Common belief in future and restricted past rationality

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Abstract. We introduce the idea that a player believes at every stage of a dynamic game that his opponents will choose rationally in the future and have chosen rationally in a restricted way in the past. This is summarized by the concept of common belief in future and restricted past rationality, which is defined epistemically. Moreover, it is shown that every properly rationalizable strategy of the normal form of a dynamic game can be chosen in the dynamic game under common belief in future and restricted past rationality. We also present an algorithm that uses strict dominance, and show that its full implementation selects exactly those strategies that can be chosen under common belief in future and restricted past rationality.

Keywords: Epistemic game theory · Dynamic games · Proper rationalizability · Belief in future rationality · Belief in restricted past rationality

JEL Classification: C72 · C73

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1 Introduction

Epistemic game theory deals with the reasoning processes of an individual about his opponents before he makes a decision. This requires a belief about the choices of his opponents, but also a belief about the opponents' beliefs about their opponents' choices, and so on.

Such reasoning processes have been studied thoroughly in the framework of static games, in various forms of the concept of common belief in rationality. However, the extension of these concepts to the framework of dynamic games is not entirely trivial. One possible way to extend the idea of common belief in rationality would require that the players believe their opponents make only rational choices, in particular that past choices have been rational, which in most cases is not possible to do.

To solve this problem some alternative concepts have been proposed. Battigalli and Siniscalchi (2002) propose the concept of common strong belief in rationality, in which players, whenever possible, must believe that their opponents are implementing rational strategies. Perea (2014) proposed the concept of common belief in future rationality, in which at each decision point a player must believe that all players are rational in the present and in the future, but allows players to believe that irrational choices have been made in the past. This concept is similar to sequential rationalizability, proposed by Dekel, Fudenberg and Levine (1999, 2002), and Asheim and Perea (2005).

Reny (1992, 1993) studies the idea of common belief in past and future rationality at all information sets, coming to the conclusion that in most games, it is not possible to reason under such concept.

However, taking as a starting point the concept of common belief in future rationality in which we allow players to believe that past choices were irrational, we consider a concept in which a restricted notion of belief in past rationality is assumed. The example that follows will be used to illustrate the concepts that are being discussed, while it also serves as one of the motivations for developing a new rationality concept.

The key idea in the new concept we propose is that a player does not only believe that his opponents choose rationally in the future, but also that the decisions made in the past were rational among a restricted set of choices. In Figure 1 we can see that at $\varnothing$ the optimal choice for player 1 is c. However, if the game were to reach h_1, player 2 must believe that a suboptimal choice was made at $\varnothing$. Under the concept of common belief in future rationality player 2 can assume either a or b was chosen at $\varnothing$, as there is no restriction on the beliefs about choices made in the past. We propose that player 2 should reason about the choice made at $\varnothing$ by considering only those choices that reach h_1 and from those find which are optimal: in this case, we can see that a is the best choice for player 1 from those that reach h_1 assuming he would choose f afterwards. Hence, under the new concept, player 2 must believe at h_1 that player 1 chose a in the past.

The concept proposed here, which we call "common belief in future and restricted past rationality" is a refinement of common belief in future rationality.

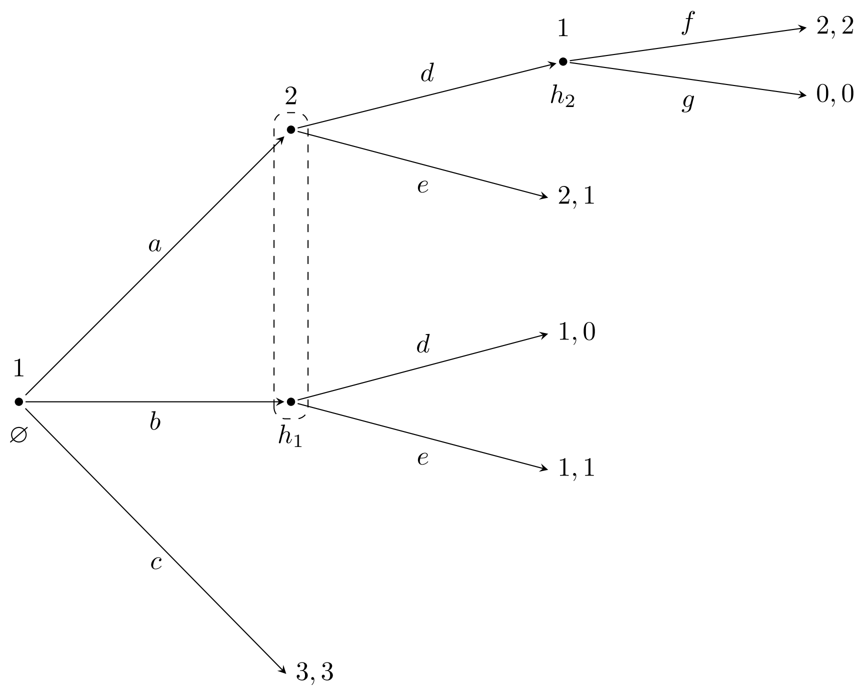

Figure 1. Example of a dynamic game.

The difference is that common belief in future rationality does not reason about the choices made in the past, while the addition of "restricted past rationality" makes players consider the subset of past choices that reach an information set and find the optimal choice in this subset.

Moreover, besides proposing the concept we show the connection between the concepts of proper rationalizability, proposed by Schuhmacher (1999) and Asheim (2001), for the normal form of a dynamic game and common belief in future and restricted past rationality for a dynamic game, namely, that properly rationalizable strategies in the normal form can rationally be chosen under common belief in future and restricted past rationality. And since we know that there are properly rationalizable strategies for every finite normal form game, then we have that there are strategies that can rationally be chosen under common belief in future and restricted past rationality for every finite dynamic game. In addition we propose an algorithm for this concept, and we show that after its full implementation we get exactly the strategies that can rationally be chosen under common belief in future and restricted past rationality.

It was shown by Asheim (2001) that every choice that has positive probability in some proper equilibrium is optimal for some properly rationalizable type. The converse is in general not true, as there are choices that are optimal for some type that is properly rationalizable, yet no proper equilibrium assigns them positive probability. Therefore, proper equilibrium is a more restrictive concept than proper rationalizability. It can be shown that to obtain proper equilibrium from proper rationalizability we need to add two requirements in the case of two players: every player fully believes that the opponent has correct beliefs; and every player fully believes that every opponent believes that he has correct beliefs (cf. Perea (2012)). A player i believes that the opponent has correct beliefs if for a given t_i and its corresponding belief about the opponent's choices, type t_i believes that his opponent believes that i holds precisely the corresponding belief for type t_i about the opponent's choices.

