarly for firm 2. By induction on k, we thus conclude that (5.7.4) holds for every $k \geq 1$.

In particular, it can be seen that the interval $P_1^k(c_1)$ becomes strictly smaller with every k, and thus the procedure does not terminate within finitely many rounds. Recall that $e < 2d$. Therefore, when k tends to infinity, the interval $P_i^k(c_i)$ reduces to

$$P_i^*(c_i) = [\tfrac{1}{2}(c_i + \tfrac{a}{d}) + \tfrac{e}{2d}\tfrac{\underline{c}+a/d}{2-e/d},\ \tfrac{1}{2}(c_i + \tfrac{a}{d}) + \tfrac{e}{2d}\tfrac{\overline{c}+a/d}{2-e/d}] \tag{5.7.7}$$

for both firms i. Hence, under common belief in rationality, firm i can rationally choose any price from $P_i^*(c_i)$ when its marginal cost is c_i. Note that the interval $P_i^*(c_i)$ becomes wider if the range $\overline{c} - \underline{c}$ of possible marginal costs becomes larger. This is to be expected, as a larger range $\overline{c} - \underline{c}$ of possible marginal costs allows for more possible beliefs about the marginal cost and price of the opponent.

An important difference with the analysis in Section 3.7.1 is that common belief in rationality no longer leads to a unique price for the firms. The reason is that both firms are uncertain about the precise marginal cost that the competitor has, which allows the firms to have a broader range of reasonable beliefs about the price of the competitor.

We finally investigate the scenario where we impose *fixed beliefs on utilities.* In our setting this means that the two firms hold a fixed belief about the competitor's marginal cost. Suppose we require both firms to believe that every marginal cost in $[\underline{c}, \overline{c}]$ is equally likely for the competitor. In mathematical terms, such a belief is called the *uniform distribution* on $[\underline{c}, \overline{c}]$. Hence, we impose fixed beliefs (r_1, r_2) on utilities where r_1 and r_2 are the uniform distribution on $[\underline{c}, \overline{c}]$.

In a similar fashion as above, the generalized iterated strict dominance procedure with fixed beliefs on utilities can be generalized to games with infinitely many choices, states and utility functions. This extension is called the *generalized iterated strict dominance procedure with fixed beliefs on utilities for infinite games.* We will now apply this procedure to our setting to derive the prices that both firms can rationally choose under common belief in rationality with fixed beliefs (r_1, r_2) about the utilities.

Round 1. This round is precisely the same as for the procedure without fixed beliefs on utilities, and leads to the set of prices

$$P_1^1(c_1) = [\tfrac{1}{2}(c_1 + \tfrac{a}{d}),\ \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{eM}{2d}] \tag{5.7.8}$$

for firm 1, for every marginal cost c_1. Similarly for firm 2.

Round 2. Focus on firm 1. By definition of the procedure, firm 1 is required to hold a belief β_1 on firm 2's price-cost pairs that (i) respects r_1, that is, deems every marginal cost $c_2 \in [\underline{c}, \overline{c}]$ equally likely, and (ii) concentrates only on price-cost pairs (p_2, c_2) where the price p_2 has survived round 1 for firm 2 at c_2, that is, where $p_2 \in P_2^1(c_2)$. By (5.7.8), the "lowest" such belief is the belief $\beta_1^{\min}$ that deems every cost $c_2 \in [\underline{c}, \overline{c}]$ equally likely, and that concentrates only on pairs (p_2, c_2) where $p_2 = \frac{1}{2}(c_2 + \frac{a}{d})$. As the price p_2 depends linearly on c_2, and every $c_2 \in [\underline{c}, \overline{c}]$ is deemed equally likely, the expected price for firm 2 under this belief is

$$E_{\beta_1^{\min}}(p_2) = \tfrac{1}{2} \cdot \tfrac{1}{2}(\underline{c} + \tfrac{a}{d}) + \tfrac{1}{2} \cdot \tfrac{1}{2}(\overline{c} + \tfrac{a}{d}) = \tfrac{1}{2}(\tfrac{1}{2}(\underline{c} + \overline{c}) + \tfrac{a}{d}). \tag{5.7.9}$$

From (5.7.8) it also follows that the "highest" such belief is the belief $\beta_1^{\max}$ that deems every cost $c_2 \in [\underline{c}, \overline{c}]$ equally likely, and that concentrates only on pairs (p_2, c_2) where $p_2 = \frac{1}{2}(c_2 + \frac{a}{d}) + \frac{eM}{2d}$. By a similar reasoning as above, the expected price for firm 2 under that belief is

$$E_{\beta_1^{\max}}(p_2) = \tfrac{1}{2}(\tfrac{1}{2}(\underline{c} + \overline{c}) + \tfrac{a}{d}) + \tfrac{eM}{2d}. \tag{5.7.10}$$

Please verify this.

By (5.7.2), (5.7.9) and (5.7.10) we conclude that the lowest price which is optimal for such a belief β_1 with the marginal cost c_1 is

$$p_1 = \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} E_{\beta_1^{\min}}(p_2) = \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d}(\tfrac{1}{2}(\tfrac{1}{2}(\underline{c} + \overline{c}) + \tfrac{a}{d})),$$

whereas the highest price which is optimal for such a belief β_1 with the marginal cost c_1 is

$$p_1 = \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} E_{\beta_1^{\max}}(p_2) = \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d}(\tfrac{1}{2}(\tfrac{1}{2}(\underline{c} + \overline{c}) + \tfrac{a}{d}) + \tfrac{eM}{2d}).$$

Hence, the set of prices that survives round 2 for firm 1 at the marginal cost c_1 is

$$P_1^2(c_1) = [\tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d}(\tfrac{1}{2}(\tfrac{1}{2}(\underline{c} + \overline{c}) + \tfrac{a}{d})),\ \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d}(\tfrac{1}{2}(\tfrac{1}{2}(\underline{c} + \overline{c}) + \tfrac{a}{d}) + \tfrac{eM}{2d}]. \tag{5.7.11}$$

Similarly for firm 2.

If we continue in this way we can derive the sets of prices $P_1^k(c_1)$ and $P_2^k(c_2)$ that survive for the firms at the various rounds and the various marginal costs. We will show, by induction on k, that

$$P_1^k(c_1) = [\tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{(\underline{c}+\overline{c})/2 + a/d}{2 - e/d}(\tfrac{e}{2d} - (\tfrac{e}{2d})^k),\ \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{(\underline{c}+\overline{c})/2 + a/d}{2 - e/d}(\tfrac{e}{2d} - (\tfrac{e}{2d})^k) + (\tfrac{e}{2d})^k M] \tag{5.7.12}$$

for every $k \geq 1$, and similarly for firm 2. For convenience, we denote the lower bound and upper bound of this interval by $l_1^k(c_1)$ and $h_1^k(c_1)$, respectively.

For round 1 we have seen that $P_1^1(c_1)$ is given by (5.7.8), which matches (5.7.12).

Consider now round $k \geq 2$ and assume that (5.7.12) holds for firm 2 and round $k-1$. Focus on firm 1. By definition, firm 1 is required to hold a belief β_1 on firm 2's price-cost pairs (p_2, c_2) that (i) deems every $c_2 \in [\underline{c}, \overline{c}]$ equally likely, and (ii) only concentrates on price-cost pairs (p_2, c_2) where $p_2 \in P_2^{k-1}(c_2)$. By (5.7.12) for firm 2 and round $k-1$, the "lowest" such belief is the belief $\beta_1^{\min}$ that deems every $c_2 \in [\underline{c}, \overline{c}]$ equally likely and concentrates only on pairs (p_2, c_2) where $p_2 = l_2^{k-1}(c_2)$. As $l_2^{k-1}(c_2)$ depends linearly on c_2 and $\beta_1^{\min}$ deems every $c_2 \in [\underline{c}, \overline{c}]$ equally likely, the expected price for firm 2 under this belief is

$$E_{\beta_1^{\min}}(p_2) = \tfrac{1}{2}(l_2^{k-1}(\underline{c}) + l_2^{k-1}(\overline{c})). \tag{5.7.13}$$

It may be verified that

$$\begin{aligned} l_2^{k-1}(\underline{c}) &= \tfrac{(\underline{c}+\overline{c})/2 + a/d}{2 - e/d}(1 - (\tfrac{e}{2d})^{k-1}) - \tfrac{\overline{c} - \underline{c}}{4} \text{ and} \\ l_2^{k-1}(\overline{c}) &= \tfrac{(\underline{c}+\overline{c})/2 + a/d}{2 - e/d}(1 - (\tfrac{e}{2d})^{k-1}) + \tfrac{\overline{c} - \underline{c}}{4}. \end{aligned}$$

Together with (5.7.13) we conclude that

$$E_{\beta_1^{\min}}(p_2) = \tfrac{(\underline{c}+\overline{c})/2 + a/d}{2 - e/d}(1 - (\tfrac{e}{2d})^{k-1}). \tag{5.7.14}$$

Note that the lowest price p_1 that is optimal for such a belief β_1 above that satisfies (i) and (ii) is the price that is optimal for $\beta_1^{\min}$. By (5.7.2) and (5.7.14), this price is

$$\begin{aligned} p_1 &= \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} E_{\beta_1^{\min}}(p_2) = \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \tfrac{(\underline{c}+\overline{c})/2 + a/d}{2 - e/d}(1 - (\tfrac{e}{2d})^{k-1}) \\ &= \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{(\underline{c}+\overline{c})/2 + a/d}{2 - e/d}(\tfrac{e}{2d} - (\tfrac{e}{2d})^k), \end{aligned}$$

which matches $l_1^k(c_1)$ in (5.7.12).

Moreover, by (5.7.12) for firm 2 and round $k-1$ we conclude that the "highest" belief β_1 for firm 1 with the properties (i) and (ii) above is the belief $\beta_1^{\max}$ that deems every $c_2 \in [\underline{c}, \overline{c}]$ equally likely and concentrates only on pairs (p_2, c_2) where $p_2 = h_2^{k-1}(c_2)$. As $h_2^{k-1}(c_2)$ depends linearly on c_2 and $\beta_1^{\max}$ deems every $c_2 \in [\underline{c}, \overline{c}]$ equally likely, the expected price for firm 2 under this belief is

$$E_{\beta_1^{\max}}(p_2) = \tfrac{1}{2}(h_2^{k-1}(\underline{c}) + h_2^{k-1}(\overline{c})). \tag{5.7.15}$$

In a similar way as above, it may be verified that

$$\begin{aligned} h_2^{k-1}(\underline{c}) &= \tfrac{(\underline{c}+\overline{c})/2+a/d}{2-e/d}(1-(\tfrac{e}{2d})^{k-1}) - \tfrac{\overline{c}-\underline{c}}{4} + (\tfrac{e}{2d})^{k-1}M \text{ and} \\ h_2^{k-1}(\overline{c}) &= \tfrac{(\underline{c}+\overline{c})/2+a/d}{2-e/d}(1-(\tfrac{e}{2d})^{k-1}) + \tfrac{\overline{c}-\underline{c}}{4} + (\tfrac{e}{2d})^{k-1}M. \end{aligned}$$

Together with (5.7.15) it follows that

$$E_{\beta_1^{\max}}(p_2) = \tfrac{(\underline{c}+\overline{c})/2+a/d}{2-e/d}(1-(\tfrac{e}{2d})^{k-1}) + (\tfrac{e}{2d})^{k-1}M. \tag{5.7.16}$$

Note that the highest price p_1 that is optimal for such a belief β_1 above that satisfies (i) and (ii) is the price that is optimal for $\beta_1^{\max}$. By (5.7.2) and (5.7.16), this price is

$$\begin{aligned} p_1 &= \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d}E_{\beta_1^{\max}}(p_2) = \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{e}{2d}(\tfrac{(\underline{c}+\overline{c})/2+a/d}{2-e/d}(1-(\tfrac{e}{2d})^{k-1}) + (\tfrac{e}{2d})^{k-1}M) \\ &= \tfrac{1}{2}(c_1 + \tfrac{a}{d}) + \tfrac{(\underline{c}+\overline{c})/2+a/d}{2-e/d}(\tfrac{e}{2d} - (\tfrac{e}{2d})^{k}) + (\tfrac{e}{2d})^{k}M, \end{aligned}$$

which matches $h_1^k(c_1)$ in (5.7.12). By induction on k, we conclude that (5.7.12) holds for every k.

We see again that the set $P_i^k(c_i)$ becomes strictly smaller with every round k, and hence the procedure does not terminate within finitely many rounds. When k tends to infinity, the set $P_i^k(c_i)$ now collapses to a single price, which is

$$p_i^*(c_i) = \tfrac{1}{2}(c_i + \tfrac{a}{d}) + \tfrac{e}{2d}\,\tfrac{(\underline{c}+\overline{c})/2+a/d}{2-e/d}. \tag{5.7.17}$$

Hence, under common belief in rationality and common belief in the fixed belief (r_1, r_2) on utilities, firm i can only rationally choose the price $p_i^*(c_i)$ above if its marginal cost is c_i.

This is fundamentally different from the scenario without fixed beliefs on utilities, where the firm could rationally choose from a whole range of prices $P_i^*(c_i)$ under common belief in rationality. See (5.7.7) above. The reason is that with fixed beliefs on utilities, the possible reasonable beliefs that both firms can hold about the competitor's prices are heavily restricted.

Note that the unique price $p_i^*(c_i)$ in (5.7.17) that can rationally be chosen under common belief in rationality in the scenario *with* fixed beliefs on utilities belongs to the range of prices $P_i^*(c_i)$ in (5.7.7) that can rationally be chosen under common belief in rationality in the scenario *without* fixed beliefs on utilities, as it should be. In fact, we can say more: The price $p_i^*(c_i)$ lies exactly in the middle of the interval $P_i^*(c_i)$. This makes intuitive sense, as the fixed beliefs on utilities deem every marginal cost for the opponent equally likely.

5.7.2 Competition in Quantities

Let us return to the Cournot competition model from Section 3.7.2. Similarly to the Bertrand model with incomplete information above, we assume that both firms have marginal costs c_1 and c_2, and that both firms are uncertain about the precise marginal cost of the competitor. Recall from Section 3.7.2 that the profit function for firm 1 is given by

$$\pi_1(q_1, q_2) = q_1 \cdot (a - c_1 - e \cdot (q_1 + q_2)), \tag{5.7.18}$$

and similarly for firm 2.

Again, suppose that the marginal cost for both firms belongs to the interval $[\underline{c}, \overline{c}]$. Moreover, assume that $2\overline{c} - \underline{c} \leq a$ and $M \in [\frac{a-\underline{c}}{2e}, \frac{a-\overline{c}}{e}]$. For every possible marginal cost $c_i \in [\underline{c}, \overline{c}]$, what quantities can firm i rationally choose under common belief in rationality? To answer this question we apply the *generalized iterated strict dominance procedure for infinite games* outlined in Section 5.7.1.

Round 1. Consider firm 1 and suppose it has a marginal cost of c_1. Which quantities are optimal for firm 1 for some probabilistic belief about firm 2's quantity and which are not? Suppose firm 1 holds a probabilistic belief β_1 about firm 2's quantity. By (3.7.12), firm 1's optimal quantity is then given by

$$q_1 = \tfrac{a-c_1}{2e} - \tfrac{1}{2} \cdot E_{\beta_1}(q_2). \tag{5.7.19}$$

Since $E_{\beta_1}(q_2)$ lies somewhere in the interval $[0, M]$, and the optimal quantity is decreasing in $E_{\beta_1}(q_2)$, the lowest quantity that is optimal for some belief is $q_1 = \frac{a-c_1}{2e} - \frac{1}{2}M$, whereas the highest such quantity is $q_1 = \frac{a-c_1}{2e}$. Thus, the set of quantities for firm 1 that survives round 1 at the marginal cost c_1 is

$$Q_1^1(c_1) = [\tfrac{a-c_1}{2e} - \tfrac{1}{2}M,\ \tfrac{a-c_1}{2e}]. \tag{5.7.20}$$

Similarly for firm 2.

Round 2. Consider firm 1 with a marginal cost of c_1. The set of states $S_1^2(c_1)$ that survives round 2 contains precisely those quantities q_2 that have survived round 1 for at least one marginal cost c_2. By (5.7.20) applied to firm 2, the lowest such quantity is $q_2 = \frac{a-\overline{c}}{2e} - \frac{1}{2}M$ whereas the highest such quantity is $q_2 = \frac{a-\underline{c}}{2e}$. Hence,

$$S_1^2(c_1) = [\tfrac{a-\overline{c}}{2e} - \tfrac{1}{2}M,\ \tfrac{a-\underline{c}}{2e}].$$

Now, take a probabilistic belief β_1 on the set of states $S_1^2(c_1)$. Then, $E_{\beta_1}(q_2)$ lies somewhere in the interval $S_1^2(c_1)$. Recall, by (5.7.19), that the optimal quantity q_1 for this belief is decreasing in $E_{\beta_1}(q_2)$. Hence, by (5.7.19), the lowest quantity for firm 1 that is optimal for such a belief β_1 is

$$q_1 = \tfrac{a-c_1}{2e} - \tfrac{1}{2}\tfrac{a-\underline{c}}{2e},$$

whereas the highest such quantity is

$$q_1 = \tfrac{a-c_1}{2e} - \tfrac{1}{2}(\tfrac{a-\overline{c}}{2e} - \tfrac{1}{2}M).$$

Hence, the set of quantities for firm 1 that survive round 2 at the marginal cost c_1 is

$$Q_1^2(c_1) = [\tfrac{a-c_1}{2e} - \tfrac{1}{2}\tfrac{a-\underline{c}}{2e},\ \tfrac{a-c_1}{2e} - \tfrac{1}{2}(\tfrac{a-\overline{c}}{2e} - \tfrac{1}{2}M)]. \tag{5.7.21}$$

Similarly for firm 2.

If we continue in this fashion we can also derive the sets of quantities $Q_i^k(c_i)$ for both firms i, every marginal cost c_i and every round $k \geq 1$. We will show, by induction on k, that for every $k \geq 1$ we have that

$$Q_1^k(c_1) = [\tfrac{a-c_1}{2e} - \tfrac{a-\underline{c}}{3e}(1-(\tfrac{1}{2})^{k-1}) + \tfrac{a-\overline{c}}{6e}(1-(\tfrac{1}{2})^{k-1}) - \tfrac{M}{2^k},\ \tfrac{a-c_1}{2e} - \tfrac{a-\overline{c}}{3e}(1-(\tfrac{1}{2})^{k-1}) + \tfrac{a-\underline{c}}{6e}(1-(\tfrac{1}{2})^{k-1})] \tag{5.7.22}$$

if k is odd, and

$$Q_1^k(c_1) = [\tfrac{a-c_1}{2e} - \tfrac{a-\underline{c}}{3e}(1-(\tfrac{1}{2})^k) + \tfrac{a-\overline{c}}{6e}(1-(\tfrac{1}{2})^{k-2}),\ \tfrac{a-c_1}{2e} - \tfrac{a-\overline{c}}{3e}(1-(\tfrac{1}{2})^k) + \tfrac{a-\underline{c}}{6e}(1-(\tfrac{1}{2})^{k-2}) + \tfrac{M}{2^k}] \quad (5.7.23)$$

if k is even. Similarly for firm 2.

It may be verified that for $k = 1$ the expression (5.7.22) matches precisely the equation (5.7.20).

Take some now even $k \geq 2$, and assume that (5.7.22) holds for firm 2 for the odd $k-1$. Consider firm 1 with a marginal cost of c_1. Then, the set of states $S_1^k(c_1)$ contains those quantities q_2 that are in $Q_2^{k-1}(c_2)$ for some marginal cost c_2. In view of (5.7.22), the lowest such q_2 is the lower bound of $Q_2^{k-1}(\overline{c})$, which we denote by $l_2^{k-1}(\overline{c})$. Similarly, the highest such q_2 is the upper bound of $Q_2^{k-1}(\underline{c})$, which we denote by $h_2^{k-1}(\underline{c})$. Hence,

$$S_1^k(c_1) = [l_2^{k-1}(\overline{c}),\ h_2^{k-1}(\underline{c})].$$

Take now a probabilistic belief β_1 on the set of states $S_1^k(c_1)$. Then, $E_{\beta_1}(q_2)$ lies somewhere in the interval $S_1^k(c_1)$. By (5.7.19), the lowest quantity q_1 that is optimal for such a belief β_1 is obtained when $E_{\beta_1}(q_2) = h_2^{k-1}(\underline{c})$. This yields the quantity

$$\begin{aligned} l_1^k(c_1) &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}h_2^{k-1}(\underline{c}) \\ &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}[\tfrac{a-\underline{c}}{2e} - \tfrac{a-\overline{c}}{3e}(1-(\tfrac{1}{2})^{k-2}) + \tfrac{a-\underline{c}}{6e}(1-(\tfrac{1}{2})^{k-2})] \\ &= \tfrac{a-c_1}{2e} - \tfrac{a-\underline{c}}{3e}(1-(\tfrac{1}{2})^k) + \tfrac{a-\overline{c}}{6e}(1-(\tfrac{1}{2})^{k-2}). \end{aligned}$$

Moreover, the highest quantity q_1 that is optimal for such a belief β_1 is obtained when $E_{\beta_1}(q_2) = l_2^{k-1}(\overline{c})$. This yields the quantity

$$\begin{aligned} h_1^k(c_1) &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}l_2^{k-1}(\overline{c}) \\ &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}[\tfrac{a-\overline{c}}{2e} - \tfrac{a-\underline{c}}{3e}(1-(\tfrac{1}{2})^{k-2}) + \tfrac{a-\overline{c}}{6e}(1-(\tfrac{1}{2})^{k-2}) - \tfrac{M}{2^{k-1}}] \\ &= \tfrac{a-c_1}{2e} - \tfrac{a-\overline{c}}{3e}(1-(\tfrac{1}{2})^k) + \tfrac{a-\underline{c}}{6e}(1-(\tfrac{1}{2})^{k-2}) + \tfrac{M}{2^k}. \end{aligned}$$

Thus, we conclude that $Q_1^k(c_1) = [l_1^k(c_1),\ h_1^k(c_1)]$, which matches (5.7.23).

Next, take some odd $k \geq 2$, and assume that (5.7.23) holds for firm 2 for the even $k-1$. Consider firm 1 with a marginal cost of c_1. Then, the set of states $S_1^k(c_1)$ contains those quantities q_2 that are in $Q_2^{k-1}(c_2)$ for some marginal cost c_2. In view of (5.7.23), the lowest such q_2 is the lower bound of $Q_2^{k-1}(\overline{c})$, which we denote by $l_2^{k-1}(\overline{c})$. Similarly, the highest such q_2 is the upper bound of $Q_2^{k-1}(\underline{c})$, which we denote by $h_2^{k-1}(\underline{c})$. Hence,

$$S_1^k(c_1) = [l_2^{k-1}(\overline{c}),\ h_2^{k-1}(\underline{c})].$$

Take now a probabilistic belief β_1 on the set of states $S_1^k(c_1)$. Then, $E_{\beta_1}(q_2)$ lies somewhere in the interval $S_1^k(c_1)$. By (5.7.19), the lowest quantity q_1 that is optimal for such a belief β_1 is obtained when $E_{\beta_1}(q_2) = h_2^{k-1}(\underline{c})$. This yields the quantity

$$\begin{aligned} l_1^k(c_1) &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}h_2^{k-1}(\underline{c}) \\ &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}[\tfrac{a-\underline{c}}{2e} - \tfrac{a-\overline{c}}{3e}(1-(\tfrac{1}{2})^{k-1}) + \tfrac{a-\underline{c}}{6e}(1-(\tfrac{1}{2})^{k-3}) + \tfrac{M}{2^{k-1}}] \\ &= \tfrac{a-c_1}{2e} - \tfrac{a-\underline{c}}{3e}(1-(\tfrac{1}{2})^{k-1}) + \tfrac{a-\overline{c}}{6e}(1-(\tfrac{1}{2})^{k-1}) - \tfrac{M}{2^k}. \end{aligned}$$

Moreover, the highest quantity q_1 that is optimal for such a belief β_1 is obtained when $E_{\beta_1}(q_2) = l_2^{k-1}(\overline{c})$. This yields the quantity

$$\begin{aligned} h_1^k(c_1) &= \tfrac{a-c_1}{2e} - \tfrac{1}{2} l_2^{k-1}(\overline{c}) \\ &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}[\tfrac{a-\overline{c}}{2e} - \tfrac{a-\underline{c}}{3e}(1-(\tfrac{1}{2})^{k-1}) + \tfrac{a-\overline{c}}{6e}(1-(\tfrac{1}{2})^{k-3})] \\ &= \tfrac{a-c_1}{2e} - \tfrac{a-\overline{c}}{3e}(1-(\tfrac{1}{2})^{k-1}) + \tfrac{a-\underline{c}}{6e}(1-(\tfrac{1}{2})^{k-1}). \end{aligned}$$

Thus, we conclude that $Q_1^k(c_1) = [l_1^k(c_1),\, h_1^k(c_1)]$, which matches (5.7.22).

By induction on k, we conclude that $Q_1^k(c_1)$ is given by (5.7.22) and (5.7.23) for every $k \geq 1$. Similarly for firm 2. In particular, we see that the set of quantities $Q_1^k(c_i)$ that survives round k of the procedure becomes strictly smaller with every round, and hence the procedure does not terminate within finitely many rounds.

When k tends to infinity, then $Q_i^k(c_i)$ approaches the interval

$$Q_i^*(c_i) = [\tfrac{a-c_i}{2e} - \tfrac{a-\underline{c}}{3e} + \tfrac{a-\overline{c}}{6e},\ \tfrac{a-c_i}{2e} - \tfrac{a-\overline{c}}{3e} + \tfrac{a-\underline{c}}{6e}] \tag{5.7.24}$$

for both firms i, and every marginal cost c_i. Hence, under common belief in rationality both firms i can only rationally choose the quantities in $Q_i^*(c_i)$ when its marginal cost is c_i. Note that this interval $Q_i^*(c_i)$ becomes wider if the range $\overline{c} - \underline{c}$ of possible marginal costs becomes larger. Again, this is to be expected, as a larger range $\overline{c} - \underline{c}$ of possible marginal costs allows for more possible beliefs about the marginal cost and quantity of the opponent.

We finally explore the scenario where there are *fixed beliefs on utilities.* Like in the Bertrand model above, assume that we require both firms to deem every marginal cost $c \in [\underline{c},\, \overline{c}]$ for the competitor equally likely. That is, we impose the fixed beliefs (r_1, r_2) on utilities, where r_1 and r_2 are the uniform distribution on $[\underline{c},\, \overline{c}]$.

To find the quantities that both firms can rationally choose under common belief in rationality and common belief in the fixed beliefs (r_1, r_2) on utilities, we use the *generalized iterated strict dominance procedure with fixed beliefs on utilities for infinite games* as outlined in Section 5.7.1 above.

Round 1. This round is exactly the same as for the generalized iterated strict dominance procedure without fixed beliefs on utilities. The set of quantities that survives for firm 1 for every marginal cost c_1 is thus given by

$$Q_1^1(c_1) = [\tfrac{a-c_1}{2e} - \tfrac{1}{2}M,\ \tfrac{a-c_1}{2e}], \tag{5.7.25}$$

and similarly for firm 2.

Round 2. Focus on firm 1. By construction of the procedure, firm 1 is required to hold a belief β_1 on the competitor's quantity-cost pairs (q_2, c_2) that (i) respects r_2, that is, deems every cost $c_2 \in [\underline{c},\, \overline{c}]$ for the competitor equally likely, and (ii) only concentrates on pairs (q_2, c_2) where $q_2 \in Q_2^1(c_2)$. In view of (5.7.25) for firm 2, the "highest" such belief β_1 is the belief $\beta_1^{\max}$ that only concentrates on the pairs (q_2, c_2) where $q_2 = \frac{a-c_2}{2e}$. As $\beta_1^{\max}$ deems every $c_2 \in [\underline{c},\, \overline{c}]$ equally likely, and q_2 is linear in the cost c_2, we have that the expected quantity for firm 2 under this belief is

$$E_{\beta_1^{\max}}(q_2) = \tfrac{1}{2} \cdot \tfrac{a-\underline{c}}{2e} + \tfrac{1}{2} \cdot \tfrac{a-\overline{c}}{2e} = \tfrac{a-(\underline{c}+\overline{c})/2}{2e}. \tag{5.7.26}$$

Similarly, the "lowest" such belief β_1 is the belief $\beta_1^{\min}$ that only concentrates on pairs (q_2, c_2) where $q_2 = \frac{a-c_2}{2e} - \frac{1}{2}M$. The expected quantity for firm 2 under that belief is

$$E_{\beta_1^{\min}}(q_2) = \tfrac{1}{2} \cdot (\tfrac{a-\underline{c}}{2e} - \tfrac{1}{2}M) + \tfrac{1}{2} \cdot (\tfrac{a-\overline{c}}{2e} - \tfrac{1}{2}M) = \tfrac{a-(\underline{c}+\overline{c})/2}{2e} - \tfrac{1}{2}M. \tag{5.7.27}$$

In view of (5.7.19), (5.7.26) and (5.7.27), the lowest quantity for firm 1 that is optimal for such a belief β_1 at the marginal cost c_1 is

$$q_1 = \frac{a-c_1}{2e} - \frac{1}{2}E_{\beta_1^{\max}}(q_2) = \frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{4e}.$$

Similarly, the highest quantity for firm 1 that is optimal for such a belief β_1 at the marginal cost c_1 is

$$q_1 = \frac{a-c_1}{2e} - \frac{1}{2}E_{\beta_1^{\min}}(q_2) = \frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{4e} + \frac{1}{4}M.$$

Hence, the set of quantities for firm 1 that survives round 2 at the marginal cost c_1 is

$$Q_1^2(c_1) = [\frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{4e},\ \frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{4e} + \frac{1}{4}M].$$

Similarly for firm 2.

If we continue in this fashion we can derive the sets of quantities $Q_i^k(c_i)$ for both firms i, for all marginal costs c_i and all rounds k. We will show, by induction on k, that for all rounds $k \geq 1$ we have that

$$Q_1^k(c_1) = [\frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-1}) - \frac{M}{2^k},\ \frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-1})] \quad (5.7.28)$$

if k is odd, and

$$Q_1^k(c_1) = [\frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-1}),\ \frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-1}) + \frac{M}{2^k}] \quad (5.7.29)$$

if k is even. Similarly for firm 2.

If $k = 1$ then (5.7.28) matches precisely (5.7.25).

Now, take some even $k \geq 2$ and assume that (5.7.28) holds for firm 2 and the odd $k-1$. In round k, firm 1 must hold a belief β_1 on quantity-cost pairs (q_2, c_2) that deems all costs $c_2 \in [\underline{c}, \overline{c}]$ equally likely, and that only concentrates on pairs (q_2, c_2) where $q_2 \in Q_2^{k-1}(c_2)$. By (5.7.28) for firm 2 and $k-1$, the "highest" such belief β_1 is the belief $\beta_1^{\max}$ that only concentrates on pairs (q_2, c_2) with

$$q_2 = \frac{a-c_2}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2}).$$

The expected quantity for firm 2 under that belief is

$$\begin{aligned} E_{\beta_1^{\max}}(q_2) &= \frac{1}{2}\cdot[\frac{a-\underline{c}}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2})] + \frac{1}{2}\cdot[\frac{a-\overline{c}}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2})] \\ &= \frac{a-(\underline{c}+\overline{c})/2}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2}). \end{aligned} \quad (5.7.30)$$

Similarly, the "lowest" such belief β_1 is the belief $\beta_1^{\min}$ that only concentrates on pairs (q_2, c_2) with

$$q_2 = \frac{a-c_2}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2}) - \frac{M}{2^{k-1}}.$$

The expected quantity for firm 2 under that belief is

$$\begin{aligned} E_{\beta_1^{\min}}(q_2) &= \frac{1}{2}\cdot[\frac{a-\underline{c}}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2}) - \frac{M}{2^{k-1}}] + \frac{1}{2}\cdot[\frac{a-\overline{c}}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2}) - \frac{M}{2^{k-1}}] \\ &= \frac{a-(\underline{c}+\overline{c})/2}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2}) - \frac{M}{2^{k-1}}. \end{aligned} \quad (5.7.31)$$

By (5.7.19), (5.7.30) and (5.7.31) we conclude that the lowest quantity q_1 that is optimal for such a belief β_1 at the marginal cost c_1 is

$$\begin{aligned} q_1 &= \frac{a-c_1}{2e} - \frac{1}{2}E_{\beta_1^{\max}}(q_2) = \frac{a-c_1}{2e} - \frac{1}{2}[\frac{a-(\underline{c}+\overline{c})/2}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2})] \\ &= \frac{a-c_1}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-1}), \end{aligned}$$

whereas the highest quantity q_1 that is optimal for such a belief β_1 at the marginal cost c_1 is

$$\begin{aligned} q_1 &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}E_{\beta_1^{\min}}(q_2) = \tfrac{a-c_1}{2e} - \tfrac{1}{2}[\tfrac{a-(\underline{c}+\overline{c})/2}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-2}) - \tfrac{M}{2^{k-1}}] \\ &= \tfrac{a-c_1}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-1}) + \tfrac{M}{2^k}. \end{aligned}$$

Thus, the set of quantities that survives round k for firm 1 at the marginal cost c_1 is

$$Q_1^k(c_1) = [\tfrac{a-c_1}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-1}),\ \tfrac{a-c_1}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-1}) + \tfrac{M}{2^k}]$$

which matches (5.7.29).

Next, take some odd $k \geq 2$, and assume that (5.7.29) holds for firm 2 and the even $k-1$. Again, firm 1 must hold a belief β_1 on quantity-cost pairs (q_2, c_2) that deems all costs $c_2 \in [\underline{c}, \overline{c}]$ equally likely, and that only concentrates on pairs (q_2, c_2) where $q_2 \in Q_2^{k-1}(c_2)$. By (5.7.29) for firm 2 and $k-1$, the "highest" such belief β_1 is the belief $\beta_1^{\max}$ that only concentrates on pairs (q_2, c_2) with

$$q_2 = \tfrac{a-c_2}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2}) + \tfrac{M}{2^{k-1}}.$$

The expected quantity for firm 2 under that belief is

$$\begin{aligned} E_{\beta_1^{\max}}(q_2) &= \tfrac{1}{2}\cdot[\tfrac{a-\underline{c}}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2}) + \tfrac{M}{2^{k-1}}] + \tfrac{1}{2}\cdot[\tfrac{a-\overline{c}}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2}) + \tfrac{M}{2^{k-1}}] \\ &= \tfrac{a-(\underline{c}+\overline{c})/2}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2}) + \tfrac{M}{2^{k-1}}. \end{aligned} \tag{5.7.32}$$

Similarly, the "lowest" such belief β_1 is the belief $\beta_1^{\min}$ that only concentrates on pairs (q_2, c_2) with

$$q_2 = \tfrac{a-c_2}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2}).$$

The expected quantity for firm 2 under that belief is

$$\begin{aligned} E_{\beta_1^{\min}}(q_2) &= \tfrac{1}{2}\cdot[\tfrac{a-\underline{c}}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2})] + \tfrac{1}{2}\cdot[\tfrac{a-\overline{c}}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2})] \\ &= \tfrac{a-(\underline{c}+\overline{c})/2}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2}). \end{aligned} \tag{5.7.33}$$

By (5.7.19), (5.7.32) and (5.7.33) we conclude that the lowest quantity q_1 that is optimal for such a belief β_1 at the marginal cost c_1 is

$$\begin{aligned} q_1 &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}E_{\beta_1^{\max}}(q_2) = \tfrac{a-c_1}{2e} - \tfrac{1}{2}[\tfrac{a-(\underline{c}+\overline{c})/2}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2}) + \tfrac{M}{2^{k-1}}] \\ &= \tfrac{a-c_1}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-1}) - \tfrac{M}{2^k}, \end{aligned}$$

whereas the highest quantity q_1 that is optimal for such a belief β_1 at the marginal cost c_1 is

$$\begin{aligned} q_1 &= \tfrac{a-c_1}{2e} - \tfrac{1}{2}E_{\beta_1^{\min}}(q_2) = \tfrac{a-c_1}{2e} - \tfrac{1}{2}[\tfrac{a-(\underline{c}+\overline{c})/2}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1+(\tfrac{1}{2})^{k-2})] \\ &= \tfrac{a-c_1}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-1}). \end{aligned}$$

Thus, the set of quantities that survives round k for firm 1 at the marginal cost c_1 is

$$Q_1^k(c_1) = [\tfrac{a-c_1}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-1}) - \tfrac{M}{2^k},\ \tfrac{a-c_1}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}(1-(\tfrac{1}{2})^{k-1})]$$

which matches (5.7.28).

By induction on k, we then conclude that for every round k, and all marginal costs c_1, the set $Q_1^k(c_1)$ is given by (5.7.28) and (5.7.29). And similarly for firm 2.

When k tends to infinity, the set $Q_i^k(c_i)$ collapses to the single quantity

$$q_i^*(c_i) = \frac{a-c_i}{2e} - \frac{a-(\underline{c}+\overline{c})/2}{6e}. \qquad (5.7.34)$$

Hence, under common belief in rationality and common belief in the fixed beliefs (r_1, r_2) on utilities, firm i can only rationally choose the quantity $q_i^*(c_i)$ if the marginal costs are c_i.

Note that this quantity $q_i^*(c_i)$ is exactly halfway the set of quantities $Q_i^*(c_i)$ in (5.7.24) that firm i can rationally choose under common belief in rationality without fixed beliefs on utilities. This makes intuitive sense, since firm i deems every possible marginal cost of competitor j equally likely. Moreover, the quantity $q_i^*(c_i)$ is decreasing in the marginal cost c_i. Also this is intuitive, as a higher marginal cost forces the firm to reduce its production.

Chapter 6

Correct and Symmetric Beliefs with Incomplete Information

6.6 Economic Applications

In this section we continue our exploration of the Bertrand model and Cournot model introduced in Section 3.7 of this online appendix. For both models of competition we investigate the possible choices that both firms can rationally make under common belief in rationality with a *simple* belief hierarchy. We also consider the scenario where the beliefs about the firms' marginal costs are fixed.

6.6.1 Competition in Prices

Recall the Bertrand competition model with incomplete information we investigated in Section 5.7.1 of this online appendix. We have seen in (5.7.7) that under common belief in rationality without fixed beliefs on utilities, both firms i can rationally choose from the set of prices

$$P_i^*(c_i) = [\tfrac{1}{2}(c_i + \tfrac{a}{d}) + \tfrac{e}{2d}\tfrac{\underline{c}+a/d}{2-e/d},\ \tfrac{1}{2}(c_i + \tfrac{a}{d}) + \tfrac{e}{2d}\tfrac{\overline{c}+a/d}{2-e/d}] \qquad (6.6.1)$$

if the marginal cost is c_i.

In Section 6.1 of the book we introduced the notion of a *simple* belief hierarchy for games with incomplete information where there are *finitely* many choices and utility functions. Moreover, it has been shown in Theorem 6.1.2 that the choices which can rationally be made under common belief in rationality with a simple belief hierarchy are precisely the choices that are optimal in a *generalized Nash equilibrium.*

The concepts of a simple belief hierarchy and a generalized Nash equilibrium, together with the result above, can be extended to games with *infinitely* many choices and utility functions, as is the case in the Bertrand model we consider. Indeed, a simple belief hierarchy for firm i is a belief hierarchy β_i that is generated by a single belief σ_1 about firm 1's price-cost pair and a single belief σ_2 about firm

2's price-cost pair. Moreover, such a pair of beliefs (σ_1, σ_2) would be a generalized Nash equilibrium if for both firms i, the belief σ_i only concentrates on price-cost pairs (p_i, c_i) where the price p_i is optimal for firm i at the marginal cost c_i under the belief about competitor j's price as induced by σ_j.

The prices that firm i can rationally choose under common belief in rationality with a simple belief hierarchy at the marginal cost c_i are exactly the prices p_i that are optimal at the marginal cost c_i in such a generalized Nash equilibrium. But what are these prices in the Bertrand model? That is the question we will investigate now.

Consider a generalized Nash equilibrium (σ_1, σ_2). Let r_1 be the belief about firm 1's costs induced by σ_1, and similarly for r_2. By (3.7.5) we know that, for every marginal cost c_1, the optimal price for firm 1 under the belief σ_2 about firm 2's price-cost pair is given by

$$p_1(c_1) = \tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot E_{\sigma_2}(p_2), \tag{6.6.2}$$

where $E_{\sigma_2}(p_2)$ is the expected price for firm 2 under the belief σ_2.

As (σ_1, σ_2) is a generalized Nash equilibrium, the belief σ_1 should only concentrate on price-cost pairs (p_1, c_1) for firm 1 where $p_1 = p_1(c_1)$ as given in (6.6.2). Since the optimal price $p_1(c_1)$ depends linearly on the cost c_1, the expected price for firm 1 under the belief σ_1 about firm 1's price-cost pairs is

$$E_{\sigma_1}(p_1) = \tfrac{1}{2} \cdot (E_{r_1}(c_1) + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot E_{\sigma_2}(p_2), \tag{6.6.3}$$

where $E_{r_1}(c_1)$ is the expected marginal cost for firm 1 under the belief r_1 about firm 1's marginal cost.

In a similar fashion it can be shown that

$$E_{\sigma_2}(p_2) = \tfrac{1}{2} \cdot (E_{r_2}(c_2) + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot E_{\sigma_1}(p_1). \tag{6.6.4}$$

If we substitute (6.6.4) into (6.6.3) we get

$$E_{\sigma_1}(p_1) = \tfrac{1}{2}(E_{r_1}(c_1) + \tfrac{a}{d}) + \tfrac{e}{2d}[\tfrac{1}{2}(E_{r_2}(c_2) + \tfrac{a}{d}) + \tfrac{e}{2d} E_{\sigma_1}(p_1)],$$

which yields

$$E_{\sigma_1}(p_1) = \frac{(E_{r_1}(c_1)+(e/2d)E_{r_2}(c_2))/(1+e/2d)+a/d}{2-e/d}. \tag{6.6.5}$$

Similarly,

$$E_{\sigma_2}(p_2) = \frac{(E_{r_2}(c_2)+(e/2d)E_{r_1}(c_1))/(1+e/2d)+a/d}{2-e/d}. \tag{6.6.6}$$

If we substitute this into (6.6.2) we conclude that

$$p_1(c_1) = \tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \frac{(E_{r_2}(c_2)+(e/2d)E_{r_1}(c_1))/(1+e/2d)+a/d}{2-e/d}. \tag{6.6.7}$$

Similarly,

$$p_2(c_2) = \tfrac{1}{2} \cdot (c_2 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \frac{(E_{r_1}(c_1)+(e/2d)E_{r_2}(c_2))/(1+e/2d)+a/d}{2-e/d} \tag{6.6.8}$$

Hence, every generalized Nash equilibrium (σ_1, σ_2) has the property that σ_1 only concentrates on pairs $(p_1(c_1), c_1)$ where $p_1(c_1)$ is given by (6.6.7), and σ_2 only concentrates on pairs $(p_2(c_2), c_2)$ where $p_2(c_2)$ is given by (6.6.8).

As a consequence, the generalized Nash equilibrium is uniquely given by the belief r_1 about firm 1's costs and the belief r_2 about firm 2's cost: Indeed, if the beliefs r_1 and r_2 about the firms' marginal costs are known, then we can compute the expected costs $E_{r_1}(c_1)$ and $E_{r_2}(c_2)$, which in turn uniquely determine the optimal prices $p_1(c_1)$ and $p_2(c_2)$ for every marginal cost by means of (6.6.7) and (6.6.8). Hence, σ_1 must be the unique belief about firm 1's price-cost pairs that (i) has the belief r_1 about firm

1's cost, and (ii) only concentrates on pairs $(p_1(c_1), c_1)$ where $p_1(c_1)$ is given by (6.6.7), and similarly for σ_2.