Van Damme (1984) proves that for every dynamic game, the proper equilibria of its normal form induce quasi-perfect equilibria of the dynamic game. In this way he shows that it is possible to reason about a dynamic game in terms of the normal form and obtain equilibria of the dynamic game by looking at the normal form only. This is precisely the driving idea behind the present paper, in which the concept of proper rationalizability, which is less restrictive than proper equilibrium, is linked to a concept for dynamic games that, in contrast to common belief in future rationality, takes into account a restricted version of rationality in the past. Also in contrast to strong belief in rationality, it makes players reason about the optimality of choices at every information set, even if an information set can only be reached by past choices that are suboptimal.

The structure of the paper is as follows. In Section 2 we introduce dynamic games. In Section 3 we present the concept of proper rationalizability for the normal form of a dynamic game. In Section 4 we introduce the notion of common belief in future and restricted past rationality for a dynamic game. In Section 5 both of these rationalizability concepts are connected, by showing that the strategies that are proper rationalizable can also be made under common belief in future and restricted past rationality. In Section 6 we describe an algorithm and show that it yields precisely those strategies that can be made under common belief in future and restricted past rationality. Section 7 has some concluding remarks and Section 8 contains all the proofs of this paper.

2 Dynamic games

In this section we define the dynamic games we consider, and some general notions that will be used throughout the paper. In what follows we assume the players have perfect recall.

Definition 1 (Dynamic game).

A dynamic game G is a tuple

$$G = (I, (C_i)_{i\in I}, X, Z, (H_i)_{i\in I}, (C_i(h))_{i\in I, h\in H_i}, (u_i)_{i\in I})$$

where

- I is the finite set of players;

- C_i is the finite set of choices for each player $i \in I$;

- X is the set of non-terminal histories, which are sequences of profiles of choices $x = (x_1, \dots, x_k)$, with $x_m = (c_i)_{i \in \hat{I}}$ in $\times_{i \in \hat{I}} C_i$ for some non-empty $\hat{I} \subseteq I$, and for all $\ell < k$, $(x_1, \dots, x_\ell)$ is also a history. As $\hat{I}$ may contain more than one player, simultaneous moves are allowed.

- Z is the set of terminal histories of the game. In this case, if $z = (x_1, \dots, x_k) \in Z$, then for every $\ell < k$, $(x_1, \dots, x_\ell) \in X$;

- H_i is a finite collection of information sets for player i. The information sets $h \in H_i$ are non-empty sets of non-terminal histories. If h contains more than one history, then player i does not know with certainty which history was realized to arrive at h.

- $C_i(h) \subseteq C_i$ is the finite set of choices available for player i at the information set $h \in H_i$. We say $c \in C_i(h)$ if there is a history $x \in X$ and $x_m = (c_j)_{j \in \hat{I}}$ such that $x \in h$, $i \in \hat{I}$, $c_i = c$ and $(x, x_m) = x' \in X \cup Z$.

- $u_i \colon Z \to \mathbb{R}$ is player i's utility function.

Example 1. In Figure 1 we have a dynamic game in its extensive form. This two-player game has the sets of histories $X = \{\varnothing, (a), (b), (a, d)\}$ and $Z = \{(c), (a, e), (b, d), (b, e), (a, d, f), (a, d, g)\}$; the collections of information sets $H_1 = \{\varnothing, h_2\}$ and $H_2 = \{h_1\}$, where $h_1 = \{(a), (b)\}$ and $h_2 = \{(a, d)\}$; and the sets of choices $C_1(\varnothing) = \{a, b, c\}$, $C_1(h_2) = \{f, g\}$, $C_2(h_1) = \{d, e\}$.

We define a partial order on the information sets of a game. An information set h' immediately follows h, or h immediately precedes h', if there exist a non-empty $\hat{I} \subseteq I$, $c_i \in C_i(h)$ for every $i \in \hat{I}$, and $x \in h$ such that $(x, (c_i)_{i \in \hat{I}}) \in h'$.

An information set h' weakly follows h, or h weakly precedes h', if $h = h'$ or there is a sequence $h_0, h_1, h_2, \dots, h_\ell$ such that h_t immediately follows h_{t-1} for $t \in \{1, 2, \dots, \ell\}$, where $h_0 = h$ and $h_\ell = h'$. If $h \neq h'$, we say h strictly precedes h'.

During the game, each player makes one or more choices, sometimes depending on his previous choices or on the choices of other players. However, if a player's choice prevents himself from making some other choices, there is no point in him making a plan that includes both the former choice and any of the latter ones. Therefore, we restrict ourselves to studying those plans that only prescribe choices at information sets that are reachable under the earlier choices: a "plan of action", as described in Rubinstein (1991). These plans we will call strategies. We also identify those strategies that can potentially reach an information set.

Looking at Example 1, the sets of strategies for each player are $S_1 = \{(a, f), (a, g), (b), (c)\}$, and $S_2 = \{(d), (e)\}$. In classical game theory, other sequences such as (b, f) would also qualify as strategies, however, player 1 prevents himself from choosing f by choosing b at an earlier information set, rendering the choice f unnecessary.

Let $h \in H$, $h' \in H_i$, where h' strictly precedes h. We say a choice $c_i \in C_i(h')$ leads to h if there exist $x \in h'$, $\hat{I} \subseteq I$ with $i \in \hat{I}$, and $c_j \in C_j(h')$ for every $j \in \hat{I} \setminus \{i\}$ such that $(x, (c_j)_{j \in \hat{I}})$ weakly precedes h.

A history $h \in H$ is reachable via $s_i \colon \tilde{H}_i \to \cup_{h \in \tilde{H}_i} C_i(h)$, with $\tilde{H}_i \subseteq H_i$, if at every history $h' \in \tilde{H}_i$ that strictly precedes h, the choice $s_i(h')$ leads to h. We say s_i is a strategy if $\tilde{H}_i$ contains exactly those histories in H_i that are reachable via s_i. A strategy s_i leads to $h \in H$ if h is reachable via s_i.

The set of strategies for player i is denoted by S_i. The set of strategy combinations for the opponents of i is denoted by $S_{-i} = \times_{j \neq i} S_j$. A strategy combination for all players is given by (s_i, s_{-i}) where $s_i \in S_i$ and $s_{-i} \in S_{-i}$.