Thus, for every pair of beliefs (r_1, r_2) about the firms marginal costs there is exactly one generalized Nash equilibrium (σ_1, σ_2) that has these beliefs (r_1, r_2). Moreover, this generalized Nash equilibrium can be derived on the basis of (6.6.7) and (6.6.8). Also, (6.6.7) tells us precisely which price is optimal for firm 1 for every possible marginal cost c_1 in such a generalized Nash equilibrium: This is exactly the price $p_1(c_1)$.

By Theorem 6.1.2, the prices that can rationally be chosen by firm 1 at marginal cost c_1 under common belief in rationality with a simple belief hierarchy are exactly the prices $p_1(c_1)$ as given by (6.6.7), where $E_{r_1}(c_1)$ and $E_{r_2}(c_2)$ can vary between $\underline{c}$ and $\overline{c}$. The lowest such price is obtained when $E_{r_1}(c_1) = E_{r_2}(c_2) = \underline{c}$, in which case

$$\begin{aligned} p_1(c_1) &= \tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \tfrac{(\underline{c}+(e/2d)\underline{c})/(1+e/2d)+a/d}{2-e/d} \\ &= \tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \tfrac{\underline{c}+a/d}{2-e/d}. \end{aligned}$$

Similarly, the highest such price is obtained when $E_{r_1}(c_1) = E_{r_2}(c_2) = \overline{c}$, in which case

$$\begin{aligned} p_1(c_1) &= \tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \tfrac{(\overline{c}+(e/2d)\overline{c})/(1+e/2d)+a/d}{2-e/d} \\ &= \tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \tfrac{\overline{c}+a/d}{2-e/d}. \end{aligned}$$

Hence, under common belief in rationality with a simple belief hierarchy at the marginal cost c_1, firm 1 can rationally choose any price in the interval

$$P_1^*(c_1) = [\tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \tfrac{\underline{c}+a/d}{2-e/d}, \tfrac{1}{2} \cdot (c_1 + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \tfrac{\overline{c}+a/d}{2-e/d}]. \tag{6.6.9}$$

Note that this is precisely the interval in (6.6.1), which contained the prices that were possible under common belief in rationality. Hence, under common belief in rationality with a simple belief hierarchy, firm 1 can rationally choose the same set of prices as under common belief in rationality without insisting on a simple belief hierarchy. The same holds for firm 2.

We next turn to the scenario where there are *fixed beliefs about utilities*. Suppose we fix some beliefs (r_1, r_2) about the firms' marginal costs. What prices can firm 1 rationally choose if it has a marginal cost of c_1, expresses common belief in rationality and common belief in (r_1, r_2), and holds a simple belief hierarchy?

The answer is given in (6.6.7): Indeed, we have seen that there is a unique generalized Nash equilibrium (σ_1, σ_2) that respects the fixed beliefs (r_1, r_2) on marginal costs, and in this generalized Nash equilibrium the optimal price for firm 1 at the marginal cost c_1 is given by (6.6.7). As such, if firm 1 has a simple belief hierarchy that expresses common belief in rationality and common belief in (r_1, r_2), then the unique optimal price at marginal cost c_1 is given by (6.6.7). Similarly for firm 2.

Note that the optimal price $p_1(c_1)$ for firm 1 in (6.6.7) is increasing in both $E_{r_1}(c_1)$ and $E_{r_2}(c_2)$. This makes intuitive sense: If firm 1's belief r_2 about firm 2's marginal cost would change by deeming higher marginal costs for firm 2 more likely, then firm 1 will believe that, in expectation, firm 2 will choose a higher price to compensate for this. In turn, this will allow firm 1 to choose a higher price as well. Similarly, if $E_{r_1}(c_1)$ rises, then firm 1 believes that firm 2 will deem higher marginal costs for firm 1 more likely. As we have seen above, this will induce firm 2 to choose a higher price in expectation. Firm 1, anticipating on this, will then also raise its price.

Consider now the special case where r_1 and r_2 are the uniform distribution on $[\underline{c}, \overline{c}]$. Then, the expected marginal costs are given by $E_{r_1}(c_1) = E_{r_2}(c_2) = (\underline{c} + \overline{c})/2$. If we substitute this into (6.6.7) we conclude that the only optimal price for firm i at marginal cost c_i is

$$p_i(c_i) = \tfrac{1}{2} \cdot (c_i + \tfrac{a}{d}) + \tfrac{e}{2d} \cdot \tfrac{(\underline{c}+\overline{c})/2+a/d}{2-e/d}. \tag{6.6.10}$$

Note that this matches precisely (5.7.17), which described the unique price that firm i can rationally choose at marginal cost c_i if it expresses common belief in rationality and common belief in (r_1, r_2). This, of course, should come as no surprise: If common belief in rationality and common belief in (r_1, r_2) already leads to a unique optimal price at c_i, then it will remain the unique optimal price if, in addition, we require the belief hierarchy to be simple.

6.6.2 Competition in Quantities

Recall the Cournot competition model with incomplete information from Section 5.7.2. We saw in (5.7.24) that under common belief in rationality, firm i, at a marginal cost of c_i, can rationally choose any quantity from the interval

$$Q_i^*(c_i) = [\tfrac{a-c_i}{2e} - \tfrac{a-\underline{c}}{3e} + \tfrac{a-\overline{c}}{6e},\ \tfrac{a-c_i}{2e} - \tfrac{a-\overline{c}}{3e} + \tfrac{a-\underline{c}}{6e}]. \tag{6.6.11}$$

What quantities can firm i rationally choose if, in addition, we require a simple belief hierarchy? That is the question we wish to address now. Similarly as for the Bertrand model above, this amounts to finding the quantities that are optimal for firm i in a *generalized Nash equilibrium.*

Consider a generalized Nash equilibrium (σ_1, σ_2), where σ_1 is a probabilistic belief about firm 1's quantity-cost pair, and σ_2 is a probabilistic belief about firm 2's quantity-cost pair. By (3.7.12) we know that in this generalized Nash equilibrium, the optimal quantity for firm 1 at a marginal cost of c_1 is given by

$$q_1(c_1) = \tfrac{a-c_1}{2e} - \tfrac{1}{2}E_{\sigma_2}(q_2), \tag{6.6.12}$$

where $E_{\sigma_2}(q_2)$ is the expected quantity for firm 2 under the belief σ_2.

Now, let r_1 be the belief about firm 1's marginal cost induced by σ_1, and similarly for r_2. Note that, by (6.6.12), the optimal quantity $q_1(c_1)$ depends linearly on the marginal cost c_1. Moreover, as (σ_1, σ_2) is a generalized Nash equilibrium, the belief σ_1 only concentrates on pairs $(q_1(c_1), c_1)$ where $q_1(c_1)$ is given by (6.6.12). As such, we conclude that the expected quantity for firm 1 under the belief σ_1 is given by

$$E_{\sigma_1}(q_1) = \tfrac{a-E_{r_1}(c_1)}{2e} - \tfrac{1}{2}E_{\sigma_2}(q_2). \tag{6.6.13}$$

Similarly, it follows that

$$E_{\sigma_2}(q_2) = \tfrac{a-E_{r_2}(c_2)}{2e} - \tfrac{1}{2}E_{\sigma_1}(q_1). \tag{6.6.14}$$

If we substitute (6.6.14) into (6.6.13) we get

$$E_{\sigma_1}(q_1) = \tfrac{a-E_{r_1}(c_1)}{2e} - \tfrac{1}{2}[\tfrac{a-E_{r_2}(c_2)}{2e} - \tfrac{1}{2}E_{\sigma_1}(q_1)].$$

Solving for $E_{\sigma_1}(q_1)$ then yields

$$E_{\sigma_1}(q_1) = \tfrac{a-2E_{r_1}(c_1)+E_{r_2}(c_2)}{3e}.$$

Similarly, we obtain that

$$E_{\sigma_2}(q_2) = \tfrac{a-2E_{r_2}(c_2)+E_{r_1}(c_1)}{3e}.$$

Together with (6.6.12) we conclude that firm 1's unique optimal quantity at marginal cost c_1 in the generalized Nash equilibrium (σ_1, σ_2) is given by

$$q_1(c_1) = \frac{a-c_1}{2e} - \frac{1}{2}\frac{a-2E_{r_2}(c_2)+E_{r_1}(c_1)}{3e}, \tag{6.6.15}$$

and similarly for firm 2.

Note that this optimal quantity only depends on the expected marginal costs $E_{r_1}(c_1)$ and $E_{r_2}(c_2)$ induced by (σ_1, σ_2). As a consequence, for every pair (r_1, r_2) of beliefs on marginal costs there is a unique generalized Nash equilibrium (σ_1, σ_2) with these beliefs, where for both firms i the belief σ_i is the unique belief on quantity-cost pairs for firm i that has the belief r_i on marginal costs, and only concentrates on pairs $(q_i(c_i), c_i)$ where $q_i(c_i)$ is given by (6.6.15).

On the basis of (6.6.15) we now know what quantities firm 1 can rationally choose at marginal cost c_1 if it has a simple belief hierarchy that expresses common belief in rationality. These would be any of the quantities $q_1(c_1)$ in (6.6.15), where $E_{r_1}(c_1)$ and $E_{r_2}(c_2)$ can vary arbitrarily between $\underline{c}$ and $\overline{c}$. The lowest such $q_1(c_1)$ is obtained when $E_{r_1}(c_1) = \overline{c}$ and $E_{r_2}(c_2) = \underline{c}$, resulting in

$$q_1(c_1) = \frac{a-c_1}{2e} - \frac{1}{2}\frac{a-2\underline{c}+\overline{c}}{3e} = \frac{a-c_1}{2e} - \frac{a-\underline{c}}{3e} + \frac{a-\overline{c}}{6e}.$$

Similarly, the highest such $q_1(c_1)$ is obtained when $E_{r_1}(c_1) = \underline{c}$ and $E_{r_2}(c_2) = \overline{c}$, resulting in

$$q_1(c_1) = \frac{a-c_1}{2e} - \frac{1}{2}\frac{a-2\overline{c}+\underline{c}}{3e} = \frac{a-c_1}{2e} - \frac{a-\overline{c}}{3e} + \frac{a-\underline{c}}{6e}.$$

Hence, with a simple belief hierarchy that expresses common belief in rationality, and with a marginal cost of c_1, firm 1 can rationally choose any quantity from the set

$$Q_1^*(c_1) = [\frac{a-c_1}{2e} - \frac{a-\underline{c}}{3e} + \frac{a-\overline{c}}{6e},\ \frac{a-c_1}{2e} - \frac{a-\overline{c}}{3e} + \frac{a-\underline{c}}{6e}]. \tag{6.6.16}$$

Similarly for firm 2.

Note that this matches precisely the set from (6.6.11), which indicated what quantities firm i could rationally choose under common belief in rationality, without requiring a simple belief hierarchy. Hence, the additional condition of a simple belief hierarchy does not alter the quantities that both firms can rationally choose under common belief in rationality.

Consider now the scenario where we impose *fixed beliefs on utilities*. Suppose we fix a pair of beliefs (r_1, r_2) on the firms' marginal costs. What quantities can both firms rationally choose, for each of the possible marginal costs, if they hold a simple belief hierarchy that expresses common belief in rationality and common belief in (r_1, r_2)?

Similarly as in the Bertrand model above, the answer is given by (6.6.15). Indeed, for a given pair (r_1, r_2) of beliefs on the firms' marginal costs, we saw that there is a unique generalized Nash equilibrium (σ_1, σ_2) that has these beliefs, and for this generalized Nash equilibrium the optimal quantity for firm 1 at marginal cost c_1 is given by (6.6.15). As a consequence, the only quantity that firm 1 can rationally choose at marginal cost c_1 if it holds a simple belief hierarchy that expresses common belief in rationality and common belief in (r_1, r_2) is the quantity $q_1(c_1)$ given by (6.6.15).

Note that this optimal quantity $q_1(c_1)$ is decreasing in the expected marginal cost $E_{r_1}(c_1)$ for firm 1, and increasing in the expected marginal cost $E_{r_2}(c_2)$ for firm 2. This has a clear economic interpretation: If $E_{r_2}(c_2)$ rises, then firm 1's belief about firm 2's marginal cost starts to assign higher probabilities to higher marginal costs. As a consequence, firm 1 expects firm 2 to decrease its quantity in expectation. Anticipating on this, firm 1 can increase its own quantity. Moreover, if $E_{r_1}(c_1)$ rises, then firm 1 believes that firm 2 starts to assign higher probabilities to higher marginal costs for firm

1. As we have seen above, this will lead firm 1 to believe that firm 2 will increase its quantity in expectation. Anticipating on this, firm 1 will then decrease its quantity.

Consider finally the special case where r_1 and r_2 are the uniform distribution on $[\underline{c}, \overline{c}]$. Then, the expected marginal costs for both firms are $E_{r_1}(c_1) = E_{r_2}(c_2) = (\underline{c} + \overline{c})/2$. If we substitute this into (6.6.15), we see that the optimal quantity for firm 1 at the marginal cost c_1 is

$$q_1(c_1) = \tfrac{a-c_1}{2e} - \tfrac{a-(\underline{c}+\overline{c})/2}{6e}, \tag{6.6.17}$$

and similarly for firm 2.

Note that this matches precisely (5.7.34), which indicated the unique quantity that firm i could rationally choose at marginal cost c_i under common belief in rationality and common belief in the beliefs (r_1, r_2) about the marginal costs. Again, this should come as no surprise: If common belief in rationality and common belief in (r_1, r_2) already leads to a unique optimal quantity $q_1(c_1)$ for every marginal cost c_1, then this will remain so if we additionally require a simple belief hierarchy.

Chapter 7

Common Belief in Rationality with Unawareness

7.9 Economic Applications

In this section we investigate variations of the Bertrand competition model and Cournot competition model from Section 3.7 in which a firm may be unaware of certain choices that the other firm can make.

7.9.1 Competition in Prices

Like in Section 3.7.1, we consider the competition in prices between two firms that produce differentiated goods. The difference is that firm 1 now has the additional option to produce a *new good* that is more similar to firm 2's good than the *standard good* it has been producing until now. More precisely, if firm 1 produces the standard good while choosing a price of p_1, and firm 2 chooses a price of p_2, then the demand for both firms is given by

$$q_1 = 24 - p_1 + p_2 \text{ and } q_2 = 24 - p_2 + p_1. \tag{7.9.1}$$

That is, in the Bertrand competition model from Section 3.7.1 we choose the parameters $a = 24$ and $d = e = 1$. If, on the other hand, firm 1 decides to produce the new good, and the prices chosen are p_1 and p_2, then the demands would be

$$q_1 = 24 - 4p_1 + 4p_2 \text{ and } q_2 = 24 - 4p_2 + 4p_1. \tag{7.9.2}$$

Hence, we would have that $d = e = 4$. In this case, the demands for both firms would change more rapidly with the prices since the goods of firms 1 and 2 are more similar now, and hence consumers start caring more about the price.

However, firm 2 need not be aware of the new good that firm 1 can produce. This leads to two different views for firm 2, which are v_2^s and v_2^n. Here, v_2^s stands for firm 2's view where it is only aware of the *standard* good of firm 1, while v_2^n is the view where it is aware of firm 1's additional option to produce the *new* good.

Of course, if firm 2's view is v_2^s, then it cannot possibly reason about firm 1 being able to produce the new good. Therefore, we need two views for firm 1, which are v_1^s and v_1^n. Here, v_1^n is firm 1's actual view, where it is aware of the additional option to produce the new good, whereas v_1^s is the restricted view where it is not aware of this option.

Suppose that both firms have a constant marginal cost of 4, and that the maximum price that can be chosen by both firms is 40. That is, we assume that $c = 4$ and $M = 40$ in the model of Section 3.7.1. What prices can firm 1 rationally choose under common belief in rationality?

As a first step towards answering this question, let us first model the situation above as a game with unawareness. As already announced above, the possible views for firm 1 are v_1^s and v_1^n, whereas the possible views for firm 2 are v_2^s and v_2^n.

In the view v_1^s, firm 1 can make any choice of the type (s, p_1), where s indicates that firm 1 produces the standard good, and p_1 can be any price from $[0, 40]$. Moreover, the states are all prices p_2 from $[0, 40]$ that can be chosen by firm 2.

In the larger view v_1^n, the possible choices for firm 1 are the pairs (s, p_1) considered above, together with any pair (n, p_1), where n indicates that firm 1 produces the new good, and p_1 can be any price from $[0, 40]$. The states are still all the prices p_2 from $[0, 40]$ that can be chosen by firm 2.

In firm 2's view v_2^s, the set of possible choices for firm 2 are all prices p_2 from $[0, 40]$ that firm 2 can choose, whereas the states are all the choices (s, p_1) that firm 1 can make at the view v_1^s.

At the larger view v_1^n, the possible choices for firm 2 are still all the prices p_2 from $[0, 40]$ that firm 2 can choose. However, the states are now all the pairs (s, p_1) and (n, p_1) that firm 1 can choose at the view v_1^n.

We thus obtain a game with unawareness with infinitely many choices and states, but with finitely many views. For such games we can adopt the procedure of *iterated strict dominance for unawareness* as follows: In round 1 we eliminate, at every view, those choices that are not optimal for any probabilistic belief about the states. This yields the 1-fold reduced decision problems at the various views.

In round 2 we start by eliminating, at a given view v, those states that involve an opponent's choice that did not survive round 1 at any view that is contained in v. In the reduced decision problem so obtained at v, we then eliminate those choices that are not optimal for any belief about the states that remain. This yields the 2-fold reduced decision problems at the various views. And so on.

Similarly as for finite games with unawareness, this procedure delivers for every view precisely those choices that can rationally be made under common belief in rationality with that particular view. We will now use this procedure to find the prices that both firms can rationally choose under common belief in rationality at both of their possible views. In fact, we opt for the *bottom-up version* of the procedure, as it significantly reduces our computations.

Views of rank 1. Clearly, the views with rank 1 are v_1^s and v_2^s. Hence, we first investigate which prices both firms can rationally choose under common belief in rationality at the views v_1^s and v_2^s. Since firm 1, with view v_1^s, must believe that firm 2 has view v_2^s, and *vice versa,* we have a standard Bertrand competition model from Section 3.7.1 where the parameters are $a = 24$, $d = e = 1$, $c = 4$ and $M = 40$. We know, from (3.7.8), that under common belief in rationality both firms can only rationally choose the price

$$p^* = \frac{c+a/d}{2-e/d} = 28.$$

Hence, at the small views v_1^s and v_2^s we expect both firms to choose a price of 28 under common belief in rationality.

Views of rank 2. We now turn to the views of rank 2, which are v_1^n and v_2^n.

Round 1. Consider first the view v_1^n for firm 1. Which prices are optimal for some belief about the states, and which are not?

Suppose that firm 1 chooses the pair (s, p_1) for the standard product and firm 2 chooses the price p_2. Then, we know from (3.7.2) in Section 3.7.1 that firm 1's profit is

$$\pi_1((s, p_1), p_2) = (p_1 - c) \cdot (a - d \cdot p_1 + e \cdot p_2) = (p_1 - 4) \cdot (24 - p_1 + p_2).$$

Similarly, if firm 1 chooses the pair (n, p_1) for the new product and firm 2 chooses the price p_2, then firm 1's profit is

$$\pi_1((n, p_1), p_2) = (p_1 - c) \cdot (a - d \cdot p_1 + e \cdot p_2) = (p_1 - 4) \cdot (24 - 4p_1 + 4p_2).$$

Assume that firm 1 has a belief β_1 about firm 2's price. Since firm 1's profit above depends linearly on firm 2's price, it can be shown in a similar way as in Section 3.7.1 that firm 1's expected profit under this belief is given by

$$\pi_1((s, p_1), \beta_1) = (p_1 - 4) \cdot (24 - p_1 + E_{\beta_1}(p_2)) \tag{7.9.3}$$

if firm 1 opts for the standard product, whereas it is

$$\pi_1((n, p_1), \beta_1) = (p_1 - 4) \cdot (24 - 4p_1 + 4E_{\beta_1}(p_2)) \tag{7.9.4}$$

if firm 1 opts for the new product. Here, $E_{\beta_1}(p_2)$ denotes the expected price for firm 2 under the belief β_1.

Suppose that firm 1 opts for the standard product under the belief β_1. Which price would then be optimal for firm 1? Note that the expected profit $\pi_1((s, p_1), \beta_1)$ is a second-degree polynomial in p_1 which becomes zero at $p_1 = 4$ and $p_1 = 24 + E_{\beta_1}(p_2)$, and obtains a maximum exactly halfway between these two points. Hence, the optimal price in this case would be

$$p_1^s(\beta_1) = \tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot (24 + E_{\beta_1}(p_2)) = 14 + \tfrac{1}{2} E_{\beta_1}(p_2). \tag{7.9.5}$$

By (7.9.3), the maximal expected profit with the standard product under the belief β_1 is

$$\pi_1^s(\beta_1) = (10 + \tfrac{1}{2} E_{\beta_1}(p_2))^2. \tag{7.9.6}$$

Now assume that firm 1 opts for the new product under the belief β_1. Then, in view of (7.9.4), the expected profit $\pi_1((s, p_1), \beta_1)$ is a second-degree polynomial in p_1 which becomes zero at $p_1 = 4$ and $p_1 = 6 + E_{\beta_1}(p_2)$, and obtains a maximum exactly halfway between these two points. Hence, the optimal price in this case would be

$$p_1^n(\beta_1) = \tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot (6 + E_{\beta_1}(p_2)) = 5 + \tfrac{1}{2} E_{\beta_1}(p_2). \tag{7.9.7}$$

By (7.9.4), the maximal expected profit with the new product under the belief β_1 is

$$\pi_1^n(\beta_1) = (2 + E_{\beta_1}(p_2))^2. \tag{7.9.8}$$

Hence, the firm will choose the standard product if $\pi_1^s(\beta_1) > \pi_1^n(\beta_1)$, it will choose the new product if $\pi_1^n(\beta_1) > \pi_1^s(\beta_1)$, and it will be indifferent between the standard and the new product if $\pi_1^s(\beta_1) = \pi_1^n(\beta_1)$. In view of (7.9.6) and (7.9.8), we have $\pi_1^s(\beta_1) > \pi_1^n(\beta_1)$ precisely when

$$(10 + \tfrac{1}{2}E_{\beta_1}(p_2))^2 > (2 + E_{\beta_1}(p_2))^2,$$

which happens exactly when

$$10 + \tfrac{1}{2}E_{\beta_1}(p_2) > 2 + E_{\beta_1}(p_2),$$

yielding $E_{\beta_1}(p_2) < 16$.

Hence, if $E_{\beta_1}(p_2) < 16$ then firm 1 will choose the standard product together with the price in (7.9.5). Similarly, if $E_{\beta_1}(p_2) > 16$ then firm 1 will choose the new product together with the price in (7.9.7). Finally, if $E_{\beta_1}(p_2) = 16$ then firm 1 will be indifferent between the standard product together with the price 22 from (7.9.5) and the new product together with the price 13 from (7.9.7). Firm 1's optimal product-price pairs for every belief β_1 are thus given by

$$c_1(\beta_1) = \begin{cases} (s, 14 + \frac{1}{2}E_{\beta_1}(p_2)), & \text{if } E_{\beta_1}(p_2) < 16 \\ (s, 22) \text{ or } (n, 13), & \text{if } E_{\beta_1}(p_2) = 16 \\ (n, 5 + \frac{1}{2}E_{\beta_1}(p_2)), & \text{if } E_{\beta_1}(p_2) > 16 \end{cases}. \tag{7.9.9}$$

As the price p_2 can vary between 0 and 40, the expected price $E_{\beta_1}(p_2)$ will also be between 0 and 40. By (7.9.9) we thus conclude that the set $P_1^1(v_1^n)$ of product-price pairs for firm 1 that can be optimal for some belief at the view v_1^n is given by

$$P_1^1(v_1^n) = \{(s, p_1) \mid p_1 \in [14, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 25]\}. \tag{7.9.10}$$

We now turn to firm 2 with the view v_2^n. If firm 1 chooses a pair (s, p_1) containing the standard product, and firm 2 chooses a price p_2, then it follows from (3.7.3) in Section 3.7.1 that firm 2's profit is

$$\pi_2((s, p_1), p_2) = (p_2 - c) \cdot (a - d \cdot p_2 + e \cdot p_1) = (p_2 - 4) \cdot (24 - p_2 + p_1). \tag{7.9.11}$$

Similarly, if firm 1 chooses a pair (n, p_1) containing the new product, then firm 2's profit is

$$\pi_2((n, p_1), p_2) = (p_2 - c) \cdot (a - d \cdot p_2 + e \cdot p_1) = (p_2 - 4) \cdot (24 - 4p_2 + 4p_1). \tag{7.9.12}$$

Suppose now that firm 2 holds the belief β_2 about firm 1's product-price pairs. Let $\beta_2(s)$ and $\beta_2(n)$ be the probability that firm 2 assigns to firm 1 choosing the standard and the new product, respectively. Moreover, let $E_{\beta_2}(p_1|s)$ be the expected price for firm 1 under the belief β_2, conditional on firm 1 choosing the standard product. Similarly, $E_{\beta_2}(p_1|n)$ denotes the expected price for firm 1 under the belief β_2, conditional on firm 1 choosing the new product.

As firm 2's profit in (7.9.11) and (7.9.12) depends linearly on firm 1's price, firm 2's expected profit of choosing the price p_2 under the belief β_2 is

$$\begin{aligned} \pi_2(p_2, \beta_2) \quad = \quad & \beta_2(s) \cdot [(p_2 - 4) \cdot (24 - p_2 + E_{\beta_2}(p_1|s))] \\ & + \beta_2(n) \cdot [(p_2 - 4) \cdot (24 - 4p_2 + 4E_{\beta_2}(p_1|n))]. \end{aligned} \tag{7.9.13}$$

Firm 2's optimal price is obtained by setting the derivative $\frac{\partial \pi_2}{\partial p_2}$ of $\pi_2(p_2, \beta_2)$ with respect to p_2 equal to zero. Thus,

$$\frac{\partial \pi_2}{\partial p_2} = \beta_2(s) \cdot [(24 - p_2 + E_{\beta_2}(p_1|s)) + (p_2 - 4) \cdot (-1)]$$
$$+ \beta_2(n) \cdot [(24 - 4p_2 + 4E_{\beta_2}(p_1|n)) + (p_2 - 4) \cdot (-4)] = 0.$$

By solving for p_2, we see that the optimal price for firm 2 under the belief β_2 is given by

$$p_2(\beta_2) = \frac{\beta_2(s)(28+E_{\beta_2}(p_1|s))+\beta_2(n)(40+4E_{\beta_2}(p_1|n))}{2\beta_2(s)+8\beta_2(n)}. \tag{7.9.14}$$

What is the minimal price $p_2(\beta_2)$ that is optimal for a belief β_2? In view of (7.9.14) this is obtained if we set $E_{\beta_2}(p_1|s) = 0$ and $E_{\beta_2}(p_1|n) = 0$. In that case we would have

$$p_2(\beta_2) = \frac{\beta_2(s)\cdot 28+\beta_2(n)\cdot 40}{2\beta_2(s)+8\beta_2(n)}.$$

Since $\beta_2(s) + \beta_2(n) = 1$, we can set $\beta_2(s) = 1 - \beta_2(n)$ and write this as

$$p_2(\beta_2) = \frac{(1-\beta_2(n))\cdot 28+\beta_2(n)\cdot 40}{2(1-\beta_2(n))+8\beta_2(n)} = \frac{28+\beta_2(n)\cdot 12}{2+6\beta_2(n)},$$

which is an expression that only depends on $\beta_2(n)$. It may be verified that the derivative of $p_2(\beta_2)$ with respect to $\beta_2(n)$ is

$$\frac{\partial p_2(\beta_2)}{\partial \beta_2(n)} = -\frac{144}{(2+6\beta_2(n))^2} < 0,$$

and hence the optimal price $p_2(\beta_2)$ is decreasing in $\beta_2(n)$. As such, the optimal price $p_2(\beta_2)$ is minimized by setting $\beta_2(n) = 1$.

Overall, we see that the minimal price $p_2(\beta_2)$ that is optimal for a belief β_2 is obtained by choosing $\beta_2(n) = 1$ and $E_{\beta_2}(p_1|n) = 0$, resulting in $p_2(\beta_2) = 5$.

Next, we are interested in the maximal price $p_2(\beta_2)$ that is optimal for a belief β_2. In view of (7.9.14) this is obtained if we set $E_{\beta_2}(p_1|s) = 40$ and $E_{\beta_2}(p_1|n) = 40$. In that case we would have

$$p_2(\beta_2) = \frac{\beta_2(s)\cdot 68+\beta_2(n)\cdot 200}{2\beta_2(s)+8\beta_2(n)}.$$

Since $\beta_2(s) + \beta_2(n) = 1$, we can set $\beta_2(s) = 1 - \beta_2(n)$ and write this as

$$p_2(\beta_2) = \frac{(1-\beta_2(n))\cdot 68+\beta_2(n)\cdot 200}{2(1-\beta_2(n))+8\beta_2(n)} = \frac{68+\beta_2(n)\cdot 132}{2+6\beta_2(n)},$$

which is an expression that only depends on $\beta_2(n)$. It may be verified that the derivative of $p_2(\beta_2)$ with respect to $\beta_2(n)$ is

$$\frac{\partial p_2(\beta_2)}{\partial \beta_2(n)} = -\frac{144}{(2+6\beta_2(n))^2} < 0,$$

and hence the optimal price $p_2(\beta_2)$ is decreasing in $\beta_2(n)$. As such, the optimal price $p_2(\beta_2)$ is maximized by setting $\beta_2(n) = 0$, and hence by choosing $\beta_2(s) = 1$.

Overall, we see that the maximal price $p_2(\beta_2)$ that is optimal for a belief β_2 is obtained by choosing $\beta_2(s) = 1$ and $E_{\beta_2}(p_1|s) = 40$, resulting in $p_2(\beta_2) = 34$.

Thus, the set of prices for firm 2 that are optimal for some belief at view v_2^n is given by

$$P_2^1(v_2^n) = [5, 34].$$

Round 2. Consider firm 1 with view v_1^n. By definition, the set of states $S_1^2(v_1^n)$ contains those prices for firm 2 that have survived so far at some view v_2 contained in v_1^n. Note that both views v_2^s and v_2^n are contained in v_1^n. Recall from above that only price $p_2 = 28$ has survived for firm 2 at the view v_2^s in the bottom-up procedure. Moreover, at the view v_2^n all prices in $[5, 34]$ have survived for firm 2 in round 1. As such, the set of states in Round 2 at view v_1^n is

$$S_1^2(v_1^n) = [5, 34].$$

Therefore, firm 1's belief β_1 should only assign positive probability to firm 2's prices in $[5, 34]$, which implies that $E_{\beta_1}(p_2) \in [5, 34]$. By (7.9.9) we know that for every expected price $E_{\beta_1}(p_2)$ in $[5, 16]$ the optimal product-price pair for firm 1 is $(s, 14 + \frac{1}{2}E_{\beta_1}(p_2))$. Hence, every product-price pair (s, p_1) with $p_1 \in [16.5, 22]$ is optimal for some belief β_1 on $S_1^2(v_1^n)$. Moreover, it follows from (7.9.9) that for every expected price $E_{\beta_1}(p_2)$ in $[16, 34]$ the optimal product-price pair for firm 1 is $(n, 5 + \frac{1}{2}E_{\beta_1}(p_2))$. Hence, every product-price pair (n, p_1) with $n \in [13, 22]$ is optimal for some belief β_1 on $S_1^2(v_1^n)$.

As such, the set of product-price pairs for firm 1 that survive Round 2 at view v_1^n is given by

$$P_1^2(v_1^n) = \{(s, p_1) \mid p_1 \in [16.5, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 22]\}. \tag{7.9.15}$$

We now turn to firm 2 with view v_2^n. The set of states $S_2^2(v_2^n)$ that survive Round 2 for firm 2 at view v_2^n contain, by definition, those product-price pairs for firm 1 that have survived all previous rounds at a view v_1 contained in v_2^n. Note that both views v_1^s and v_1^n are contained in v_2^n. We have seen that at the view v_1^s only the product-price pair $(s, 28)$ has survived for firm 1 in the bottom-up procedure so far. Moreover, we know from (7.9.10) that at the view v_1^n, the set of product-price pairs that survived Round 1 was

$$P_1^1(v_1^n) = \{(s, p_1) \mid p_1 \in [14, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 25]\}.$$

Taken together, we conclude that the set of states $S_2^2(v_2^n)$ for firm 2 in Round 2 at view v_2^n is given by

$$S_2^2(v_2^n) = \{(s, p_1) \mid p_1 \in [14, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 25]\} \cup \{(s, 28)\}. \tag{7.9.16}$$

Here, the first two sets contain the choices for firm 1 that survived Round 1 at the view v_1^n, whereas the last set contains the unique choice that survived for firm 1 at the view v_1^s.

Firm 2 is thus required to hold a belief β_2 on this set of states $S_2^2(v_2^n)$. From the first and the last set in (7.9.16) we conclude that $E_{\beta_2}(p_1|s) \in [14, 28]$. Indeed, every expected price between 22 and 28 can be induced by a belief β_2 that assigns a positive probability to $(s, 22)$ and a positive probability to $(s, 28)$. Moreover, from the second set in (7.9.16) we know that $E_{\beta_2}(p_1|n) \in [13, 25]$.

In view of (7.9.14) it can be verified, similarly to what we have done in Round 1, that the lowest price $p_2(\beta_2)$ that is optimal for such a belief β_2 is obtained by choosing $\beta_2(n) = 1$ and $E_{\beta_2}(p_1|n) = 13$, resulting in the optimal price $p_2(\beta_2) = 11.5$. From (7.9.14) it also follows, in a similar way as in Round 1, that the highest price $p_2(\beta_2)$ that is optimal for such a belief β_2 is obtained by choosing $\beta_2(s) = 1$ and $E_{\beta_2}(p_1|s) = 28$, resulting in the optimal price $p_2(\beta_2) = 28$.

Hence, the set of prices that survive Round 2 for firm 2 at view v_2^n is

$$P_2^2(v_2^n) = [11.5, 28]. \tag{7.9.17}$$

Round 3. Consider firm 1 with view v_1^n. By definition, the set of states $S_1^3(v_1^n)$ contains those prices for firm 2 that have survived so far at some view v_2 contained in v_1^n. Recall that both views v_2^s and v_2^n are contained in v_1^n. From above we know that only price $p_2 = 28$ has survived for firm 2 at the view v_2^s in the bottom-up procedure. Moreover, we know by (7.9.17) that at the view v_2^n all prices in $[11.5, 28]$ have survived for firm 2 in round 2. As such, the set of states in Round 3 at view v_1^n is

$$S_1^3(v_1^n) = [11.5, 28].$$

Therefore, firm 1's belief β_1 should only assign positive probability to firm 2's prices in $[11.5, 28]$, which implies that $E_{\beta_1}(p_2) \in [11.5, 28]$. By (7.9.9) we know that for every expected price $E_{\beta_1}(p_2)$ in $[11.5, 16]$ the optimal product-price pair for firm 1 is $(s, 14 + \frac{1}{2}E_{\beta_1}(p_2))$. Hence, every product-price pair (s, p_1) with $p_1 \in [19.75, 22]$ is optimal for some belief β_1 on $S_1^3(v_1^n)$. Moreover, it follows from (7.9.9) that for every expected price $E_{\beta_1}(p_2)$ in $[16, 28]$ the optimal product-price pair for firm 1 is $(n, 5 + \frac{1}{2}E_{\beta_1}(p_2))$. Hence, every product-price pair (n, p_1) with $p_1 \in [13, 19]$ is optimal for some belief β_1 on $S_1^3(v_1^n)$.

As such, the set of product-price pairs for firm 1 that survive Round 3 at view v_1^n is given by

$$P_1^3(v_1^n) = \{(s, p_1) \mid p_1 \in [19.75, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 19]\}. \tag{7.9.18}$$

We now turn to firm 2 with view v_2^n. The set of states $S_2^3(v_2^n)$ that survive Round 3 for firm 2 at view v_2^n contain, by definition, those product-price pairs for firm 1 that have survived all previous rounds at a view v_1 contained in v_2^n. Recall that both views v_1^s and v_1^n are contained in v_2^n. We have seen that at the view v_1^s only the product-price pair $(s, 28)$ has survived for firm 1 in the bottom-up procedure so far. Moreover, we know from (7.9.15) that at the view v_1^n, the set of product-price pairs that survived Round 2 was

$$P_1^2(v_1^n) = \{(s, p_1) \mid p_1 \in [16.5, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 22]\}.$$

Taken together, we conclude that the set of states $S_2^3(v_2^n)$ for firm 2 in Round 3 at view v_2^n is given by

$$S_2^3(v_2^n) = \{(s, p_1) \mid p_1 \in [16.5, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 22]\} \cup \{(s, 28)\}. \tag{7.9.19}$$

Here, the first two sets contain the choices for firm 1 that survived Round 2 at the view v_1^n, whereas the last set contains the unique choice that survived for firm 1 at the view v_1^s.

Firm 2 is thus required to hold a belief β_2 on this set of states $S_2^3(v_2^n)$. From the first and the last set in (7.9.19) we conclude that $E_{\beta_2}(p_1|s) \in [16.5, 28]$. Indeed, as before, every expected price between 22 and 28 can be induced by a belief β_2 that assigns a positive probability to $(s, 22)$ and a positive probability to $(s, 28)$. Moreover, from the second set in (7.9.19) we know that $E_{\beta_2}(p_1|n) \in [13, 22]$.

In view of (7.9.14) it may be verified, similarly to what we have done in Round 1, that the lowest price $p_2(\beta_2)$ that is optimal for such a belief β_2 is obtained by choosing $\beta_2(n) = 1$ and $E_{\beta_2}(p_1|n) = 13$, resulting in the optimal price $p_2(\beta_2) = 11.5$. From (7.9.14) it also follows, similarly to Round 1, that the highest price $p_2(\beta_2)$ that is optimal for such a belief β_2 is obtained by choosing $\beta_2(s) = 1$ and $E_{\beta_2}(p_1|s) = 28$, resulting in the optimal price $p_2(\beta_2) = 28$.

Hence, the set of prices that survive Round 3 for firm 2 at view v_2^n is

$$P_2^3(v_2^n) = [11.5, 28], \tag{7.9.20}$$

which is the same as $P_2^2(v_2^n)$.

Round 4. Consider firm 1 with view v_1^n. As before, $S_2^4(v_1^n)$ contains all prices for firm 2 that have survived so far at the views v_2^s and v_2^n. Since only price $p_2 = 28$ has survived for firm 2 at view v_2^s, and $P_2^3(v_2^n) = P_2^2(v_2^n)$, we conclude that $S_1^4(v_1^n) = S_1^3(v_1^n)$. As a consequence,

$$P_1^4(v_1^n) = P_1^3(v_1^n) = \{(s, p_1) \mid p_1 \in [19.75, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 19]\}. \tag{7.9.21}$$

We next turn to firm 2 with view v_2^n. The set of states $S_2^4(v_2^n)$ that survive Round 4 for firm 2 at view v_2^n contain, by definition, those product-price pairs for firm 1 that have survived all previous rounds at a view v_1 contained in v_2^n. Recall that both views v_1^s and v_1^n are contained in v_2^n. We have seen that at the view v_1^s only the product-price pair $(s, 28)$ has survived for firm 1 in the bottom-up procedure so far. Moreover, we know from (7.9.18) that at the view v_1^n, the set of product-price pairs that survived Round 3 was

$$P_1^3(v_1^n) = \{(s, p_1) \mid p_1 \in [19.75, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 19]\}.$$

Taken together, we conclude that the set of states $S_2^4(v_2^n)$ for firm 2 in Round 4 at view v_2^n is given by

$$S_2^4(v_2^n) = \{(s, p_1) \mid p_1 \in [19.75, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 19]\} \cup \{(s, 28)\}. \tag{7.9.22}$$

Here, the first two sets contain the choices for firm 1 that survived Round 3 at the view v_1^n, whereas the last set contains the unique choice that survived for firm 1 at the view v_1^s.

Firm 2 is thus required to hold a belief β_2 on this set of states $S_2^4(v_2^n)$. From the first and the last set in (7.9.22) we conclude that $E_{\beta_2}(p_1|s) \in [19.75, 28]$. Indeed, as before, every expected price between 22 and 28 can be induced by a belief β_2 that assigns a positive probability to $(s, 22)$ and a positive probability to $(s, 28)$. Moreover, from the second set in (7.9.22) we know that $E_{\beta_2}(p_1|n) \in [13, 19]$.

In view of (7.9.14) it can be verified, similarly to what we have done in Round 1, that the lowest price $p_2(\beta_2)$ that is optimal for such a belief β_2 is obtained by choosing $\beta_2(n) = 1$ and $E_{\beta_2}(p_1|n) = 13$, resulting in the optimal price $p_2(\beta_2) = 11.5$. From (7.9.14) it also follows, similarly to Round 1, that the highest price $p_2(\beta_2)$ that is optimal for such a belief β_2 is obtained by choosing $\beta_2(s) = 1$ and $E_{\beta_2}(p_1|s) = 28$, resulting in the optimal price $p_2(\beta_2) = 28$.

Hence, the set of prices that survive Round 4 for firm 2 at view v_2^n is

$$P_2^4(v_2^n) = [11.5, 28],$$

which is the same as $P_2^2(v_2^n)$ and $P_2^3(v_2^n)$.

As $P_1^4(v_1^n) = P_1^3(v_1^n)$ and $P_2^4(v_2^n) = P_2^3(v_2^n)$, the bottom-up procedure terminates at this round. The surviving choices for firm 1 and firm 2 at the views v_1^n and v_2^n are thus

$$P_1^*(v_1^n) = \{(s, p_1) \mid p_1 \in [19.75, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 19]\} \tag{7.9.23}$$

and

$$P_2^*(v_2^n) = [11.5, 28]. \tag{7.9.24}$$

Hence, under common belief in rationality at view v_1^n, firm 1 can rationally choose the standard product together with any price between 19.75 and 22, or the new product together with any price between 13 and 19. Moreover, at the view v_2^n firm 2 can rationally choose any price between 11.5 and 28 under common belief in rationality.

At the view v_1^s, firm 1 can only rationally choose the product-price pair $(s, 28)$ under common belief in rationality, whereas firm 2 can only rationally choose the price 28 under common belief in rationality at the view v_2^s.

We now turn to a scenario where there are *fixed beliefs about the views.* Consider the fixed belief combination $p = (p_1, p_2)$ on views as given by Figure 7.9.1. Hence, if firm 1 holds the larger view v_1^n, then it believes that firm 2 will have either of the two views v_2^n and v_2^s with equal probability. In other words, if firm 1 can offer the new product, then it believes that with probability 0.5 firm 2 will also be aware of the new product, and with probability 0.5 firm 2 will not be aware of the new product. However, if firm 2 is aware of the possibility that firm 1 can offer the new product, then it believes

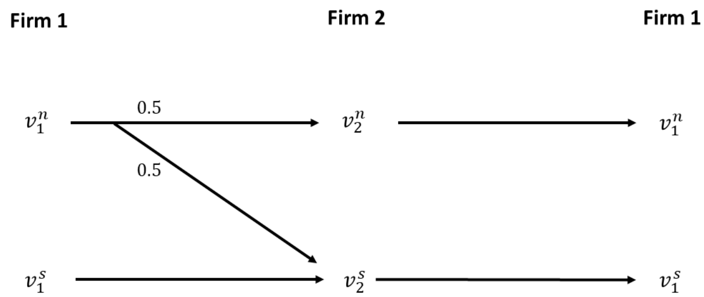

Figure 7.9.1 Fixed beliefs about views for price competition

that, with probability 1, firm 1 can actually offer the new product. This makes intuitive sense. What product-price pairs can firm 1 rationally choose at the view v_1^n under common belief in rationality and common belief in the fixed beliefs p on views? Similarly as before, we will use the *bottom-up* version of *iterated strict dominance for unawareness with fixed beliefs p on views.*

Views of rank 1. At the views v_1^s and v_2^s, both firms are only aware of the standard product for firm 1, and believe that the other firm is also aware of only the standard product for firm 1. From the analysis above without fixed beliefs on views we know that under common belief in rationality, firm 1 can only rationally choose the product-price pair $(s, 28)$, and firm 2 can only rationally choose the price 28.