The set of strategies for player i that lead to h is denoted by $S_i(h)$. In Example 1, $S_1(h_1) = \{(a, f), (a, g), b\}$, $S_1(h_2) = \{(a, f), (a, g)\}$, $S_2(h_2) = \{d\}$.

The set of strategy combinations for the opponents of i that lead to h is denoted by $S_{-i}(h)$. The set of information sets for player i that strategy s_i leads to is denoted by $H_i(s_i)$.

Finally we identify those strategy combinations that reach a particular information set. Let $(s_i, s_{-i}) \in S_i \times S_{-i}$ be a strategy combination for all players. We define $H(s_i, s_{-i})$ as the class of information sets h such that $s_i \in S_i(h)$ and $s_{-i} \in S_{-i}(h)$. $H(s_i, s_{-i})$ are the information sets that can be reached with the strategy combination (s_i, s_{-i}).

3 Proper rationalizability

To connect the rationalizability concepts in dynamic games with related rationalizability concepts in normal form games, we also need to connect a dynamic game with a related game in its normal form.

Definition 2 (Normal form of a dynamic game)**.**
Let G be a dynamic game. The normal form of G is the game $G' = (I, (S_i)_{i \in I}, (v_i)_{i \in I})$ in which all players i choose simultaneously a strategy $s_i \in S_i$, and each player i receives the utility $v_i(s_i, s_{-i}) = u_i(z(s_i, s_{-i}))$ where $z(s_i, s_{-i})$ is the terminal history reached by (s_i, s_{-i}).

We define a structure called an epistemic model, which serves as a compact way to encode belief hierarchies, so we can easily check the levels of belief for each player by looking at the epistemic model. Then we define strategy-type combinations, which are the objects on which beliefs are constructed, and lexicographic beliefs.

A lexicographic belief b_i for player i on a finite set A is a sequence $(b_i^1; \ldots; b_i^m)$ where each b_i^k is a probability distribution on A. The belief b_i^k is called the level k of the lexicographic belief.

Definition 3 (Epistemic model for a normal form game)**.**
An epistemic model $M = (T_i, b_i)_{i \in I}$ for a normal form game $G' = (I, (S_i)_{i \in I}, (v_i)_{i \in I})$ consists of a finite set of types T_i for each player i, and for each type $t_i \in T_i$

we define a lexicographic belief $b_i(t_i) = (b_i^1(t_i); \ldots ; b_i^m(t_i))$ on $S_{-i} \times T_{-i} = \times_{k \neq i}(S_k \times T_k)$, which is the set of strategy-type combinations of i's opponents.

To derive a lexicographic belief hierarchy for every type, consider a type t_i and its lexicographic belief $b_i(t_i) = (b_i^1(t_i); \cdots ; b_i^m(t_i))$.

For the first order of the lexicographic belief hierarchy of t_i, we have that player i deems the strategies in the support of $b_i^1(t_i)$ infinitely more likely than the strategies that are in the support of $b_i^2(t_i)$ but not in the support of $b_i^1(t_i)$; and deems the strategies in the support of $b_i^2(t_i)$ infinitely more likely than the strategies that are in the support of $b_i^3(t_i)$ but not in the supports of $b_i^1(t_i)$ or $b_i^2(t_i)$; and so on.

On the second order of the lexicographic belief hierarchy of t_i, we have that player i deems the lexicographic beliefs of each type that appears in $b_i^1(t_i)$ infinitely more likely than the lexicographic beliefs of each type that appears in $b_i^2(t_i)$ but didn't appear in $b_i^1(t_i)$; and deems the lexicographic beliefs of each type that appears in $b_i^2(t_i)$ but didn't appear in $b_i^1(t_i)$ infinitely more likely than the lexicographic beliefs of each type that appears in $b_i^3(t_i)$ but didn't appear in a previous level; and so on. Continuing this way it is possible to obtain the full lexicographic belief hierarchy.

We say type t_j is deemed possible by type t_i for the lexicographic belief $b_i(t_i) = (b_i^1(t_i); \ldots ; b_i^m(t_i))$ if there exists a strategy-type combination $(s_{-i}, t_{-i}) \in (S_j \times \{t_j\}) \times \times_{k \neq i,j}(S_k \times T_k)$ such that $b_i^\ell(t_i)(s_{-i}, t_{-i}) > 0$ for some $\ell \in \{1, \ldots, m\}$. The set of types for player j deemed possible by $b_i(t_i)$ is denoted by $T_j(t_i)$.

If positive probability is assigned to a strategy-type combination in level ℓ, earlier than another strategy-type combination in a level k, with $\ell < k$, we say that the first combination is deemed infinitely more likely than the second one.

Definition 4 (Strategy-type combinations deemed infinitely more likely)**.**

Let $b_i(t_i) = (b_i^1(t_i); \ldots , b_i^m(t_i))$ be a lexicographic belief for type t_i for player i. We say t_i deems a strategy-type combination (s_{-i}, t_{-i}) infinitely more likely than (s'_{-i}, t'_{-i}) if there exists $k \in \{1, \ldots , m\}$ such that

1. for all $\ell \leq k$, $b_i^\ell(t_i)(s'_{-i}, t'_{-i}) = 0$; and
2. $b_i^k(t_i)(s_{-i}, t_{-i}) > 0$.

We focus on a particular type of lexicographic beliefs, which are such that for every type combination for i's opponents that is deemed possible in the belief, every strategy combination for i's opponents must receive positive probability at some level k.

Definition 5 (Cautious lexicographic belief)**.**

Consider an epistemic model $M = (T_i, b_i)_{i \in I}$. Let $b_i(t_i) = (b_i^1(t_i); \ldots ; b_i^m(t_i))$ be a lexicographic belief for type $t_i \in T_i$ for player i. We say $b_i(t_i)$ is cautious if for each $(s_{-i}, t_{-i}) \in \times_{j \neq i}(S_j \times T_j(t_i))$ there is a $k \in \{1, \ldots , m\}$ such that

$$b_i^k(t_i)(s_{-i}, t_{-i}) > 0.$$

In order to compare strategies for a player we define the expected utility for a given lexicographic belief. Note that it is defined by levels, and the comparison is made at the first level in which two strategies disagree in their expected utility.

Given a type t_i for player i and a lexicographic belief $b_i(t_i) = (b_i^1(t_i); \ldots ; b_i^m(t_i))$ we define the expected utility of choosing strategy s_i at level k as

$$v_i^k(s_i, b_i(t_i)) = \sum_{(s_{-i},t_{-i}) \in S_{-i} \times T_{-i}} b_i^k(t_i)(s_{-i}, t_{-i}) v_i(s_i, s_{-i}).$$

A type t_i with a lexicographic belief $b_i(t_i) = (b_i^1(t_i); \ldots , b_i^m(t_i))$ for player i prefers strategy s_i to s_i' if there exists $k \in \{1, \ldots , m\}$ such that

1. for all $\ell < k$, $v_i^\ell(s_i, b_i(t_i)) = v_i^\ell(s_i', b_i(t_i))$; and
2. $v_i^k(s_i, b_i(t_i)) > v_i^k(s_i', b_i(t_i))$.

Given a lexicographic belief $b_i(t_i)$ for type t_i, a strategy s_i is optimal for t_i if there is no other $s_i' \in S_i$ such that t_i prefers s_i' to s_i.

Now we define the notion of rationalizability that will be used for normal form games: respect of preferences, due to Asheim (2001), which in turn defines the concept of proper rationalizability.

Definition 6 (Respect of preferences)**.**

Consider an epistemic model $M = (T_i, b_i)_{i \in I}$. Let $b_i(t_i) = (b_i^1(t_i); \ldots ; b_i^m(t_i))$ be a lexicographic belief for type t_i for player i. We say t_i respects j's preferences if for every type t_j of player j deemed possible by t_i, and strategies $s_j, s_j' \in S_j$ such that t_j prefers s_j to s_j', t_i deems at least one strategy-type combination in $\times_{k \in I \setminus \{i,j\}}(S_k \times T_k(t_i)) \times \{(s_j, t_j)\}$ infinitely more likely than every strategy-type combination in $\times_{k \in I \setminus \{i,j\}}(S_k \times T_k(t_i)) \times \{(s_j', t_j)\}$.

We say t_i respects the opponents' preferences if t_i respects j's preferences for all $j \in I \setminus \{i\}$.

Definition 7 (k-fold and common full belief in caution)**.**

1. Type t_i expresses 1-fold full belief in caution if t_i only deems possible opponents' types that are cautious.
2. For every $k > 1$, type t_i expresses k-fold full belief in caution if t_i only deems possible opponents' types that express $(k-1)$-fold full belief in caution.
3. Type t_i expresses common full belief in caution if t_i expresses k-fold full belief in caution for all $k \in \mathbb{N}$.

In a similar way we can define k-fold and common full belief in respect of preferences. Now we can define proper rationalizability, which was introduced by Schuhmacher (1999) and Asheim (2001).

$T_1 = \{t_1\}$, $T_2 = \{t_2\}$

$b_1(t_1) = ((d, t_2); (e, t_2))$

$b_2(t_2) = ((c, t_1); ((a, f), t_1); (b, t_1); ((a, g), t_1))$

Table 1. An epistemic model for the normal form of Example 1.

Definition 8 (Proper rationalizability)**.**
Type t_i is properly rationalizable if t_i is cautious, respects the opponents' preferences and expresses common full belief in caution and common full belief in respect of preferences.

A strategy s_i for player i is properly rationalizable if there exists an epistemic model $M = (T_i, b_i)_{i \in I}$ and some type $t_i \in T_i$ such that t_i is properly rationalizable, and strategy s_i is optimal for type t_i.

For Example 1, consider the epistemic model given in Table 1. We shall check that each type is properly rationalizable.

Type t_1 only deems possible type t_2, and the strategy-type combinations (d, t_2) and (e, t_2) appear at some level of $b_1(t_1)$, so t_1 is cautious. Similarly t_2 only deems possible type t_1, and the strategy-type combinations $((a, f), t_1)$, $((a, g), t_1)$, (b, t_1) and (c, t_1) appear at some level of $b_2(t_2)$, so t_2 is cautious.

Type t_1 believes player 2 is of type t_2, which believes at the first level of $b_2(t_2)$ that player 1 will choose c, and at the second level that player 1 will choose (a, f), in which case the order of preference for player 2 is d, then e, so t_1 respects the opponent's preferences.

Type t_2 believes player 1 is of type t_1, which believes at the first level of $b_1(t_1)$ that player 2 will choose d, in which case the order of preference for player 1 is c, then (a, f), followed by b and finally (a, g), so t_2 respects the opponent's preferences.

Since all the types in the epistemic model are cautious and respect the opponent's preferences, all the types are properly rationalizable. For player 1, c is a strategy that is optimal for t_1, and for player 2, d is a strategy that is optimal for t_2. Therefore c and d are properly rationalizable.

4 Common belief in future and restricted past rationality

Now we turn to dynamic games, and we will define the concept of common belief in future and restricted past rationality. In Section 5 we will connect the concept to proper rationalizability of the normal form.

We first define an epistemic model for a dynamic game, which is rather similar to the definition for normal form games, except the beliefs depend on the information set.

Definition 9 (Epistemic model for a dynamic game).

An epistemic model $\hat{M} = (\hat{T}_i, \beta_i)_{i\in I}$ for a dynamic game G consists of a finite set of types $\hat{T}_i$ for each player i, and for each type $\hat{t}_i \in \hat{T}_i$ and each information set $h \in H_i$ of player i we define a conditional belief $\beta_i(\hat{t}_i, h)$ which is a probability distribution over $S_{-i}(h) \times \hat{T}_{-i}$, the set of strategy-type combinations of i's opponents that lead to $h \in H_i$.

Given a type $\hat{t}_i$, an information set h for player i, and a conditional belief $\beta_i(\hat{t}_i, h)$ we define the expected utility of choosing strategy $s_i \in S_i(h)$ as

$$u_i(s_i, \beta_i(\hat{t}_i, h)) = \sum_{(s_{-i},\hat{t}_{-i})\in S_{-i}\times\hat{T}_{-i}} \beta_i(\hat{t}_i, h)(s_{-i}, \hat{t}_{-i}) u_i(z(s_i, s_{-i})),$$

where $z(s_i, s_{-i})$ is the terminal history reached by (s_i, s_{-i}).