Views of rank 2. We now consider the views of rank 2, which are v_1^n and v_2^n.

Round 1. This round is the same as for the procedure without fixed beliefs on views. We thus obtain the set of product-price pairs

$$P_1^1(v_1^n) = \{(s, p_1) \mid p_1 \in [14, 22]\} \cup \{(n, p_1) \mid p_1 \in [13, 25]\} \tag{7.9.25}$$

for firm 1, and the set of prices

$$P_2^1(v_2^n) = [5, 34] \tag{7.9.26}$$

for firm 2.

Round 2. Firm 1 is required to hold a belief β_1 about firm 2's choice-view pairs that (i) assigns probability 0.5 to the views v_2^n and v_2^s, (ii) for the view v_2^n only assigns positive probability to price-view pairs (p_2, v_2^n) where $p_2 \in P_2^1(v_2^n)$, and (iii) for the view v_2^s only assigns positive probability to the price-view pair $(28, v_2^s)$. But then, in view of (7.9.26), the lowest expected price $E_{\beta_1}(p_2)$ for firm 2 under such a belief β_1 is $(0.5) \cdot 5 + (0.5) \cdot 28 = 16.5$, whereas the highest expected price $E_{\beta_1}(p_2)$ for firm 2 under such a belief β_1 is $(0.5) \cdot 34 + (0.5) \cdot 28 = 31$. Hence, the expected price $E_{\beta_1}(p_2)$ under such a belief β_1 lies in the interval $[16.5, 31]$.

By (7.9.9), the optimal product-price pair for firm 1 under such a belief β_1 is $(n, 5 + \frac{1}{2} E_{\beta_1}(p_2))$. As $E_{\beta_1}(p_2)$ lies in the interval $[16.5, 31]$, the optimal price $5 + \frac{1}{2} E_{\beta_1}(p_2)$ lies in the interval $[13.25, 20.5]$. Hence, the set of product-price pairs for firm 1 that are optimal at the view v_1^n for such a belief β_1 is

$$P_1^2(v_1^n) = \{(n, p_1) \mid p_1 \in [13.25, 20.5]\}. \tag{7.9.27}$$

We now turn to firm 2. By definition, firm 2 is required to hold a belief β_2 about firm 1's choice-view pairs that (i) assigns probability 1 to the view v_1^n, and (ii) for the view v_1^n only assigns positive probability to choice-view pairs (c_1, v_1^n) where $c_1 \in P_1^1(v_1^n)$. By (7.9.25) we then conclude that

$$E_{\beta_2}(p_1|s) \in [14, 22] \text{ and } E_{\beta_2}(p_1|n) \in [13, 25]. \tag{7.9.28}$$

By (7.9.14) and (7.9.28), the lowest price $p_2(\beta_2)$ that is optimal for such a belief β_2 is obtained by choosing $E_{\beta_2}(p_1|s) = 14$ and $E_{\beta_2}(p_1|n) = 13$ in (7.9.14). We then get

$$p_2(\beta_2) = \frac{\beta_2(s)(28+14)+\beta_2(n)(40+4\cdot 13)}{2\beta_2(s)+8\beta_2(n)} = \frac{42\beta_2(s)+92\beta_2(n)}{2\beta_2(s)+8\beta_2(n)}$$

As $\beta_2(s) + \beta_2(n) = 1$, we may substitute $\beta_2(s) = 1 - \beta_2(n)$ in the equation above, and obtain

$$p_2(\beta_2) = \frac{42+50\beta_2(n)}{2+6\beta_2(n)}. \tag{7.9.29}$$

It may be verified that the derivative of $p_2(\beta_2)$ with respect to $\beta_2(n)$ is

$$\frac{\partial p_2(\beta_2)}{\partial \beta_2(n)} = -\frac{152}{(2+6\beta_2(n))^2} < 0,$$

which implies that the optimal price $p_2(\beta_2)$ is decreasing in $\beta_2(n)$.

Therefore, the lowest optimal price $p_2(\beta_2)$ for such a belief β_2 is obtained when $\beta_2(n)$ is as large as possible. We thus see that the lowest possible optimal price $p_2(\beta_2)$ for such a belief is reached by choosing $\beta_2(n) = 1$ and $E_{\beta_2}(p_1|n) = 13$. By (7.9.29), the associated optimal price is $p_2(\beta_2) = \frac{42+50}{2+6} = 11.5$.

By (7.9.14) and (7.9.28), the highest price $p_2(\beta_2)$ that is optimal for such a belief β_2 is obtained by choosing $E_{\beta_2}(p_1|s) = 22$ and $E_{\beta_2}(p_1|n) = 25$ in (7.9.14). We then get

$$p_2(\beta_2) = \frac{\beta_2(s)(28+22)+\beta_2(n)(40+4\cdot 25)}{2\beta_2(s)+8\beta_2(n)} = \frac{50\beta_2(s)+140\beta_2(n)}{2\beta_2(s)+8\beta_2(n)}$$

As $\beta_2(s) + \beta_2(n) = 1$, we may substitute $\beta_2(s) = 1 - \beta_2(n)$ in the equation above, and obtain

$$p_2(\beta_2) = \frac{50+90\beta_2(n)}{2+6\beta_2(n)}. \tag{7.9.30}$$

It may be verified that the derivative of $p_2(\beta_2)$ with respect to $\beta_2(n)$ is

$$\frac{\partial p_2(\beta_2)}{\partial \beta_2(n)} = -\frac{120}{(2+6\beta_2(n))^2} < 0,$$

which implies that the optimal price $p_2(\beta_2)$ is decreasing in $\beta_2(n)$.

Therefore, the highest optimal price $p_2(\beta_2)$ for such a belief β_2 is obtained when $\beta_2(n)$ is as low as possible. We thus see that the highest possible optimal price $p_2(\beta_2)$ for such a belief is reached by choosing $\beta_2(n) = 0$ and $E_{\beta_2}(p_1|s) = 22$. By (7.9.30), the associated optimal price is $p_2(\beta_2) = \frac{50}{2} = 25$.

We thus conclude that the set of prices for firm 2 that are optimal for such a belief β_2 is given by

$$P_2^2(v_2^n) = [11.5, 25]. \tag{7.9.31}$$

Round 3. Firm 1 is required to hold a belief β_1 about firm 2's choice-view pairs that (i) assigns probability 0.5 to the views v_2^n and v_2^s, (ii) for the view v_2^n only assigns positive probability to price-view pairs (p_2, v_2^n) where $p_2 \in P_2^2(v_2^n)$, and (iii) for the view v_2^s only assigns positive probability to the price-view pair $(28, v_2^s)$. But then, in view of (7.9.31), the lowest expected price $E_{\beta_1}(p_2)$ for firm 2 under such a belief β_1 is $(0.5) \cdot 11.5 + (0.5) \cdot 28 = 19.75$, whereas the highest expected price $E_{\beta_1}(p_2)$

for firm 2 under such a belief β_1 is $(0.5) \cdot 25 + (0.5) \cdot 28 = 26.5$. Hence, the expected price $E_{\beta_1}(p_2)$ under such a belief β_1 lies in the interval $[19.75, 26.5]$.

By (7.9.9), the optimal product-price pair for firm 1 under such a belief β_1 is $(n, 5 + \frac{1}{2} E_{\beta_1}(p_2))$. As $E_{\beta_1}(p_2)$ lies in the interval $[19.75, 26.5]$, the optimal price $5 + \frac{1}{2} E_{\beta_1}(p_2)$ lies in the interval $[14.875, 18.25]$. Hence, the set of product-price pairs for firm 1 that are optimal at the view v_1^n for such a belief β_1 is

$$P_1^3(v_1^n) = \{(n, p_1) \mid p_1 \in [14.875, 18.25]\}. \tag{7.9.32}$$

We now turn to firm 2. By definition, firm 2 is required to hold a belief β_2 about firm 1's choice-view pairs that (i) assigns probability 1 to the view v_1^n, and (ii) for the view v_1^n only assigns positive probability to choice-view pairs (c_1, v_1^n) where $c_1 \in P_1^2(v_1^n)$. By (7.9.27) we conclude that β_2 only assigns positive probability to pairs (n, p_1) with $p_1 \in [13.25, 20.5]$.

Together with (i) and (ii) we see that

$$\beta_2(s) = 0, \ \beta_2(n) = 1, \text{ and } E_{\beta_2}(p_1|n) \in [13.25, 20.5]. \tag{7.9.33}$$

If we substitute this into (7.9.14), we see that the optimal price $p_2(\beta_2)$ under this belief β_2 is

$$p_2(\beta_2) = 5 + \tfrac{1}{2} E_{\beta_2}(p_1|n). \tag{7.9.34}$$

In view of (7.9.33) and (7.9.34), the lowest price $p_2(\beta_2)$ that is optimal for such a belief is

$$5 + \tfrac{1}{2} \cdot (13.25) = 11.625,$$

whereas the highest price $p_2(\beta_2)$ that is optimal for such a belief is

$$5 + \tfrac{1}{2} \cdot (20.5) = 15.25.$$

Hence, the set of prices for firm 2 that is optimal for such a belief β_2 at the view v_2^n is

$$P_2^3(v_2^n) = [11.625, 15.25]. \tag{7.9.35}$$

If we continue in this fashion we can derive $P_1^k(v_1^n)$ and $P_2^k(v_2^n)$ for every round $k \geq 4$ as well. We will show that, for every round $k \geq 3$,

$$P_1^k(v_1^n) = \{(n, p_1) \mid p_1 \in [l_1^k, h_1^k]\} \text{ and } P_2^k(v_2^n) = [l_2^k, h_2^k]$$

where

$$l_1^k = \begin{cases} \frac{106}{7} - (\frac{1}{8})^{(k-3)/2} \cdot \frac{1.875}{7}, & \text{if } k \text{ is odd} \\ \frac{106}{7} - \frac{1}{4} \cdot (\frac{1}{8})^{(k-4)/2} \cdot \frac{6.625}{7}, & \text{if } k \text{ is even} \end{cases}, \tag{7.9.36}$$

$$h_1^k = \begin{cases} \frac{106}{7} + (\frac{1}{8})^{(k-3)/2} \cdot \frac{21.75}{7}, & \text{if } k \text{ is odd} \\ \frac{106}{7} + \frac{1}{4} \cdot (\frac{1}{8})^{(k-4)/2} \cdot \frac{18.75}{7}, & \text{if } k \text{ is even} \end{cases}, \tag{7.9.37}$$

$$l_2^k = \begin{cases} \frac{88}{7} - (\frac{1}{8})^{(k-3)/2} \cdot \frac{6.625}{7}, & \text{if } k \text{ is odd} \\ \frac{88}{7} - \frac{1}{2} \cdot (\frac{1}{8})^{(k-4)/2} \cdot \frac{1.875}{7}, & \text{if } k \text{ is even} \end{cases} \tag{7.9.38}$$

and

$$h_2^k = \begin{cases} \frac{88}{7} + (\frac{1}{8})^{(k-3)/2} \cdot \frac{18.75}{7}, & \text{if } k \text{ is odd} \\ \frac{88}{7} + \frac{1}{2} \cdot (\frac{1}{8})^{(k-4)/2} \cdot \frac{21.75}{7}, & \text{if } k \text{ is even} \end{cases}. \tag{7.9.39}$$

We will prove this by induction on k, for $k \geq 3$. Start with $k = 3$. Then, it follows by (7.9.36), (7.9.37), (7.9.38) and (7.9.39) that

$$l_1^3 = 14.875,\ h_1^3 = 17.25,\ l_2^3 = 11.625 \text{ and } h_2^3 = 15.25,$$

which matches (7.9.32) and (7.9.35).

Assume next that $k \geq 4$, and that (7.9.36), (7.9.37), (7.9.38) and (7.9.39) hold for $k-1$. We distinguish two cases: (1) k is even, and (2) k is odd.

Case 1. Suppose that k is even. Firm 1 is required to hold a belief β_1 about firm 2's choice-view pairs that (i) assigns probability 0.5 to the views v_2^n and v_2^s, (ii) for the view v_2^n only assigns positive probability to price-view pairs (p_2, v_2^n) where $p_2 \in P_2^{k-1}(v_2^n)$, and (iii) for the view v_2^s only assigns positive probability to the price-view pair $(28, v_2^s)$. But then, in view of (7.9.38) and (7.9.39) for $k-1$, the lowest expected price $E_{\beta_1}(p_2)$ for firm 2 under such a belief β_1 is

$$(0.5) \cdot (\tfrac{88}{7} - (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{6.625}{7}) + (0.5) \cdot 28 = \tfrac{142}{7} - \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{6.625}{7},$$

whereas the highest expected price $E_{\beta_1}(p_2)$ for firm 2 under such a belief β_1 is

$$(0.5) \cdot (\tfrac{88}{7} + (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{18.75}{7}) + (0.5) \cdot 28 = \tfrac{142}{7} + \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{18.75}{7}.$$

Hence, the expected price $E_{\beta_1}(p_2)$ under such a belief β_1 lies in the interval

$$[\tfrac{142}{7} - \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{6.625}{7},\ \tfrac{142}{7} + \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{18.75}{7}]. \tag{7.9.40}$$

Moreover, the lowest expected price in this interval is at least

$$\tfrac{142}{7} - \tfrac{1}{2} \cdot \tfrac{6.625}{7} > 16.$$

By (7.9.9), the optimal product-price pair for firm 1 under such a belief β_1 is $(n, 5 + \frac{1}{2}E_{\beta_1}(p_2))$. As $E_{\beta_1}(p_2)$ lies in the interval given by (7.9.40), the optimal price $5 + \frac{1}{2}E_{\beta_1}(p_2)$ lies in the interval

$$\begin{aligned} & [5 + \tfrac{1}{2} \cdot (\tfrac{142}{7} - \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{6.625}{7}),\ 5 + \tfrac{1}{2} \cdot (\tfrac{142}{7} + \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{18.75}{7})] \\ = & [\tfrac{106}{7} - \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{6.625}{7},\ \tfrac{106}{7} + \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{18.75}{7}]. \end{aligned}$$

Hence, the set of product-price pairs for firm 1 that are optimal at the view v_1^n for such a belief β_1 is

$$P_1^k(v_1^n) = \{(n, p_1) \mid p_1 \in [\tfrac{106}{7} - \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{6.625}{7},\ \tfrac{106}{7} + \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{18.75}{7}]\},$$

which matches (7.9.36) and (7.9.37).

We now turn to firm 2. By definition, firm 2 is required to hold a belief β_2 about firm 1's choice-view pairs that (i) assigns probability 1 to the view v_1^n, and (ii) for the view v_1^n only assigns positive probability to choice-view pairs (c_1, v_1^n) where $c_1 \in P_1^{k-1}(v_1^n)$. By (7.9.36) and (7.9.37) for $k-1$ we conclude that β_2 only assigns positive probability to pairs (n, p_1) with $p_1 \in [l_1^{k-1}, h_1^{k-1}]$.

Together with (i) and (iii) we see that

$$\beta_2(s) = 0,\ \beta_2(n) = 1, \text{ and } E_{\beta_2}(p_1|n) \in [l_1^{k-1}, h_1^{k-1}]. \tag{7.9.41}$$

If we substitute this into (7.9.14), we see that the optimal price $p_2(\beta_2)$ under this belief β_2 is

$$p_2(\beta_2) = 5 + \tfrac{1}{2}E_{\beta_2}(p_1|n). \tag{7.9.42}$$

In view of (7.9.41) and (7.9.42), the lowest price $p_2(\beta_2)$ that is optimal for such a belief is

$$\begin{aligned}5 + \tfrac{1}{2} \cdot l_1^{k-1} &= 5 + \tfrac{1}{2} \cdot (\tfrac{106}{7} - (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{1.875}{7}) \\ &= \tfrac{88}{7} - \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{1.875}{7},\end{aligned}$$

whereas the highest price $p_2(\beta_2)$ that is optimal for such a belief is

$$\begin{aligned}5 + \tfrac{1}{2} \cdot h_1^{k-1} &= 5 + \tfrac{1}{2} \cdot (\tfrac{106}{7} + (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{21.75}{7}) \\ &= \tfrac{88}{7} + \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{21.75}{7}.\end{aligned}$$

Hence, the set of prices for firm 2 that is optimal for such a belief β_2 at the view v_2^n is

$$P_2^k(v_2^n) = [\tfrac{88}{7} - \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{1.875}{7},\ \tfrac{88}{7} + \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-4)/2} \cdot \tfrac{21.75}{7}],$$

which matches (7.9.38) and (7.9.39).

Case 2. Suppose that k is odd. Firm 1 is required to hold a belief β_1 about firm 2's choice-view pairs that (i) assigns probability 0.5 to the views v_2^n and v_2^s, (ii) for the view v_2^n only assigns positive probability to price-view pairs (p_2, v_2^n) where $p_2 \in P_2^{k-1}(v_2^n)$, and (iii) for the view v_2^s only assigns positive probability to the price-view pair $(28, v_2^s)$. But then, in view of (7.9.38) and (7.9.39) for $k-1$, the lowest expected price $E_{\beta_1}(p_2)$ for firm 2 under such a belief β_1 is

$$(0.5) \cdot (\tfrac{88}{7} - \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{1.875}{7}) + (0.5) \cdot 28 = \tfrac{142}{7} - \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{1.875}{7},$$

whereas the highest expected price $E_{\beta_1}(p_2)$ for firm 2 under such a belief β_1 is

$$(0.5) \cdot (\tfrac{88}{7} + \tfrac{1}{2} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{21.75}{7}) + (0.5) \cdot 28 = \tfrac{142}{7} + \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{21.75}{7}.$$

Hence, the expected price $E_{\beta_1}(p_2)$ under such a belief β_1 lies in the interval

$$[\tfrac{142}{7} - \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{1.875}{7},\ \tfrac{142}{7} + \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{21.75}{7}]. \tag{7.9.43}$$

Moreover, the lowest expected price in this interval is at least

$$\tfrac{142}{7} - \tfrac{1}{4} \cdot \tfrac{1.875}{7} > 16.$$

By (7.9.9), the optimal product-price pair for firm 1 under such a belief β_1 is $(n, 5 + \frac{1}{2}E_{\beta_1}(p_2))$. As $E_{\beta_1}(p_2)$ lies in the interval given by (7.9.43), the optimal price $5 + \frac{1}{2}E_{\beta_1}(p_2)$ lies in the interval

$$\begin{aligned}&[5 + \tfrac{1}{2} \cdot (\tfrac{142}{7} - \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{1.875}{7}),\ 5 + \tfrac{1}{2} \cdot (\tfrac{142}{7} + \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{21.75}{7})] \\ = \ &[\tfrac{106}{7} - \tfrac{1}{8} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{1.875}{7},\ \tfrac{106}{7} + \tfrac{1}{8} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{21.75}{7}] \\ = \ &[\tfrac{106}{7} - (\tfrac{1}{8})^{(k-3)/2} \cdot \tfrac{1.875}{7},\ \tfrac{106}{7} + (\tfrac{1}{8})^{(k-3)/2} \cdot \tfrac{21.75}{7}].\end{aligned}$$

Hence, the set of product-price pairs for firm 1 that are optimal at the view v_1^n for such a belief β_1 is

$$P_1^k(v_1^n) = \{(n, p_1) \mid p_1 \in [\tfrac{106}{7} - (\tfrac{1}{8})^{(k-3)/2} \cdot \tfrac{1.875}{7},\ \tfrac{106}{7} + (\tfrac{1}{8})^{(k-3)/2} \cdot \tfrac{21.75}{7}]\},$$

which matches (7.9.36) and (7.9.37).

We now turn to firm 2. By definition, firm 2 is required to hold a belief β_2 about firm 1's choice-view pairs that (i) assigns probability 1 to the view v_1^n, and (ii) for the view v_1^n only assigns positive probability to choice-view pairs (c_1, v_1^n) where $c_1 \in P_1^{k-1}(v_1^n)$. By (7.9.36) and (7.9.37) for $k-1$ we conclude that β_2 only assigns positive probability to pairs (n, p_1) with $p_1 \in [l_1^{k-1}, h_1^{k-1}]$.

Together with (i) and (ii) we see that

$$\beta_2(s) = 0,\ \beta_2(n) = 1,\ \text{and } E_{\beta_2}(p_1|n) \in [l_1^{k-1}, h_1^{k-1}]. \tag{7.9.44}$$

If we substitute this into (7.9.14), we see that the optimal price $p_2(\beta_2)$ under this belief β_2 is

$$p_2(\beta_2) = 5 + \tfrac{1}{2} E_{\beta_2}(p_1|n). \tag{7.9.45}$$

In view of (7.9.44) and (7.9.45), the lowest price $p_2(\beta_2)$ that is optimal for such a belief is

$$\begin{aligned} 5 + \tfrac{1}{2} \cdot l_1^{k-1} &= 5 + \tfrac{1}{2} \cdot (\tfrac{106}{7} - \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{6.625}{7}) \\ &= \tfrac{88}{7} - \tfrac{1}{8} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{6.625}{7} \\ &= \tfrac{88}{7} - (\tfrac{1}{8})^{(k-3)/2} \cdot \tfrac{6.625}{7}, \end{aligned}$$

whereas the highest price $p_2(\beta_2)$ that is optimal for such a belief is

$$\begin{aligned} 5 + \tfrac{1}{2} \cdot h_1^{k-1} &= 5 + \tfrac{1}{2} \cdot (\tfrac{106}{7} + \tfrac{1}{4} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{18.75}{7}) \\ &= \tfrac{88}{7} + \tfrac{1}{8} \cdot (\tfrac{1}{8})^{(k-5)/2} \cdot \tfrac{18.75}{7} \\ &= \tfrac{88}{7} + (\tfrac{1}{8})^{(k-3)/2} \cdot \tfrac{18.75}{7}. \end{aligned}$$

Hence, the set of prices for firm 2 that is optimal for such a belief β_2 at the view v_2^n is

$$P_2^k(v_2^n) = [\tfrac{88}{7} - (\tfrac{1}{8})^{(k-3)/2} \cdot \tfrac{6.625}{7},\ \tfrac{88}{7} + (\tfrac{1}{8})^{(k-3)/2} \cdot \tfrac{18.75}{7}],$$

which matches (7.9.38) and (7.9.39).