Given a conditional belief $\beta_i(\hat{t}_i, h)$ for type $\hat{t}_i$ at the information set h, a strategy $s_i \in S_i(h)$ is optimal for $\hat{t}_i$ at h if for all $s_i' \in S_i(h)$

$$u_i(s_i, \beta_i(\hat{t}_i, h)) \geq u_i(s_i', \beta_i(\hat{t}_i, h))$$

Now we define the key conditions that will be used: belief in future rationality as has been defined in Perea (2014), and a new notion that we propose, which requires players to think about the past rationality of the opponents, insofar as it concerns the strategies that reach the information set at which the player is. We define both notions separately, then we define common belief in future rationality and common belief in restricted past rationality in an iterative way, to combine them into one concept that refines common belief in future rationality.

Definition 10 (Belief in the opponents' future rationality).

We say that a type $\hat{t}_i$ believes in j's future rationality if at every $h \in H_i$, $\beta_i(\hat{t}_i, h)(s_j, \hat{t}_j) > 0$ only if for every $h' \in H_j(s_j)$ that weakly follows h:

$$u_j(s_j, \beta_j(\hat{t}_j, h')) \geq u_j(s_j', \beta_j(\hat{t}_j, h'))$$

for every $s_j' \in S_j(h')$.

Type $\hat{t}_i$ believes in the opponents' future rationality if $\hat{t}_i$ believes in j's future rationality for all players $j \in I \setminus \{i\}$.

Definition 11 (k-fold and common belief in future rationality).

1. Type $\hat{t}_i$ expresses 1-fold belief in future rationality if $\hat{t}_i$ believes in the opponents' future rationality.

2. For every $k > 1$, type $\hat{t}_i$ expresses k-fold belief in future rationality if at every information set $h \in H_i$, $\hat{t}_i$ only assigns positive probability to opponents' types that express $(k-1)$-fold belief in future rationality.

3. Type $\hat{t}_i$ expresses common belief in future rationality if $\hat{t}_i$ expresses k-fold belief in future rationality for every $k \in \mathbb{N}$.

$\hat{T}_1 = \{\hat{t}_1\}$, $\hat{T}_2 = \{\hat{t}_2\}$

$b_1(\hat{t}_1, \varnothing) = (d, \hat{t}_2)$
$b_1(\hat{t}_1, h_2) = (d, \hat{t}_2)$

$b_2(\hat{t}_2, h_1) = ((a, f), \hat{t}_1)$

Table 2. An epistemic model for the dynamic form of Example 1.

Definition 12 (Belief in the opponents' restricted past rationality)**.**
We say that a type $\hat{t}_i$ believes in j's restricted past rationality if at every $h \in H_i$, $\beta_i(\hat{t}_i, h)(s_j, \hat{t}_j) > 0$ only if for every $h' \in H_j(s_j)$ such that h' weakly precedes h:

$$u_j(s_j, \beta_j(\hat{t}_j, h')) \geq u_j(s'_j, \beta_j(\hat{t}_j, h'))$$

for every $s'_j \in S_j(h') \cap S_j(h)$.

Type $\hat{t}_i$ believes in the opponents' restricted past rationality if $\hat{t}_i$ believes in j's restricted past rationality for all players $j \in I \setminus \{i\}$.

The previous definition establishes that type $\hat{t}_i$ must reason at h about those strategies of his opponents that can be chosen at a previous information set, h', but only if those strategies can reach the information set h too. That is, i considers at h only those strategies at h' that give the highest utility to the opponent at h' from those strategies that actually reach h.

We can define k-fold and common belief in restricted past rationality in an analogous way to the definition of k-fold and common belief in future rationality.

A strategy s_i for player i can rationally be chosen under common belief in future and restricted past rationality if there exists an epistemic model $\hat{M} = (\hat{T}_i, \beta_i)_{i \in I}$ and some type $\hat{t}_i \in \hat{T}_i$ such that $\hat{t}_i$ expresses common belief in future and restricted past rationality, and strategy s_i is optimal for type $\hat{t}_i$ at every information set $h \in H_i(s_i)$.

Returning to Example 1, consider the epistemic model given in Table 2, for which we check that every type expresses common belief in future and restricted past rationality.

At $\varnothing \in H_1$, $\hat{t}_1$ believes that player 2 chooses d and is of type $\hat{t}_2$. Type $\hat{t}_2$ believes at h_1, which weakly follows $\varnothing$, that player 1 chooses (a, f), so the optimal strategy in $S_2(h_1) = \{d, e\}$ for player 2 is d. Therefore $\hat{t}_1$ believes in the opponent's future rationality at $\varnothing$. Since there are no information sets for player 2 that weakly precede $\varnothing$, $\hat{t}_1$ believes in the opponent's restricted past rationality at $\varnothing$.

At $h_2 \in H_1$ there are no information sets for player 2 that weakly follow h_2, so $\hat{t}_1$ believes in the opponent's future rationality at h_2. Now, type $\hat{t}_1$ believes at h_2 that player 2 chooses d and is of type t_2; in fact $S_2(h_1) \cap S_2(h_2) = \{d\}$. Therefore $\hat{t}_1$ believes in the opponent's restricted past rationality at h_2.

At $h_1 \in H_2$, $\hat{t}_2$ believes that player 1 chooses (a, f) and is of type $\hat{t}_1$. Type $\hat{t}_1$ believes at h_2, which weakly follows h_1, that player 2 chooses d at h_1, so the optimal strategy in $S_1(h_2)$ for player 1 is (a, f). Therefore $\hat{t}_2$ believes in the opponent's future rationality. Type $\hat{t}_1$ believes at $\varnothing$, which weakly precedes h_1, that player 2 chooses d at h_1, so the optimal strategy in $S_1(\varnothing) \cap S_1(h_1) = \{(a, f), (a, g), b\}$ for player 1 is (a, f). Therefore $\hat{t}_2$ believes in the opponent's restricted past rationality. We can see that among all strategies in $S_1(\varnothing)$, (a, f) is not optimal for $\hat{t}_1$ at $\varnothing$, as c gives a higher utility.

Since all the types in the epistemic model believe in the opponent's future and restricted past rationality, then all the types express common belief in future and restricted past rationality. For player 1, c is optimal for type $\hat{t}_1$ at information set $\varnothing$, and for player 2, d is optimal for type $\hat{t}_2$ at information set h_1. Therefore c and d can rationally be chosen under common belief in future and restricted past rationality.

5 Connection with proper rationalizability

In this section we prove one of our main theorems, which states that proper rationalizability of a strategy in the normal form implies optimality of the same strategy under common belief in future and restricted past rationality in the dynamic game.

In order to do so, we break down the proof into four smaller parts. We start by showing that optimality of a strategy for a cautious type in the normal form of the game implies optimality of the same strategy for the induced type in the dynamic game. Then we go on to show that respect of the opponent's preferences in the normal form implies belief in the opponent's future and restricted past rationality in the dynamic game. As a consequence, proper rationalizability in the normal form implies common belief in future and restricted past rationality in the dynamic game. This finally implies that every strategy which is properly rationalizable in the normal form can rationally be chosen under common belief in future and restricted past rationality in the dynamic game.

Theorem 1. Consider a dynamic game G. If a strategy s_i is properly rationalizable in the normal form of G, then s_i can rationally be chosen under common belief in future and restricted past rationality in the dynamic game G.

This result has a connection with van Damme (1984), in which proper equilibria in the normal form are shown to imply quasi-perfect equilibria in the dynamic game. The theorem above makes a similar link between proper rationalizability and common belief in future and restricted past rationality.

As a first step, we define a way to transform an epistemic model of the normal form into an epistemic model for the dynamic game.

Let $M = (T_i, b_i)_{i \in I}$ be an epistemic model for the normal form of the game where every type $t_i \in T_i$ is cautious for all $i \in I$. We define the induced epistemic model for the dynamic game $\hat{M} = (\hat{T}_i, \beta_i)_{i \in I}$ in the following way: for each player i take the bijective mapping $f_i \colon T_i \to \hat{T}_i$, effectively a renaming

$T_1 = \{t_1, t_1'\}$, $T_2 = \{t_2, t_2'\}$

$b_1(t_1) = ((d, t_2); (e, t_2))$
$b_1(t_1') = (\frac{1}{4}(d, t_2') + \frac{3}{4}(e, t_2'))$

$b_2(t_2) = ((c, t_1'); ((a, f), t_1'); (b, t_1'); ((a, g), t_1')))$
$b_2(t_2') = (\frac{1}{4}(c, t_1) + \frac{1}{4}(b, t_1) + \frac{1}{2}((a, f), t_1); ((a, g), t_1))$

Table 3. An epistemic model for the normal form.

$\hat{T}_1 = \{\hat{t}_1, \hat{t}_1'\}$, $\hat{T}_2 = \{\hat{t}_2, \hat{t}_2'\}$

$b_1(\hat{t}_1, \varnothing) = (d, \hat{t}_2)$
$b_1(\hat{t}_1, h_2) = (d, \hat{t}_2)$
$b_1(\hat{t}_1', \varnothing) = (\frac{1}{4}(d, \hat{t}_2') + \frac{3}{4}(e, \hat{t}_2'))$
$b_1(\hat{t}_1', h_2) = ((d, \hat{t}_2'))$

$b_2(\hat{t}_2, h_1) = ((a, f), \hat{t}_1')$
$b_2(\hat{t}_2', h_1) = (\frac{1}{3}(b, \hat{t}_1) + \frac{2}{3}((a, f), \hat{t}_1))$

Table 4. The epistemic model of the dynamic game induced by Table 3.

of the types, and let the conditional belief of type $f_i(t_i)$ at the information set $h \in H_i$ be defined as

$$\beta_i(f_i(t_i), h)(s_{-i}, f_{-i}(t_{-i})) = \frac{b_i^k(t_i)(s_{-i}, t_{-i})}{b_i^k(t_i)(S_{-i}(h) \times T_{-i})}$$

where k is the smallest number for which $b_i^k(t_i)(S_{-i}(h) \times T_{-i}) > 0$. Here,

$$b_i^k(t_i)(S_{-i}(h) \times T_{-i}) = \sum_{(s_{-i}, t_{-i}) \in S_{-i}(h) \times T_{-i}} b_i^k(s_{-i}, t_{-i}),$$

that is, we take the first level k of the lexicographic belief for t_i in which there is at least one strategy combination for i's opponents that reaches h, and normalize the probabilities accordingly.

To illustrate how to transform cautious lexicographic beliefs into conditional beliefs, we the game from Figure 1. Suppose the epistemic model for its normal form is the one in Table 3, then the epistemic model induced for the dynamic game is the one in Table 4.

Now that we have a way to relate epistemic models of the normal form with those of the dynamic game, we will see how the rationalizability concepts relate to each other. First we show that optimality of a strategy for a cautious type in the normal form of the game implies optimality of the same strategy for the induced type in the dynamic game. This is presented in the following lemma.

Lemma 1. Let M be an epistemic model of the normal form in which all types are cautious, $h \in H_i$, h' an information set that weakly follows or weakly precedes h, and t_i a type for player i in M. If $s_i \in S_i(h) \cap S_i(h')$ is not optimal for $f_i(t_i)$ among strategies in $S_i(h) \cap S_i(h')$ at $h \in H_i$, then there exists $\hat{s}_i \in S_i(h) \cap S_i(h')$ such that t_i prefers $\hat{s}_i$ to s_i.

The optimality implication described above will be very useful to show the relations between the rationalizability concepts that we are studying. The next step is to show that respect of preferences in the normal form of the game implies belief in future and restricted past rationality.

Lemma 2. If t_i respects player j's preferences, then $f_i(t_i)$ believes in j's future and restricted past rationality.

And also, the notion of proper rationalizability implies common belief in future and restricted past rationality.

Lemma 3. If t_i is properly rationalizable, then $f_i(t_i)$ expresses common belief in future and restricted past rationality.

Since we know that for every normal form game there exists at least one properly rationalizable type for every player (cf. Asheim (2001), Perea (2012)), then Lemma 3 implies the following result.

Corollary 1. For every dynamic game G there exists for every player i an epistemic model M and a type $\hat{t}_i$ in it that expresses common belief in future and restricted past rationality.

Once we have all of these results, Lemma 1 and Lemma 3 imply Theorem 1. Therefore, if we transform a dynamic game into its normal form and proceed to find an epistemic model in which the types express proper rationalizability, we can find an induced epistemic model for the dynamic game in which the types express common belief in future and restricted past rationality.

We can check that the epistemic model in Table 2 is induced by the epistemic model in Table 1 via the transformation described before, and we have seen that all types in Table 1 are properly rationalizable. Since strategy c is optimal for type t_1 and strategy d is optimal for type t_2, both strategies can rationally be chosen under common belief in future and restricted past rationality by Theorem 1.

As we can see, at information sets $\varnothing$ and h_2, type t_1 of player 1 believes type t_2 of player 2 will be and has been rational. However, if the game reaches information set h_1, this means player 1 was not rational before, nevertheless, player 2 believes that if h_1 was reached, then player 1 is choosing optimally among strategies that lead to h_1, therefore, type t_2 believes that player 1 will choose (a, f). Hence, player 2 can only rationally choose d under common belief in future and restricted past rationality.