By induction on k we thus conclude that (7.9.36), (7.9.37), (7.9.38) and (7.9.39) hold for every $k \geq 3$. In particular, when k tends to infinity, the sets $P_1^k(v_1^n)$ and $P_2^k(v_2^n)$ collapse to the single choices

$$p_1^*(v_1^n) = (n,\ \tfrac{106}{7}) \approx (n,\ 15.14) \text{ and } p_2^*(v_2^n) = \tfrac{88}{7} \approx 12.57.$$

Hence, under common belief in rationality with the view v_1^n we expect firm 1 to offer the new product at the price 15.14. Moreover, under common belief in rationality with the view v_2^n we expect firm 2 to choose the price 12.57.

The asymmetry in prices can be explained as follows: Firm 1, with the view v_1^n, believes that with probability 0.5 firm 2 is not aware of the new product. In this case, firm 2 would believe to compete with the more different standard good, which leads firm 2 to choose the higher price 28. As firm 1 believes this to happen with probability 0.5, firm 1 will also choose a relatively high price itself.

Firm 2, in contrast, believes with the view v_2^n that firm 1 will definitely have the view v_1^n, and hence firm 2 will believe that firm 1 will definitely offer the new good. As the new good is more similar to firm 2's good than the standard good, firm 2 will choose a relatively low price.

7.9.2 Competition in Quantities

Consider a Cournot competition model between two firms as discussed in Section 3.7.2. Assume that both firms have a constant marginal cost equal to 5. Moreover, if both firms choose the quantities q_1 and q_2, the market price for the good is given by

$$p = 20 - q_1 - q_2.$$

Under the standard production technology, both firms are able to produce at most 4 units of the good. However, there is a new production technology which has only recently been developed, and which would allow the firm to produce up to 10 units of the good, at the same marginal costs as before.

Firm 1 has recently incorporated the new production technology, but is uncertain whether firm 2 is aware of this new technology or not. What quantities can firm 1 then rationally choose under common belief in rationality?

To answer this question we first model the scenario above as a game with unawareness. The possible views for firm 1 are v_1^n and v_1^s, where v_1^n denotes its actual view by which it is aware of the new technology, whereas v_1^s denotes the smaller view where it is not aware of the new technology. Similarly, we can define the possible views v_2^n and v_2^s for firm 2.

Under the view v_1^n the set of choices for firm 1 is $C_1(v_1^n) = [0, 10]$, because it can produce any amount between 0 and 10. Moreover, the set of states under this view is $C_2(v_1^n) = [0, 10]$, as firm 1 is allowed to believe that firm 2 is aware of the new production technology, and that firm 2 is actually using this new technology.

Under the view v_1^s firm 1's set of choices is $C_1(v_1^s) = [0, 4]$, since firm 1 can only produce up to 4 units with the standard technology. Moreover, the set of states is $C_2(v_1^s) = [0, 4]$, as firm 1 must believe that firm 2 is only aware of the standard technology, which allows firm 2 to produce at most 4 units. Similarly for the view of firm 2.

To see which amounts firm 1 can rationally produce under common belief in rationality we use the extension of *iterated strict dominance with unawareness* to games with infinitely many choices and states, like we did in the previous subsection. As before, we will use the *bottom-up* version of this procedure. We will thus start with the views of rank 1.

Views of rank 1. Consider the views of rank 1, which are v_1^s and v_2^s. This gives rise to a Cournot competition model in Section 3.7.2 where $a = 20$, $c = 5$, $e = 1$ and $M = 4$. Note, however, that the condition $M \in [\frac{a-c}{2e}, \frac{a-c}{e}]$ is violated since $M < \frac{a-c}{2e}$. Hence, we cannot use the analysis in Section 3.7.2 to obtain the quantities that both firms can rationally choose under common belief in rationality. We have to apply the procedure round by round to this scenario.

Round 1. Consider firm 1 with view v_1^s. Then, firm 1 must believe that firm 2's view is v_2^s. By (3.7.11) it follows that firm 1's profit under the belief β_1 about firm 2's quantity is

$$\pi_1(q_1, \beta_1) = q_1 \cdot (a - c - e \cdot (q_1 + E_{\beta_1}(q_2)) = q_1 \cdot (15 - (q_1 + E_{\beta_1}(q_2)).$$

The derivative of this profit with respect to q_1 is

$$\frac{\partial \pi_1}{\partial q_1} = 15 - 2q_1 - E_{\beta_1}(q_2) > 0$$

since $q_1 \leq 4$ and $E_{\beta_1}(q_2) \leq 4$. That is, firm 1's profit is always increasing in its output q_1. The unique optimal quantity is therefore the maximal possible quantity, which is $q_1 = 4$. Thus, the set of quantities for firm 1 that survive Round 1 at the view v_1^s is

$$Q_1^1(v_1^s) = \{4\}.$$

Similarly,

$$Q_2^1(v_2^s) = \{4\}.$$

Then, the procedure at the views of rank 1 terminates. Thus, under common belief in rationality, both firms will rationally choose the quantity

$$q_1^*(v_1^s) = q_2^*(v_2^s) = 4 \tag{7.9.46}$$

at the views v_1^s and v_2^s, respectively.

Views of rank 2. Consider next the views of rank 2, which are v_1^n and v_2^n.

Round 1. Focus on firm 1 with view v_1^n. Suppose that firm 1 holds a belief β_1 about firm 2's quantity. As firm 1 may believe that firm 2 holds the view v_2^n, the belief β_1 may attach positive probability to all possible prices in $[0, 10]$, and hence $E_{\beta_1}(p_2) \in [0, 10]$. By (3.7.12) we know that firm 1's optimal quantity under this belief β_1 is

$$q_1(\beta_1) = \tfrac{a-c}{2e} - \tfrac{1}{2}E_{\beta_1}(p_2) = 7.5 - \tfrac{1}{2}E_{\beta_1}(p_2). \tag{7.9.47}$$

As $E_{\beta_1}(p_2) \in [0, 10]$, the quantities that are optimal for some belief β_1 are given by

$$Q_1^1(v_1^n) = [7.5 - \tfrac{1}{2} \cdot 10,\ 7.5 - \tfrac{1}{2} \cdot 0] = [2.5,\ 7.5]. \tag{7.9.48}$$

Similarly for firm 2.

Round 2. By definition, the set of states $S_1^2(v_1^n)$ for firm 1 at the view v_1^n are those quantities that survived so far for firm 2 at the views which are contained in v_1^n, which are the views v_2^s and v_2^n. By (7.9.46) and (7.9.48) we know that quantity 4 survived at the view v_2^s, whereas all quantities in [2.5, 7.5] survived Round 1 at the view v_2^n. Hence,

$$S_1^2(v_1^n) = [2.5,\ 7.5].$$

Since firm 1 is required to hold a belief β_1 on $S_1^2(v_1^n)$, we know that $E_{\beta_1}(p_2) \in$ [2.5, 7.5]. By (7.9.47), the set of quantities that are optimal for firm 1 for such a belief β_1 is given by

$$Q_1^2(v_1^n) = [7.5 - \tfrac{1}{2} \cdot (7.5),\ 7.5 - \tfrac{1}{2} \cdot (2.5)] = [3.75,\ 6.25]. \tag{7.9.49}$$

Similarly for firm 2.

Round 3. By definition, the set of states $S_1^3(v_1^n)$ for firm 1 at the view v_1^n are those quantities that survived so far for firm 2 at the views v_2^s and v_2^n. By (7.9.46) and (7.9.49) we know that quantity 4 survived at the view v_2^s, whereas all quantities in [3.75, 6.25] survived Round 2 at the view v_2^n. Hence,

$$S_1^3(v_1^n) = [3.75,\ 6.25].$$

Since firm 1 is required to hold a belief β_1 on $S_1^3(v_1^n)$, we know that $E_{\beta_1}(p_2) \in$ [3.75, 6.25]. By (7.9.47), the set of quantities that are optimal for firm 1 for such a belief β_1 is given by

$$Q_1^3(v_1^n) = [7.5 - \tfrac{1}{2} \cdot (6.25),\ 7.5 - \tfrac{1}{2} \cdot (3.75)] = [4.375,\ 5.625]. \tag{7.9.50}$$

Similarly for firm 2.

Round 4. By definition, the set of states $S_1^4(v_1^n)$ for firm 1 at the view v_1^n are those quantities that survived so far for firm 2 at the views v_2^s and v_2^n. By (7.9.46) and (7.9.50) we know that quantity 4 survived at the view v_2^s, whereas all quantities in [4.375, 5.625] survived Round 3 at the view v_2^n. Hence,

$$S_1^4(v_1^n) = \{4\} \cup [4.375,\ 5.625].$$

Since firm 1 is required to hold a belief β_1 on $S_1^4(v_1^n)$, we know that $E_{\beta_1}(p_2) \in$ [4, 5.625]. Indeed, every expected price between 4 and 4.375 can be obtained by a belief β_1 that assigns positive probability to the quantities 4 and 4.375. By (7.9.47), the set of quantities that are optimal for firm 1 for such a belief β_1 is given by

$$Q_1^4(v_1^n) = [7.5 - \tfrac{1}{2} \cdot (5.625),\ 7.5 - \tfrac{1}{2} \cdot 4] = [4.6875,\ 5.5]. \tag{7.9.51}$$

Similarly for firm 2.

Round 5. By definition, the set of states $S_1^5(v_1^n)$ for firm 1 at the view v_1^n are those quantities that survived so far for firm 2 at the views v_2^s and v_2^n. By (7.9.46) and (7.9.51) we know that quantity 4 survived at the view v_2^s, whereas all quantities in [4.6875, 5.5] survived Round 4 at the view v_2^n. Hence,

$$S_1^5(v_1^n) = \{4\} \cup [4.6875,\ 5.5].$$

Since firm 1 is required to hold a belief β_1 on $S_1^5(v_1^n)$, we know that $E_{\beta_1}(p_2) \in [4,\ 5.5]$. Indeed, every expected price between 4 and 4.6875 can be obtained by a belief β_1 that assigns positive probability to the quantities 4 and 4.6875. By (7.9.47), the set of quantities that are optimal for firm 1 for such a belief β_1 is given by

$$Q_1^5(v_1^n) = [7.5 - \tfrac{1}{2} \cdot (5.5),\ 7.5 - \tfrac{1}{2} \cdot 4] = [4.75,\ 5.5]. \tag{7.9.52}$$

Similarly for firm 2.

Round 6. By definition, the set of states $S_1^6(v_1^n)$ for firm 1 at the view v_1^n are those quantities that survived so far for firm 2 at the views v_2^s and v_2^n. By (7.9.46) and (7.9.52) we know that quantity 4 survived at the view v_2^s, whereas all quantities in [4.75, 5.5] survived Round 5 at the view v_2^n. Hence,

$$S_1^6(v_1^n) = \{4\} \cup [4.75, 5.5].$$

Since firm 1 is required to hold a belief β_1 on $S_1^5(v_1^n)$, we know that $E_{\beta_1}(p_2) \in [4,\ 5.5]$. Indeed, every expected price between 4 and 4.75 can be obtained by a belief β_1 that assigns positive probability to the quantities 4 and 4.75. By (7.9.47), the set of quantities that are optimal for firm 1 for such a belief β_1 is given by

$$Q_1^6(v_1^n) = [7.5 - \tfrac{1}{2} \cdot (5.5),\ 7.5 - \tfrac{1}{2} \cdot 4] = [4.75,\ 5.5]. \tag{7.9.53}$$

Similarly for firm 2.

As $Q_1^6(v_1^n) = Q_1^5(v_1^n)$ and $Q_2^6(v_2^n) = Q_2^5(v_2^n)$, we conclude that the procedure for the views of rank 2 terminates here. As such, the sets of quantities that firms 1 and 2 can rationally choose under common belief in rationality at the views v_1^n and v_2^n are given by

$$Q_1^*(v_1^n) = Q_2^*(v_2^n) = [4.75,\ 5.5]. \tag{7.9.54}$$

We finally turn to a scenario with *fixed beliefs on views.* Assume that the fixed belief combination $p = (p_1, p_2)$ on views is given by Figure 7.9.2. Hence, if a firm is aware of the new production technology, then it believes that with probability 0.5 the other firm will also be aware of it, and with probability 0.5 the other firm is not aware of it. Suppose that firm 1 is aware of the new technology. What quantities can it then rationally choose under common belief in rationality and common belief in the fixed belief combination p about views?

We will use the extension of *iterated strict dominance for unawareness with fixed beliefs on views* to games with infinitely many choices and states. As for the case of price competition, we will use the *bottom-up* version of this procedure. Hence, we will start with the views of rank 1.

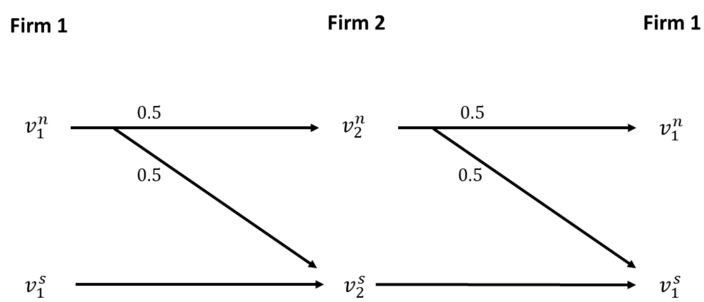

Figure 7.9.2 Fixed beliefs about views for quantity competition

Views of rank 1. Consider the views of rank 1, which are v_1^s and v_2^s. For these views the analysis is exactly the same as for the case without fixed beliefs about views. Hence, the only quantities that survive the procedure at these views are

$$q_1^*(v_1^s) = q_2^*(v_2^s) = 4. \tag{7.9.55}$$

Views of rank 2. Consider next the views of rank 2, which are v_1^n and v_2^n.

Round 1. This round is exactly the same as for the case without fixed beliefs about views, and leads to the set of quantities

$$Q_1^1(v_1^n) = [2.5,\ 7.5] \tag{7.9.56}$$

for firm 1. Similarly for firm 2.

Round 2. Firm 1 is required to hold a belief β_1 about firm 2's quantity-view pairs that (i) assigns probability 0.5 to the views v_2^n and v_2^s, (ii) for the view v_2^n only assigns positive probability to quantity-view pairs (q_2, v_2^n) where $q_2 \in Q_2^1(v_2^n)$, and (iii) for the view v_2^s only assigns positive probability to the quantity-view pair $(4, v_2^s)$. In view of (7.9.56), the belief β_1 will then assign probability 0.5 to the quantity 4, and probability 0.5 to some quantity in [2.5, 7.5]. Hence, the expected quantity $E_{\beta_1}(q_2)$ will be somewhere in the interval

$$[\tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot 2.5,\ \tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot 7.5] = [3.25,\ 5.75].$$

By (7.9.47) we know that the optimal quantity $q_1(\beta_1)$ for such a belief is given by $7.5 - \frac{1}{2} \cdot E_{\beta_1}(q_2)$. As $E_{\beta_1}(q_2) \in [3.25, 5.75]$, the optimal quantity for such a belief will be in the interval

$$[7.5 - \tfrac{1}{2} \cdot (5.75),\ 7.5 - \tfrac{1}{2} \cdot (3.25)] = [4.625,\ 5.875].$$

Hence, the set of quantities that survive Round 2 for firm 1 at view v_1^n is

$$Q_1^2(v_1^n) = [4.625,\ 5.875]. \tag{7.9.57}$$

Similarly for firm 2.

By continuing in this fashion we can compute the sets $Q_1^k(v_1^n)$ and $Q_2^k(v_2^n)$ for all rounds $k \geq 3$ as well. It can be shown, for every round $k \geq 2$, that

$$Q_1^k(v_1^n) = Q_2^k(v_2^n) = [l^k, h^k],$$

where

$$l^k = \begin{cases} 5.2 - (\frac{1}{4})^{k-1} \cdot (2.3), & \text{if } k \text{ is even} \\ 5.2 - (\frac{1}{4})^{k-1} \cdot (2.7), & \text{if } k \text{ is odd} \end{cases} \tag{7.9.58}$$

and

$$h^k = \begin{cases} 5.2 + (\frac{1}{4})^{k-1} \cdot (2.7), & \text{if } k \text{ is even} \\ 5.2 + (\frac{1}{4})^{k-1} \cdot (2.3), & \text{if } k \text{ is odd} \end{cases}. \tag{7.9.59}$$

We prove this by induction on k, for $k \geq 2$.

If $k = 2$, then we know from (7.9.57) that $l^2 = 4.625$ and $h^2 = 5.875$, which matches (7.9.58) and (7.9.59).

Take now some $k \geq 3$, and assume that (7.9.58) and (7.9.59) hold for $k-1$. We distinguish two cases: (1) k is odd, and (2) k is even.

Case 1. Suppose that k is odd. Firm 1 is required to hold a belief β_1 about firm 2's quantity-view pairs that (i) assigns probability 0.5 to the views v_2^n and v_2^s, (ii) for the view v_2^n only assigns positive probability to quantity-view pairs (q_2, v_2^n) where $q_2 \in Q_2^{k-1}(v_2^n)$, and (iii) for the view v_2^s only assigns positive probability to the quantity-view pair $(4, v_2^s)$. In view of (7.9.58) and (7.9.59), the belief β_1 will then assign probability 0.5 to the quantity 4, and probability 0.5 to some quantity in $[l^{k-1}, h^{k-1}]$. Hence, the expected quantity $E_{\beta_1}(q_2)$ will be somewhere in the interval

$$\begin{aligned} & [\tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot l^{k-1},\ \tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot h^{k-1}] \\ = \ & [\tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot (5.2 - (\tfrac{1}{4})^{k-2} \cdot (2.3)),\ \tfrac{1}{2} \cdot 4 + \tfrac{1}{2} \cdot (5.2 + (\tfrac{1}{4})^{k-2} \cdot (2.7))] \\ = \ & [4.6 - \tfrac{1}{2} \cdot (\tfrac{1}{4})^{k-2} \cdot (2.3),\ 4.6 + \tfrac{1}{2} \cdot (\tfrac{1}{4})^{k-2} \cdot (2.7)] \end{aligned} \tag{7.9.60}$$

By (7.9.47) we know that the optimal quantity $q_1(\beta_1)$ for such a belief is given by $7.5 - \frac{1}{2} \cdot E_{\beta_1}(q_2)$. As $E_{\beta_1}(q_2)$ is in the interval given by (7.9.60), the lowest quantity $q_1(\beta_1)$ that is optimal for such a belief β_1 is

$$\begin{aligned} q_1(\beta_1) &= 7.5 - \tfrac{1}{2} \cdot (4.6 + \tfrac{1}{2} \cdot (\tfrac{1}{4})^{k-2} \cdot (2.7)) \\ &= 5.2 - \tfrac{1}{4} \cdot (\tfrac{1}{4})^{k-2} \cdot (2.7) = 5.2 - (\tfrac{1}{4})^{k-1} \cdot (2.7) = l^k. \end{aligned}$$

Similarly, the highest quantity $q_1(\beta_1)$ that is optimal for such a belief β_1 is

$$\begin{aligned} q_1(\beta_1) &= 7.5 - \tfrac{1}{2} \cdot (4.6 - \tfrac{1}{2} \cdot (\tfrac{1}{4})^{k-2} \cdot (2.3)) \\ &= 5.2 + \tfrac{1}{4} \cdot (\tfrac{1}{4})^{k-2} \cdot (2.3) = 5.2 + (\tfrac{1}{4})^{k-1} \cdot (2.3) = h^k. \end{aligned}$$

Hence, the set of quantities that survives Round k for firm 1 at view v_1^n is

$$Q_1^k(v_1^n) = [l^k,\ h^k]$$

as given by (7.9.58) and (7.9.59). Similarly for firm 2.

Case 2. Suppose that k is even. Firm 1 is required to hold a belief β_1 about firm 2's quantity-view pairs that (i) assigns probability 0.5 to the views v_2^n and v_2^s, (ii) for the view v_2^n only assigns positive probability to quantity-view pairs (q_2, v_2^n) where $q_2 \in Q_2^{k-1}(v_2^n)$, and (iii) for the view v_2^s only assigns positive probability to the quantity-view pair $(4, v_2^s)$. In view of (7.9.58) and (7.9.59), the belief β_1

will then assign probability 0.5 to the quantity 4, and probability 0.5 to some quantity in $[l^{k-1}, h^{k-1}]$. Hence, the expected quantity $E_{\beta_1}(q_2)$ will be somewhere in the interval

$$\begin{aligned} & [\tfrac{1}{2}\cdot 4+\tfrac{1}{2}\cdot l^{k-1},\ \tfrac{1}{2}\cdot 4+\tfrac{1}{2}\cdot h^{k-1}] \\ = & [\tfrac{1}{2}\cdot 4+\tfrac{1}{2}\cdot (5.2-(\tfrac{1}{4})^{k-2}\cdot (2.7)),\ \tfrac{1}{2}\cdot 4+\tfrac{1}{2}\cdot (5.2+(\tfrac{1}{4})^{k-2}\cdot (2.3))] \\ = & [4.6-\tfrac{1}{2}\cdot (\tfrac{1}{4})^{k-2}\cdot (2.7),\ 4.6+\tfrac{1}{2}\cdot (\tfrac{1}{4})^{k-2}\cdot (2.3)]. \end{aligned} \tag{7.9.61}$$

By (7.9.47) we know that the optimal quantity $q_1(\beta_1)$ for such a belief is given by $7.5-\frac{1}{2}\cdot E_{\beta_1}(q_2)$. As $E_{\beta_1}(q_2)$ is in the interval given by (7.9.61), the lowest quantity $q_1(\beta_1)$ that is optimal for such a belief β_1 is

$$\begin{aligned} q_1(\beta_1) &= 7.5-\tfrac{1}{2}\cdot (4.6+\tfrac{1}{2}\cdot (\tfrac{1}{4})^{k-2}\cdot (2.3)) \\ &= 5.2-\tfrac{1}{4}\cdot (\tfrac{1}{4})^{k-2}\cdot (2.3) = 5.2-(\tfrac{1}{4})^{k-1}\cdot (2.3) = l^k. \end{aligned}$$

Similarly, the highest quantity $q_1(\beta_1)$ that is optimal for such a belief β_1 is

$$\begin{aligned} q_1(\beta_1) &= 7.5-\tfrac{1}{2}\cdot (4.6-\tfrac{1}{2}\cdot (\tfrac{1}{4})^{k-2}\cdot (2.7)) \\ &= 5.2+\tfrac{1}{4}\cdot (\tfrac{1}{4})^{k-2}\cdot (2.7) = 5.2+(\tfrac{1}{4})^{k-1}\cdot (2.7) = h^k. \end{aligned}$$

Hence, the set of quantities that survive Round k for firm 1 at view v_1^n is

$$Q_1^k(v_1^n) = [l^k,\ h^k]$$

as given by (7.9.58) and (7.9.59). Similarly for firm 2.