Under common strong belief in rationality, if player 2 sees that h_1 has been reached, then, if possible, he must believe that player 1 made a choice that is rational at $\varnothing$. But choosing c at $\varnothing$ gives the highest utility for player 1, so it is not possible for player 2 to believe that player 1 made a rational choice under common strong belief in rationality. Therefore, player 2 can believe player 1 chose any strategy that leads to h_1, so both d and e can rationally be chosen at h_1 under common strong belief in rationality.

Under common belief in future rationality, if player 2 sees that h_1 was reached, then he may believe that player 1 chose irrationally at $\varnothing$, but he must believe that from now on, player 1 will choose rationally. Therefore, player 2 can believe player 1 chose a or b at $\varnothing$, so both d and e can rationally be chosen under common belief in future rationality.

6 Algorithm

In order to find the strategies that can rationally be chosen under common belief in future and restricted past rationality, we propose an algorithm based on the backward dominance procedure proposed in Perea (2014). Then we show that the strategies that survive the full implementation of the algorithm are exactly those strategies that can be chosen under common belief in future and restricted past rationality.

Definition 13 (Full and reduced decision problems at an information set).

Let $h \in H_i$ be an information set for player i. The pair $\Gamma^0(h) = (S_i^0(h), S_{-i}^0(h))$ is called the full decision problem for player i at h, where $S_i^0(h) = S_i(h)$ and $S_{-i}^0(h) = S_{-i}(h)$. A pair $\Gamma^k(h) = (S_i^k(h), S_{-i}^k(h))$ is a reduced decision problem for player i at h, with $S_i^k(h) \subseteq S_i^0(h)$ and $S_{-i}^k(h) \subseteq S_{-i}(h)$.

Definition 14 (Strict dominance by a randomization).

Let $h \in H_i$ be an information set for player i, and $\Gamma^k(h) = (S_i^k(h), S_{-i}^k(h))$ be a reduced decision problem for player i at h. A strategy $s_i \in S_i^k$ is strictly dominated on $S_{-i}^k(h)$ by a randomization on $A_i \subseteq S_i^k(h)$ if there is $\rho_i \in \Delta(A_i)$ such that

$$\sum_{s_i' \in A_i} \rho_i(s_i') u_i(z(s_i', s_{-i})) > u_i(z(s_i, s_{-i}))$$

for all $s_{-i} \in S_{-i}^k(h)$.

Algorithm 1. Set $S_i^0(h) = S_i(h)$ and $S_{-i}^0(h) = S_{-i}(h)$ for all $i \in I$ and all $h \in H_i$. For every $k \geq 1$ we have:

Step k: For every player i and every information set $h \in H_i$, we define

$$S_i^k(h) = \{s_i \in S_i^{k-1}(h) \mid s_i \text{ is not strictly dominated on } S_{-i}^{k-1}(h) \text{ by a randomization on } S_i(h)\},$$

$$S_{-i}^k(h) = \{(s_j)_{j \neq i} \in S_{-i}^{k-1}(h) \mid \text{for all } j \neq i,\ s_j \text{ is not strictly dominated on } S_{-j}^{k-1}(h') \text{ by a randomization on } S_j(h')$$

for every $h' \in H_j(s_j)$ weakly following h,
and s_j is not strictly dominated on $S^{k-1}_{-j}(h'')$
by a randomization on $S_j(h) \cap S_j(h'')$
at every $h'' \in H_j(s_j)$ weakly preceding $h\}$.

The algorithm ends after K steps if $S_i^{K+1}(h) = S_i^K(h)$ and $S_{-i}^{K+1}(h) = S_{-i}^K(h)$ for every $i \in I$ and every $h \in H_i$.

Now we have the following result showing that the algorithm identifies the strategies that can be chosen under k-fold belief in future and restricted past rationality, and those that can be chosen under common belief in future and restricted past rationality.

Theorem 2. For every $k \geq 1$ the strategies that can rationally be chosen by a type that expresses up to k-fold belief in future and restricted past rationality are exactly the strategies s_i such that $s_i \in S_i^{k+1}(h)$ for all $h \in H_i(s_i)$, surviving the first $k+1$ steps of the algorithm.

The strategies that can rationally be chosen by a type that expresses common belief in future and restricted past rationality are exactly the strategies that survive the full algorithm, that is, the strategies s_i such that $s_i \in S_i^k(h)$ for all $k \geq 1$ and all $h \in H_i(s_i)$.

To illustrate the algorithm, we use the game from Figure 1. We have that $H_1 = \{\varnothing, h_2\}$ and $H_2 = \{h_1\}$ and the initial sets of strategies:

$$
\begin{array}{ll}
S_1^0(\varnothing) = \{(a,f),(a,g),b,c\} & S_{-1}^0(\varnothing) = \{d,e\} \\
S_2^0(h_1) = \{d,e\} & S_{-2}^0(h_1) = \{(a,f),(a,g),b\} \\
S_1^0(h_2) = \{(a,f),(a,g)\} & S_{-1}^0(h_2) = \{d\}
\end{array}
$$

After the first step is applied, we obtain the following reduced decision problems:

$$
\begin{array}{ll}
S_1^1(\varnothing) = \{c\} & S_{-1}^1(\varnothing) = \{d,e\} \\
S_2^1(h_1) = \{d,e\} & S_{-2}^1(h_1) = \{(a,f)\} \\
S_1^1(h_2) = \{(a,f)\} & S_{-1}^1(h_2) = \{d\}
\end{array}
$$

Observe that at $\varnothing$, b is strictly dominated by $(a,f) \in S_1^0(h_1) \cap S_1^0(\varnothing)$. We also have that at h_2, (a,g) is strictly dominated by $(a,f) \in S_1^0(h_2)$. Therefore the only strategy that remains in $S_{-2}^1(h_1)$ is (a,f).

At the second iteration of the algorithm we obtain:

$$
\begin{array}{ll}
S_1^2(\varnothing) = \{c\} & S_{-1}^2(\varnothing) = \{d\} \\
S_2^2(h_1) = \{d\} & S_{-2}^2(h_1) = \{(a,f)\} \\
S_1^2(h_2) = \{(a,f)\} & S_{-1}^2(h_2) = \{d\}
\end{array}
$$

We see that at h_1, e is strictly dominated on $S^1_{-2}(h_1)$ by d, so the only strategy in $S^2_{-1}(\varnothing)$ and $S^2_2(h_1)$ is d.