By induction on k, we conclude that (7.9.58) and (7.9.59) hold for every $k \geq 2$. In particular, when k tends to infinity, the sets of quantities $Q_1^k(v_1^n)$ and $Q_2^k(v_2^n)$ collapse to the single quantity

$$q_1^*(v_1^n) = q_2^*(v_2^n) = 5.2.$$

Hence, under common belief in rationality and common belief in the fixed belief combination p on views, we expect both firms to choose the quantity 5.2 at the views v_1^n and v_2^n, respectively.

Compare this to the situation where both firms are aware of the new technology and both firms would believe, with probability 1, that the other firm is also aware of the new technology. In this case, we would be back to a standard Cournot competition model in Section 3.7.2 where $a = 20$, $c = 5$, $e = 1$ and $M = 10$. Since $M \in [\frac{a-c}{2e}, \frac{a-c}{e}]$, we know by (3.7.14) that under common belief in rationality, both firms would rationally choose the quantity

$$q = \tfrac{a-c}{3e} = 5.$$

Above we have seen that, if both firms believe there is a 50% chance that the other firm is unaware of the new technology, then both firms would opt for a higher quantity, which is 5.2. The intuition is clear: In the latter case, the firm believes there is a 50% chance that the other firm will choose the low quantity 4 because it is only aware of the standard technology. The firm that is aware of the new technology will then respond by choosing a quantity higher than 5.

Chapter 8

Common Belief in Rationality in Psychological Games

8.8 Economic Applications

In this section we take as a starting point the Bertrand competition model and the Cournot competition model from Section 3.7. In both models we assume that firm i receives a mental *bonus* if it chooses a price or quantity that is *more cooperative* than the price or quantity that firm j expects firm i to choose. This turns both scenarios into a *psychological game,* as the utility for firm i depends on what firm i believes that firm j believes that i will do. For both scenarios we investigate which prices or quantities the firms can rationally choose under common belief in rationality.

8.8.1 Competition in Prices

Consider the Bertrand competition model from Section 3.7.1. As we did for one of the views in Section 7.9.1, we choose the parameters $a = 24$, $c = 4$, $d = 1$, $e = 1$ and $M = 40$. By following the arguments in Section 3.7.1 we then know that firm 1's profit is given by

$$\pi_1(p_1, p_2) = (p_1 - 4) \cdot (24 - p_1 + p_2) \tag{8.8.1}$$

if the firms choose the prices p_1 and p_2.

Suppose now that firm 1 not only cares about its profit, but that, in addition, it receives a mental bonus if it chooses a price which is higher – and hence more cooperative – than the price that firm 2 expects firm 1 to choose. More precisely, if firm 1 chooses a price p_1, and believes that firm 2 expects firm 1 to choose a price of p_1' with $p_1 \geq p_1'$, then firm 1 receives the mental bonus

$$f \cdot (p_1 - p_1')^2. \tag{8.8.2}$$

Here, f is a parameter that measures how strongly firm 1 wishes to exceed firm 2's expectations by the price it chooses. We choose f such that $f \leq 0.15$.

Assume that firm 1's utility is the sum of its profit and (possibly) the mental bonus from exceeding firm 2's expectations. Then, in view of (8.8.1) and (8.8.2), firm 1's utility function is given by

$$u_1(p_1,(p_2,p_1')) = (p_1-4)\cdot(24-p_1+p_2) + \begin{cases} f\cdot(p_1-p_1')^2, & \text{if } p_1 \geq p_1' \\ 0, & \text{if } p_1 < p_1' \end{cases}, \qquad (8.8.3)$$

where p_1' is the price that firm 1 believes that firm 2 believes that firm 1 will choose. Similarly for firm 2. We thus obtain a psychological game where the choices for firm 1 are the prices p_1, the states are the pairs (p_2,p_1'), and similarly for firm 2.

What prices can both firms rationally choose under common belief in rationality? To answer that question we apply the *states-first procedure* from Section 8.5 where we first perform the *iterated elimination of choices and states* and subsequently apply the *iterated elimination of choices and second-order expectations.*

Let us start with the iterated elimination of choices and states. Note that the psychological game at hand is *infinite,* because we have infinitely many choices and states for both firms. However, the iterated elimination of choices and states can be extended to such infinite psychological games as follows: In round 1 we start by eliminating, for every decision problem, those choices that are not optimal for any second-order expectation. At the beginning of round 2 we eliminate, for every decision problem, all states that contain a choice that has already been eliminated in round 1. Within the reduced decision problem so obtained, we then eliminate all choices that are not optimal for any second-order expectation within the reduced decision problem. For further rounds we procede in a similar fashion. Let us now apply this iterated elimination of choices and states to our model above.

Round 1. Focus on firm 1. Which prices p_1 are optimal for some second-order expectation e_1 and which are not? Consider a second-order expectation e_1 which assigns positive probability to finitely many states (p_2,p_1'). By $E_{e_1}(p_2)$ we denote the expected price for firm 2 under e_1, whereas $e_1(p_1')$ denotes the probability that e_1 assigns to firm 1's price p_1'. By following the steps in Section 3.7.1 it can then be shown, based on (8.8.3), that the expected utility for firm 1 if it chooses the price p_1 under the second-order expectation e_1 is given by

$$u_1(p_1,e_1) = (p_1-4)\cdot(24-p_1+E_{e_1}(p_2)) + f\cdot \sum_{p_1'\leq p_1: e_1(p_1')>0} e_1(p_1')\cdot(p_1-p_1')^2.$$

As a function of p_1, the expected utility $u_1(p_1,e_1)$ achieves its maximum at the unique point where the derivative $\frac{\partial u_1(p_1,e_1)}{\partial p_1}$ is equal to 0. It may be verified that

$$\frac{\partial u_1(p_1,e_1)}{\partial p_1} = 28 - 2p_1 + E_{e_1}(p_2) + 2f\cdot \sum_{p_1'\leq p_1: e_1(p_1')>0} e_1(p_1')\cdot(p_1-p_1').$$

Hence, $\frac{\partial u_1(p_1,e_1)}{\partial p_1} = 0$ precisely when

$$p_1 = 14 + \tfrac{1}{2}E_{e_1}(p_2) + f\cdot \sum_{p_1'\leq p_1: e_1(p_1')>0} e_1(p_1')\cdot(p_1-p_1'). \qquad (8.8.4)$$

Let the function on the right-hand side of (8.8.4) be denoted by $g(p_1,e_1)$. Hence, the optimal price $p_1(e_1)$ under the second-order expectation e_1 is the unique price p_1 for which $p_1 = g(p_1,e_1)$.

Note that the function $g(p_1,e_1)$ is increasing in p_1. Graphically, the unique price $p_1(e_1)$ with $p_1(e_1) = g(p_1(e_1),e_1)$ is the intersection point between the 45 degree line and the increasing curve

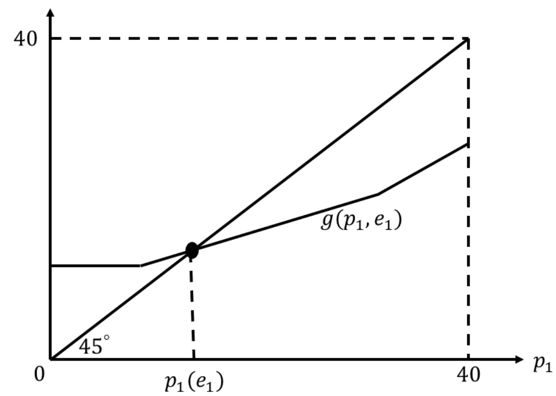

Figure 8.8.1 Optimal price for a second-order expectation

of $g(p_1, e_1)$ in Figure 8.8.1. Suppose now that the second-order expectation starts to assign higher probabilities to *higher* prices p_1'. Then, by (8.8.4), the curve of $g(p_1, e_1)$ in Figure 8.8.1 will shift downwards, which means that the optimal price $p_1(e_1)$ will become smaller. If, on the other hand, the second-order expectation starts to assign higher probabilities to *lower* prices p_1', then the curve of $g(p_1, e_1)$ in Figure 8.8.1 will shift upwards, which means that the optimal price $p_1(e_1)$ will become larger.

In view of (8.8.4), the lowest price p_1 that is optimal for a second-order expectation is thus obtained when e_1 assigns probability 1 to the lowest possible price $p_2 = 0$ for firm 2, and to the highest possible price $p_1' = 40$ for firm 1. In that case, $E_{e_1}(p_2) = 0$ and

$$\sum_{p_1' \leq p_1 : e_1(p_1') > 0} e_1(p_1') \cdot (p_1 - p_1') = 0$$

which means that

$$p_1 = 14. \tag{8.8.5}$$

At the same time, the highest price p_1 that is optimal for a second-order expectation is thus obtained when e_1 assigns probability 1 to the highest possible price $p_2 = 40$ for firm 2, and to the lowest possible price $p_1' = 0$ for firm 1. In that case, $E_{e_1}(p_2) = 40$ and

$$\sum_{p_1' \leq p_1 : e_1(p_1') > 0} e_1(p_1') \cdot (p_1 - p_1') = p_1 - 0 = p_1,$$

which implies that

$$p_1 = 14 + \tfrac{1}{2} \cdot 40 + f \cdot p_1.$$

Hence,

$$p_1 = \tfrac{34}{1-f}. \tag{8.8.6}$$

Note that $\frac{34}{1-f} \leq 40$ as we assume that $f \leq 0.15$.

On the basis of (8.8.5) and (8.8.6) we conclude that the set P_1^1 of prices that is optimal for firm 1 for some second-order expectation is given by

$$P_1^1 = [14, \tfrac{34}{1-f}]. \tag{8.8.7}$$

Similarly for firm 2.

Round 2. We focus on firm 1. In firm 1's decision problem we eliminate all states (p_2, p_1') where either p_2 or p_1' did not survive round 1. Hence, we concentrate on those second-order expectations e_1 which only assign positive probability to pairs (p_2, p_1') where $p_2 \in P_2^1$ and $p_1' \in P_1^1$. Let this set of second-order expectations be E_1^2.

By (8.8.4) and our arguments in round 1 we conclude that the lowest possible price p_1 that is optimal for some second-order expectation e_1 in E_1^2 is obtained by letting e_1 assign probability 1 to the lowest possible price $p_2 \in P_2^1$, which is $p_2 = 14$, and to the highest possible price $p_1' \in P_1^1$, which is $p_1' = \frac{34}{1-f}$. Hence, we have that $E_{e_1}(p_2) = 14$. By construction, every price p_1 that is optimal for this second-order expectation $e_1 \in E_1^2$ must be in P_1^1, which means that $p_1 \leq \frac{34}{1-f}$. Hence,

$$\sum_{p_1' \leq p_1 : e_1(p_1') > 0} e_1(p_1') \cdot (p_1 - p_1') = 0$$

for every price $p_1 \in P_1^1 = [14, \frac{34}{1-f}]$. In view of (8.8.4) we conclude that

$$p_1 = 14 + \tfrac{1}{2} \cdot 14 = 21 \tag{8.8.8}$$

in this case.

Moreover, the highest possible price p_1 that is optimal for some second-order expectation e_1 in E_1^2 is obtained by letting e_1 assign probability 1 to the highest possible price $p_2 \in P_2^1$, which is $p_2 = \frac{34}{1-f}$, and to the lowest possible price $p_1' \in P_1^1$, which is $p_1' = 14$. Hence, we have that $E_{e_1}(p_2) = \frac{34}{1-f}$ and

$$\sum_{p_1' \leq p_1 : e_1(p_1') > 0} e_1(p_1') \cdot (p_1 - p_1') = p_1 - 14$$

for every price $p_1 \in P_1^1 = [14, \frac{34}{1-f}]$. In view of (8.8.4) we conclude that

$$p_1 = 14 + \tfrac{1}{2} \cdot \tfrac{34}{1-f} + f \cdot (p_1 - 14)$$

which yields

$$p_1 = 14 + \tfrac{17}{(1-f)^2} \tag{8.8.9}$$

in this case. On the basis of (8.8.8) and (8.8.9) we see that the set of prices for firm 1 that survive round 2 is given by

$$P_1^2 = [21, 14 + \tfrac{17}{(1-f)^2}]. \tag{8.8.10}$$

Similarly for firm 2.

By continuing in this fashion we can derive the sets of prices P_1^k and P_2^k for every round $k \geq 3$ as well. We will show, by induction on k, that

$$P_1^k = P_2^k = [28 - \tfrac{1}{2^k} \cdot 28,\ 28 + \tfrac{1}{2^k} \cdot (\tfrac{68}{(1-f)^k} - 56)] \tag{8.8.11}$$

for every $k \geq 1$.

For $k = 1$ we see that (8.8.11) matches (8.8.7), and hence (8.8.11) holds for $k = 1$.

Take now some $k \geq 2$, and assume that (8.8.11) holds for $k - 1$. Concentrate on firm 1. In round k we restrict to first-order expectations e_1 which assign positive probability only to pairs (p_2, p_1') where $p_2 \in P_2^{k-1}$ and $p_1' \in P_1^{k-1}$. Let this set of first-order expectations be E_1^k.

By (8.8.4) and our arguments in round 1 we conclude that the lowest possible price p_1 that is optimal for some second-order expectation e_1 in E_1^k is obtained by letting e_1 assign probability 1 to the lowest possible price $p_2 \in P_2^{k-1}$, which is $28 - \frac{1}{2^{k-1}} \cdot 28$, and to the highest possible price $p_1' \in P_1^{k-1}$, which is

$$p_1' = 28 + \tfrac{1}{2^{k-1}} \cdot (\tfrac{68}{(1-f)^{k-1}} - 56).$$

Hence, we have that $E_{e_1}(p_2) = 28 - \frac{1}{2^{k-1}} \cdot 28$ and

$$\sum_{p_1' \leq p_1 : e_1(p_1') > 0} e_1(p_1') \cdot (p_1 - p_1') = 0$$

for every price $p_1 \in P_1^{k-1}$. In view of (8.8.4) we conclude that

$$p_1 = 14 + \tfrac{1}{2} \cdot (28 - \tfrac{1}{2^{k-1}} \cdot 28) = 28 - \tfrac{1}{2^k} \cdot 28 \qquad (8.8.12)$$

in this case.

Moreover, the highest possible price p_1 that is optimal for some second-order expectation e_1 in E_1^k is obtained by letting e_1 assign probability 1 to the highest possible price $p_2 \in P_2^{k-1}$, which is

$$p_2 = 28 + \tfrac{1}{2^{k-1}} \cdot (\tfrac{68}{(1-f)^{k-1}} - 56),$$

and to the lowest possible price $p_1' \in P_1^{k-1}$, which is $p_1' = 28 - \frac{1}{2^{k-1}} \cdot 28$. Hence, we have that

$$E_{e_1}(p_2) = 28 + \tfrac{1}{2^{k-1}} \cdot (\tfrac{68}{(1-f)^{k-1}} - 56)$$

and

$$\sum_{p_1' \leq p_1 : e_1(p_1') > 0} e_1(p_1') \cdot (p_1 - p_1') = p_1 - (28 - \tfrac{1}{2^k - 1} \cdot 28)$$

for every price $p_1 \in P_1^{k-1}$. In view of (8.8.4) we conclude that

$$p_1 = 14 + \tfrac{1}{2} \cdot (28 + \tfrac{1}{2^{k-1}} \cdot (\tfrac{68}{(1-f)^{k-1}} - 56)) + f \cdot (p_1 - (28 - \tfrac{1}{2^{k-1}} \cdot 28)),$$

which yields

$$p_1 = 28 + \tfrac{1}{2^k} \cdot (\tfrac{68}{(1-f)^k} - 56) \qquad (8.8.13)$$

in this case. On the basis of (8.8.12) and (8.8.13) we see that the set of prices for firm 1 that survive round k is given by

$$P_1^k = [28 - \tfrac{1}{2^k} \cdot 28,\ 28 + \tfrac{1}{2^k} \cdot (\tfrac{68}{(1-f)^k} - 56)],$$

which matches (8.8.11). Similarly for firm 2.

By induction on k we thus conclude that (8.8.11) holds for every k.

Recall the assumption that $f \leq 0.15$. This guarantees that $2(1-f) \geq 1.7$, and hence $\frac{1}{2^k} \cdot \frac{68}{(1-f)^k}$ tends to 0 when k tends to infinity. As such, the sets P_1^k and P_2^k collapse to the single price

$$p^* = 28 \qquad (8.8.14)$$

when k tends to infinity. Hence, we see that the iterated elimination of choices and states already leads to a unique price for both firms, which is $p^* = 28$. As such, the states-first procedure will lead to that same unique price. Hence, under common belief in rationality both firms can only rationally choose the price 28. In particular, both firms will not be able to exceed the other firm's expectations under common belief in rationality, as firm 1 believes that firm 2 believes that firm 1 will choose the price 28, and similarly for firm 2.

From (8.8.11) we can conclude, however, that the highest price which survives round k of the iterated elimination of choices and states becomes larger as f increases. This makes intuitive sense, since a larger f means that firm 1 has a stronger preference for choosing a price that exceeds firm 2's expectation, which pushes the highest price in P_1^k upwards.

8.8.2 Competition in Quantities

Consider the Cournot competition model from Section 3.7.2. Like we did for one of the views in Section 7.9.2, we choose the parameters $a = 20$, $c = 5$, $e = 1$ and $M = 10$. Following (3.7.10), the profit for firm 1 is then equal to

$$\pi_1(q_1, q_2) = q_1 \cdot (a - c - e \cdot (q_1 + q_2)) = q_1 \cdot (15 - q_1 - q_2), \tag{8.8.15}$$

where q_1 and q_2 are the quantities chosen by the two firms. Similarly for firm 2.

Assume now that firm 1 also has a preference for choosing a quantity that is lower – and hence more cooperative – than the quantity that firm 1 believes that firm 2 expects firm 1 to choose. More concretely, if firm 1 chooses a quantity q_1 and believes that firm 2 believes that firm 1 chooses a quantity q_1' that is higher than q_1, then firm 1 receives a mental bonus equal to

$$f \cdot (q_1' - q_1)^2. \tag{8.8.16}$$

Here, f is a parameter which measures how strongly firm 1 wishes to exceed firm 2's expectations. We assume that $f \leq 0.25$.

Suppose that the utility for firm 1 is the sum of its profit and (possibly) the mental bonus above. Then, in view of (8.8.15) and (8.8.16), firm 1's utility is equal to

$$u_1(q_1, (q_2, q_1')) = q_1 \cdot (15 - q_1 - q_2) + \left\{ \begin{array}{rl} f \cdot (q_1' - q_1)^2, & \text{if } q_1' \geq q_1 \\ 0, & \text{if } q_1' < q_1 \end{array} \right. , \tag{8.8.17}$$

where q_1' is the quantity that firm 1 believes that firm 2 believes that firm 1 will choose. Similarly for firm 2.

We hereby have modelled this scenario as a psychological game, where the choices for firm 1 are the possible quantities q_1, and the states for firm 1 are the possible pairs (q_2, q_1'), and similarly for firm 2.

The question we wish to answer is: What quantities can both firms rationally choose under common in rationality? To this purpose we will use the *states-first* procedure adapted to infinite psychological games, as described in the previous subsection. Hence, we will start with the iterated elimination of choices and states, and subsequently apply the iterated elimination of choices and second-order expectations.

Let us first apply the iterated elimination of choices and states.

Round 1. Consider firm 1. In round 1 we must identify all quantities that are optimal for some second-order expectation e_1, and eliminate all quantities that are not. Take some second-order expectation e_1 which assigns positive probability to finitely many pairs (q_2, q_1'). Let $E_{e_1}(q_2)$ be the expected quantity for firm 2 under e_1, and let $e_1(q_1')$ be the probability that e_1 assigns to firm 1's quantity q_1'. Similarly

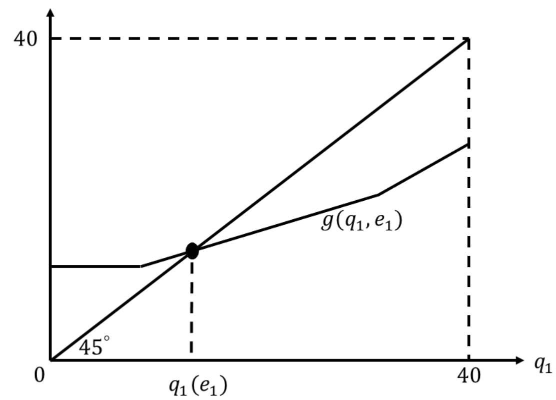

Figure 8.8.2 Optimal quantity for a second-order expectation

to Sections 3.7.2 and 8.8.1, and building on (8.8.17), it may be verified that the expected utility for firm 1 from choosing the quantity q_1 under the second-order expectation e_1 is given by

$$u_1(q_1, e_1) = q_1 \cdot (15 - q_1 - E_{e_1}(q_2)) + \sum_{q_1' \geq q_1 : e_1(q_1') > 0} e_1(q_1') \cdot f \cdot (q_1' - q_1)^2.$$

As a function of q_1, the expected utility $u_1(q_1, e_1)$ achieves the unique maximum precisely where the derivative $\frac{\partial u_1(q_1, e_1)}{\partial q_1}$ is equal to 0. It may be verified that

$$\frac{\partial u_1(q_1, e_1)}{\partial q_1} = 15 - 2q_1 - E_{e_1}(q_2) - 2f \cdot \sum_{q_1' \geq q_1 : e_1(q_1') > 0} e_1(q_1') \cdot (q_1' - q_1).$$

By putting this derivative equal to 0 we find that

$$q_1 = 7.5 - \tfrac{1}{2} E_{e_1}(q_2) - f \cdot \sum_{q_1' \geq q_1 : e_1(q_1') > 0} e_1(q_1') \cdot (q_1' - q_1). \qquad (8.8.18)$$

Hence, the only quantity $q_1(e_1)$ that is optimal under the second-order expectation e_1 is the unique quantity q_1 that satisfies (8.8.18).

Let us denote the right-hand side in (8.8.18) by $g(q_1, e_1)$. Hence, the optimal quantity $q_1(e_1)$ is the unique quantity q_1 where $q_1 = g(q_1, e_1)$. Note that $g(q_1, e_1)$ is increasing in q_1. Graphically, the optimal quantity $q_1(e_1)$ has been depicted in Figure 8.8.2 as the unique intersection point between the 45 degree line and the increasing curve of $g(q_1, e_1)$.

Suppose now that in the second-order expectation e_1 we start assigning higher probabilities to higher quantities q_1'. Then, we conclude from (8.8.18) that the curve of $g(q_1, e_1)$ will shift downwards, and hence we conclude on the basis of Figure 8.8.2 that the optimal quantity $q_1(e_1)$ will decrease as well. Similarly, if e_1 starts to assign higher probabilities to lower quantities q_1', then the curve of $g(q_1, e_1)$ will shift upwards, and therefore the optimal quantity $q_1(e_1)$ will increase.

In view of (8.8.18), the lowest quantity $q_1(e_1)$ that is optimal for a second-order expectation e_1 is obtained if e_1 assigns probability 1 to the highest possible quantity q_2, which is $q_2 = 10$, and e_1 assigns probability 1 to the highest possible q_1', which is $q_1' = 10$. Then,

$$\sum_{q_1' \geq q_1 : e_1(q_1') > 0} e_1(q_1') \cdot (q_1' - q_1) = 10 - q_1.$$

By (8.8.18) we thus see that

$$q_1 = 7.5 - \tfrac{1}{2} \cdot 10 - f \cdot (10 - q_1)$$

which yields

$$q_1 = \tfrac{2.5 - 10f}{1 - f}. \tag{8.8.19}$$

Note that $q_1 \geq 0$ since we assume that $f \leq 0.25$.

Similarly, the highest quantity $q_1(e_1)$ that is optimal for a second-order expectation e_1 is obtained if e_1 assigns probability 1 to the lowest possible quantity q_2, which is $q_2 = 0$, and e_1 assigns probability 1 to the lowest possible q_1', which is $q_1' = 0$. Then,

$$\sum_{q_1' \geq q_1 : e_1(q_1') > 0} e_1(q_1') \cdot (q_1' - q_1) = 0.$$

By (8.8.18) we thus see that

$$q_1 = 7.5 - \tfrac{1}{2} \cdot 0 - f \cdot 0 = 7.5. \tag{8.8.20}$$

On the basis of (8.8.19) and (8.8.20) we conclude that the set Q_1^1 of quantities surviving the first round is

$$Q_1^1 = [\tfrac{2.5 - 10f}{1 - f},\ 7.5]. \tag{8.8.21}$$

Similarly for firm 2.

Round 2. Consider firm 1. At the beginning of round 2 we eliminate all states (q_2, q_1') where either q_2 or q_1' did not survive round 1. Hence, we concentrate on second-order expectations e_1 that only assign positive probability to states (q_2, q_1') where $q_2 \in Q_2^1$ and $q_1' \in Q_1^1$. Let us denote by E_1^2 the set of such second-order expectations.

Which quantities q_1 can be optimal for a second-order expectation e_1 in E_1^2? Similarly to round 1, the smallest quantity q_1 that is optimal for such an e_1 is obtained by choosing the e_1 that assigns probability 1 to the highest possible quantity q_2 in Q_2^1, which is $q_2 = 7.5$, and that assigns probability 1 to the highest possible quantity q_1' in Q_1^1, which is $q_1' = 7.5$. By (8.8.18), and the fact that the optimal quantity q_1 must be in Q_1^1, the optimal quantity q_1 must satisfy

$$q_1 = 7.5 - \tfrac{1}{2} \cdot (7.5) - f \cdot (7.5 - q_1)$$

which yields

$$q_1 = \tfrac{3.75 - (7.5)f}{1 - f}. \tag{8.8.22}$$

Similarly, the highest quantity q_1 that is optimal for such an e_1 is obtained by choosing the e_1 that assigns probability 1 to the lowest possible quantity q_2 in Q_2^1, which is $q_2 = \frac{2.5 - 10f}{1 - f}$, and that assigns probability 1 to the lowest possible quantity q_1' in Q_1^1, which is $q_1' = \frac{2.5 - 10f}{1 - f}$. By (8.8.18), and the fact that the optimal quantity q_1 must be in Q_1^1, the optimal quantity q_1 must satisfy

$$q_1 = 7.5 - \tfrac{1}{2} \cdot \tfrac{2.5 - 10f}{1 - f} - f \cdot 0$$

which yields

$$q_1 = \tfrac{6.25 - (2.5)f}{1 - f}. \tag{8.8.23}$$

On the basis of (8.8.22) and (8.8.23), we see that the set Q_1^2 of quantities that survive round 2 is given by

$$Q_1^2 = [\tfrac{3.75-(7.5)f}{1-f},\ \tfrac{6.25-(2.5)f}{1-f}]. \tag{8.8.24}$$

Similarly for firm 2.

If we continue in this manner we can derive the sets Q_1^k and Q_2^k for every round $k \geq 3$. We will show, by induction on k, that

$$Q_1^k = Q_2^k = \begin{cases} [5\cdot(1-(\tfrac{1}{2})^{\frac{k-1}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k+1}{2}}),\ 5\cdot(1+(\tfrac{1}{2})^{\frac{k+1}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}})], & \text{if } k \text{ is odd} \\ [5\cdot(1-(\tfrac{1}{2})^{\frac{k}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k}{2}}),\ 5\cdot(1+(\tfrac{1}{2})^{\frac{k}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k}{2}})], & \text{if } k \text{ is even} \end{cases} \tag{8.8.25}$$

for every $k \geq 1$.

For $k = 1$ the expression for Q_1^k in (8.8.25) matches (8.8.21), and hence (8.8.25) holds for $k = 1$.

Suppose now that $k \geq 2$, and that (8.8.25) holds for $k-1$. We distinguish two cases: (1) k is even, and (2) k is odd.

Case 1. Suppose that k is even. Consider firm 1. At the beginning of round k we eliminate all states (q_2, q_1') where either q_2 or q_1' did not survive the odd round $k-1$. Hence, we concentrate on second-order expectations e_1 that only assign positive probability to states (q_2, q_1') where $q_2 \in Q_2^{k-1}$ and $q_1' \in Q_1^{k-1}$. Let us denote by E_1^k the set of such second-order expectations.

Which quantities q_1 can be optimal for a second-order expectation e_1 in E_1^k? Similarly to above, the smallest quantity q_1 that is optimal for such an e_1 is obtained by choosing the e_1 that assigns probability 1 to the highest possible quantity q_2 in Q_2^{k-1}, which is

$$q_2 = 5\cdot(1+(\tfrac{1}{2})^{\frac{k}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k-2}{2}}),$$

and that assigns probability 1 to the highest possible quantity q_1' in Q_1^{k-1}, which is

$$q_1' = 5\cdot(1+(\tfrac{1}{2})^{\frac{k}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k-2}{2}}).$$

By (8.8.18), and the fact that the optimal quantity q_1 must be in Q_1^{k-1}, the optimal quantity q_1 must satisfy

$$q_1 = 7.5 - \tfrac{1}{2}\cdot 5\cdot(1+(\tfrac{1}{2})^{\frac{k}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k-2}{2}}) - f\cdot(5\cdot(1+(\tfrac{1}{2})^{\frac{k}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k-2}{2}}) - q_1).$$

Solving for q_1 yields

$$q_1 = 5\cdot(1-(\tfrac{1}{2})^{\frac{k}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k}{2}}). \tag{8.8.26}$$

Similarly, the highest quantity q_1 that is optimal for such an e_1 is obtained by choosing the e_1 that assigns probability 1 to the lowest possible quantity q_2 in Q_2^{k-1}, which is

$$q_2 = 5\cdot(1-(\tfrac{1}{2})^{\frac{k-2}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k}{2}}),$$

and that assigns probability 1 to the lowest possible quantity q_1' in Q_1^{k-1}, which is

$$q_1' = 5\cdot(1-(\tfrac{1}{2})^{\frac{k-2}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k}{2}}).$$

By (8.8.18), and the fact that the optimal quantity q_1 must be in Q_1^{k-1}, the optimal quantity q_1 must satisfy

$$q_1 = 7.5 - \tfrac{1}{2}\cdot 5\cdot(1-(\tfrac{1}{2})^{\frac{k-2}{2}}(\tfrac{1/2+f}{1-f})^{\frac{k}{2}}) - f\cdot 0$$

which yields

$$q_1 = 5 \cdot (1 + (\tfrac{1}{2})^{\frac{k}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k}{2}}). \tag{8.8.27}$$

On the basis of (8.8.26) and (8.8.27), we see that the set Q_1^k of quantities that survive round k is given by

$$Q_1^k = [5 \cdot (1 - (\tfrac{1}{2})^{\frac{k}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k}{2}}),\ 5 \cdot (1 + (\tfrac{1}{2})^{\frac{k}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k}{2}})],$$

which matches (8.8.25). Similarly for firm 2.

Case 2. Suppose now that k is odd. Consider firm 1. At the beginning of round k we eliminate all states (q_2, q_1') where either q_2 or q_1' did not survive the even round $k-1$. Hence, we concentrate on second-order expectations e_1 that only assign positive probability to states (q_2, q_1') where $q_2 \in Q_2^{k-1}$ and $q_1' \in Q_1^{k-1}$. Let us denote by E_1^k the set of such second-order expectations.

Which quantities q_1 can be optimal for a second-order expectation e_1 in E_1^k? Similarly to above, the smallest quantity q_1 that is optimal for such an e_1 is obtained by choosing the e_1 that assigns probability 1 to the highest possible quantity q_2 in Q_2^{k-1}, which is

$$q_2 = 5 \cdot (1 + (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}}),$$

and that assigns probability 1 to the highest possible quantity q_1' in Q_1^{k-1}, which is

$$q_1' = 5 \cdot (1 + (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}}).$$

By (8.8.18), and the fact that the optimal quantity q_1 must be in Q_1^{k-1}, the optimal quantity q_1 must satisfy

$$q_1 = 7.5 - \tfrac{1}{2} \cdot 5 \cdot (1 + (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}}) - f \cdot (5 \cdot (1 + (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}}) - q_1).$$

Solving for q_1 yields

$$q_1 = 5 \cdot (1 - (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k+1}{2}}). \tag{8.8.28}$$

Similarly, the highest quantity q_1 that is optimal for such an e_1 is obtained by choosing the e_1 that assigns probability 1 to the lowest possible quantity q_2 in Q_2^{k-1}, which is

$$q_2 = 5 \cdot (1 - (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}}),$$

and that assigns probability 1 to the lowest possible quantity q_1' in Q_1^{k-1}, which is

$$q_1' = 5 \cdot (1 - (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}}).$$

By (8.8.18), and the fact that the optimal quantity q_1 must be in Q_1^{k-1}, the optimal quantity q_1 must satisfy

$$q_1 = 7.5 - \tfrac{1}{2} \cdot 5 \cdot (1 - (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}}) - f \cdot 0$$

which yields

$$q_1 = 5 \cdot (1 + (\tfrac{1}{2})^{\frac{k+1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}}). \tag{8.8.29}$$

On the basis of (8.8.28) and (8.8.29), we see that the set Q_1^k of quantities that survive round k is given by

$$Q_1^k = [5 \cdot (1 - (\tfrac{1}{2})^{\frac{k-1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k+1}{2}}),\ 5 \cdot (1 + (\tfrac{1}{2})^{\frac{k+1}{2}} (\tfrac{1/2+f}{1-f})^{\frac{k-1}{2}})],$$

which matches (8.8.25). Similarly for firm 2.

By induction on k we conclude that (8.8.25) holds for every round k and for both firms. It may be verified that $\frac{1/2+f}{1-f} \leq 1$ since we assume that $f \leq 0.25$. Therefore, the sets Q_1^k and Q_2^k of quantities that survive round k of the iterated elimination of choices and states collapse to the single quantity

$$q^* = 5 \tag{8.8.30}$$

when k tends to infinity. But then, the states-first procedure will only yield the quantity $q^* = 5$ at the end. As such, both firms can only rationally choose the quantity 5 under common belief in rationality. In particular, it will be impossible for both firms to exceed the other firm's expectation under common belief in rationality.

But we can say a little more: Consider the set of quantities Q_1^k in (8.8.25) that survive round k of the iterated elimination of choices and states. If f increases, then the number $\frac{1/2+f}{1-f}$ will increase as well, and therefore the interval Q_1^k of quantities that survives round k becomes wider. This makes intuitive sense, since a larger f means that firm 1 has a stronger preference for choosing a low quantity that exceeds firm 2's expectation. If firm 2 indeed believes that firm 1 chooses a low quantity, then firm 2 will choose a high quantity in response. This will make the lower bound of Q_1^k lower, and the upper bound of Q_2^{k+1} higher. As the same holds if we reverse the roles of firms 1 and 2, the intervals Q_1^k and Q_2^k become wider.

Chapter 9

Correct and Symmetric Beliefs in Psychological Games

9.5 Economic Applications

In this section we reconsider the Bertrand competition model and Cournot competition model from Sections 8.8.1 and 8.8.2. For both models we investigate which prices or quantities the firms can rationally choose under common belief in rationality with a simple belief hierarchy, a symmetric belief hierarchy, and a symmetric belief hierarchy using one theory per choice.

9.5.1 Competition in Prices

Let us return to the Bertrand competition model from Section 8.8.1, which we modelled as a psychological game. We saw in that section that under common belief in rationality, both firms can only rationally choose the price $p^* = 28$. As such, this will be the only price that both firms can rationally choose under common belief in rationality with a simple belief hierarchy, a symmetric belief hierarchy, and a symmetric belief hierarchy using one theory per choice.

But suppose we would not have the information from Section 8.8.1, but would nevertheless wish to find those prices that both firms can rationally choose under common belief in rationality with a simple belief hierarchy. Is there a quick way to do this? The answer is "yes", by trying to find the psychological Nash equilibria in this game. By Theorem 9.1.2 we know that for finite psychological games, the choices that can rationally be made under common belief in rationality with a simple belief hierarchy are precisely those that are optimal in a psychological Nash equilibrium. The same is true for infinite psychological games.

In the Betrand competition model from Section 8.8.1, suppose that (σ_1, σ_2) is a psychological Nash equilibrium where σ_1 is a belief about firm 1's price that assigns positive probability to finitely many prices for firm 1, and similarly for σ_2. Let $e_1[\sigma_1, \sigma_2]$ and $e_2[\sigma_1, \sigma_2]$ be the second-order expectation for firms 1 and 2, respectively, induced by the beliefs σ_1 and σ_2. Then, we know from (8.8.4) that the unique price p_1^* that is optimal for firm 1 under the second-order expectation $e_1[\sigma_1, \sigma_2]$ satisfies

$$p_1^* = 14 + \tfrac{1}{2} E_{e_1[\sigma_1,\sigma_2]}(p_2) + f \cdot \sum_{p_1' \leq p_1^* : e_1[\sigma_1,\sigma_2](p_1') > 0} e_1[\sigma_1, \sigma_2](p_1') \cdot (p_1^* - p_1'). \tag{9.5.1}$$

Similarly, the unique price p_2^* that is optimal for firm 2 under the second-order expectation $e_2[\sigma_1, \sigma_2]$ satisfies

$$p_2^* = 14 + \tfrac{1}{2} E_{e_2[\sigma_1,\sigma_2]}(p_1) + f \cdot \sum_{p_2' \leq p_2^* : e_2[\sigma_1,\sigma_2](p_2') > 0} e_2[\sigma_1, \sigma_2](p_2') \cdot (p_2^* - p_2'). \tag{9.5.2}$$

But then, by definition of a psychological Nash equilibrium, σ_1 must assign probability 1 to p_1^* and σ_2 must assign probability 1 to p_2^*. Hence, $e_1[\sigma_1, \sigma_2]$ assigns probability 1 to the pair (p_2^*, p_1^*), whereas $e_2[\sigma_1, \sigma_2]$ assigns probability 1 to the pair (p_1^*, p_2^*). In particular, $E_{e_1[\sigma_1,\sigma_2]}(p_2) = p_2^*$, the second-order expectation $e_1[\sigma_1, \sigma_2]$ assigns probability 1 to p_1^*, and similarly for $e_2[\sigma_1, \sigma_2]$. As such, (9.5.1) and (9.5.2) can be reduced to

$$p_1^* = 14 + \tfrac{1}{2} p_2^* \text{ and } p_2^* = 14 + \tfrac{1}{2} p_1^*.$$

By substituting the second equation in the first we obtain

$$p_1^* = 14 + \tfrac{1}{2} \cdot (14 + \tfrac{1}{2} p_1^*),$$

which yields $p_1^* = 28$. Hence, $p_2^* = 28$ as well.

We thus conclude that there is a unique psychological Nash equilibrium (σ_1, σ_2) where σ_1 assigns probability 1 to firm 1's price 28, and σ_2 assigns probability 1 to firm 2's price 28. By (9.5.1) and (9.5.2), the unique price that is optimal for firm 1 in this psychological Nash equilibrium is $p_1^* = 28$, and similarly for firm 2. Hence, we see that under common belief in rationality with a simple belief hierarchy, both firms can only rationally choose the price 28.

9.5.2 Competition in Quantities

Let us finally go back to the Cournot competition model from Section 8.8.2. We have seen in that section that under common belief in rationality both firms can only rationally choose the quantity $q^* = 5$. Hence, we conclude that under common belief in rationality with a simple belief hierarchy, a symmetric belief hierarchy, or a symmetric belief hierarchy using one theory per choice, both firms can still only rationally choose the quantity 5. Similarly to what we have done above, we will now directly compute the quantities that both firms can rationally choose under common belief in rationality with a simple belief hierarchy, by looking for the psychological Nash equilibria.

Suppose that (σ_1, σ_2) is a psychological Nash equilibrium, where σ_1 is a belief about firm 1's quantities that assigns positive probability to finitely many quantities, and similarly for σ_2. By $e_1[\sigma_1, \sigma_2]$ and $e_2[\sigma_1, \sigma_2]$ we denote the second-order expectation for firm 1 and firm 2, respectively, induced by the beliefs σ_1 and σ_2. From (8.8.18) we know that the unique quantity q_1^* that is optimal for firm 1 under the second-order expectation $e_1[\sigma_1, \sigma_2]$ satisfies

$$q_1^* = 7.5 - \tfrac{1}{2} E_{e_1[\sigma_1,\sigma_2]}(q_2) - f \cdot \sum_{q_1' \geq q_1^* : e_1[\sigma_1,\sigma_2](q_1') > 0} e_1[\sigma_1, \sigma_2](q_1') \cdot (q_1' - q_1^*). \tag{9.5.3}$$

Similarly, the unique quantity q_2^* that is optimal for firm 2 under the second-order expectation $e_2[\sigma_1, \sigma_2]$ satisfies

$$q_2^* = 7.5 - \tfrac{1}{2} E_{e_2[\sigma_1,\sigma_2]}(q_1) - f \cdot \sum_{q_2' \geq q_2^* : e_2[\sigma_1,\sigma_2](q_2') > 0} e_2[\sigma_1, \sigma_2](q_2') \cdot (q_2' - q_2^*). \tag{9.5.4}$$

As (σ_1, σ_2) is a psychological Nash equilibrium, the belief σ_1 must assign probability 1 to the unique optimal quantity q_1^* that satisfies (9.5.3), whereas σ_2 must assign probability 1 to the unique optimal quantity q_2^* that satisfies (9.5.4). But then, we know that $E_{e_1[\sigma_1,\sigma_2]}(q_2) = q_2^*$, that $e_1[\sigma_1, \sigma_2]$ assigns probability 1 to firm 1's quantity q_1^*, and similarly for $e_2[\sigma_1, \sigma_2]$. Hence, (9.5.3) and (9.5.4) can then be reduced to

$$q_1^* = 7.5 - \tfrac{1}{2}q_2^* \text{ and } q_2^* = 7.5 - \tfrac{1}{2}q_1^*.$$

By substituting the second equation into the first we obtain

$$q_1^* = 7.5 - \tfrac{1}{2}(7.5 - \tfrac{1}{2}q_1^*),$$

which yields $q_1^* = 5$. As $q_2^* = 7.5 - \frac{1}{2}q_1^*$ it follows that $q_2^* = 5$ as well.

Thus, the unique psychological Nash equilibrium (σ_1, σ_2) is such that σ_1 assigns probability 1 to firm 1's quantity 5, and σ_2 assigns probability 1 to firm 2's quantity 5. By (9.5.3) and (9.5.4), the unique optimal quantity for firm 1 and firm 2 in this psychological Nash equilibrium is 5. Therefore, both firms can only rationally choose the quantity 5 under common belief in rationality with a simple belief hierarchy.