Since all the sets are singletons, the algorithm stops. Therefore the surviving strategies are c for player 1 and d for player 2, which are exactly the strategies that we found in Section 4 as those that can be chosen under common belief in future and restricted past rationality.

7 Concluding Remarks

A new reasoning concept for dynamic games was introduced, which not only assumes rationality of the opponents in the future, but also assumes players reason about what happened in the past in the following way: if the game reaches an information set, players should consider only those strategies that actually reach that information set and believe that the opponent has chosen rationally in the past among that restricted set of strategies. In this way, players are reasoning at every information set about the past, but only a restricted part of it. We have also presented the fact that common belief in future and restricted past rationality can be obtained from using proper rationalizability in the normal form of the dynamic game, which shows that some reasoning concepts for dynamic games can be obtained if we study particular concepts used for normal form games. Additionally, it was possible to define a procedure that starts from the decision problems in the dynamic game, and using strict dominance, selects the strategies that can be chosen under common belief in future and restricted past rationality.

Some future research that can stem from the results obtained here would include the application of this concept to other classes of games such as including infinite games, repeated games and stochastic games, as well as finding an algorithm in each case that finds the choices that can be made under common belief in future and restricted past rationality.

Another problem that could be investigated in future work is whether we can find an equilibrium analogue to common belief in future and restricted past rationality, and how it would relate to existing equilibrium concepts for dynamic games. Such a search for an equilibrium analogue could be based on Perea and Predtetchinski (2017) who have shown that for stochastic dynamic games with perfect information, subgame perfect equilibrium is equivalent to common belief in future rationality with a correct beliefs assumption. Since players have perfect information, the addition of restricted past rationality does not affect the result, so a natural extension would be to study the case of dynamic games with imperfect information.

Perea (2017) has proven that for finite dynamic games, the outcomes obtained under common strong belief in rationality are also reachable under common belief in future rationality, proving that common strong belief in rationality is a more restrictive concept in terms of outcomes. It would be interesting to study the relation in terms of outcomes of the concept of common strong belief in rationality and common belief in future and restricted past rationality.

8 Proofs

8.1 Proofs for Section 5

Proof (Lemma 1)**.** Let $s_i \in S_i(h) \cap S_i(h')$ be a suboptimal choice for $f_i(t_i)$ among strategies in $S_i(h) \cap S_i(h')$ at h. Then there is at least one $s'_i \in S_i(h) \cap S_i(h')$ such that

$$u_i(s'_i, \beta_i(f_i(t_i), h)) > u_i(s_i, \beta_i(f_i(t_i), h)). \tag{$*$}$$

Define $\hat{s}_i$ as

$$\hat{s}_i(h'') = s_i(h'') \text{ for all } h'' \in H_i(s_i) \text{ if } h'' \text{ does not weakly follow } h, \tag{1a}$$

$$\hat{s}_i(h'') = s'_i(h'') \text{ for all } h'' \in H_i(s'_i) \text{ if } h'' \text{ weakly follows } h. \tag{1b}$$

First we show that $\hat{s}_i \in S_i(h) \cap S_i(h')$.

Since $s_i \in S_i(h)$, there is $s_{-i} \in S_{-i}(h)$ such that (s_i, s_{-i}) reaches h. Then at every $h'' \in H(s_i, s_{-i})$ such that h follows h'', we have $\hat{s}_i(h'') = s_i(h'')$. Hence $h \in H(\hat{s}_i, s_{-i})$ and $\hat{s}_i \in S_i(h)$.

To show that $\hat{s}_i \in S_i(h')$ we distinguish two cases: whether h' weakly precedes h or h' weakly follows h.

If h' weakly precedes h, then $\hat{s}_i \in S_i(h')$ since $\hat{s}_i \in S_i(h)$.

Assume now that h' weakly follows h. Since $s'_i \in S_i(h')$, there is $s_{-i} \in S_{-i}(h')$ such that (s'_i, s_{-i}) reaches h'. Then at every $h'' \in H(s'_i, s_{-i})$ weakly following h and weakly followed by h' we have by definition $\hat{s}_i(h'') = s'_i(h'')$, and at every $h'' \in H(s'_i, s_{-i})$ such that h follows h'' we know that $\hat{s}_i(h'') = s_i(h'')$. But by perfect recall of player i, there exists a unique choice $c^*_i(h'')$ at the information set h'' such that h can be reached. Since both $s_i, s'_i \in S_i(h)$, both strategies must choose $c^*_i(h'')$. Therefore $s_i(h'') = s'_i(h'')$ for all h'' such that h follows h''.

Hence, $\hat{s}_i(h'') = s'_i(h'')$ at every $h'' \in H(s'_i, s_{-i})$ such that h weakly follows h''. Since we have seen that $\hat{s}_i(h'') = s'_i(h'')$ for all $h'' \in H(s'_i, s_{-i})$ weakly following h and weakly preceding h', the strategy combination $(\hat{s}_i, s_{-i})$ reaches h', and $\hat{s}_i \in S_i(h')$.

By the two results above, we have that $\hat{s}_i \in S_i(h) \cap S_i(h')$.

Now we will show that t_i prefers $\hat{s}_i$ to s_i. Let $b_i(t_i) = (b^1_i(t_i); b^2_i(t_i); \ldots; b^m_i(t_i))$ be the cautious lexicographic belief for type t_i. Let k be the smallest number such that $b^k_i(t_i)(S_{-i}(h) \times T_{-i}) > 0$.

For $\ell < k$, $b^\ell_i(t_i)(S_{-i}(h)) = 0$. Hence by (1a):

$$v^\ell_i(\hat{s}_i, b_i(t_i)) = v^\ell_i(s_i, b_i(t_i))$$

for all $\ell < k$. Moreover

$$\begin{aligned} v^k_i(\hat{s}_i, b_i(t_i)) &= \sum_{(s_{-i}, t_{-i}) \in S_{-i} \times T_{-i}} b^k_i(t_i)(s_{-i}, t_{-i}) v_i(\hat{s}_i, s_{-i}) \\ &= \sum_{(s_{-i}, t_{-i}) \in S_{-i}(h) \times T_{-i}} b^k_i(t_i)(s_{-i}, t_{-i}) v_i(\hat{s}_i, s_{-i}) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{(s_{-i},t_{-i})\in(S_{-i}\setminus S_{-i}(h))\times T_{-i}} b_i^k(t_i)(s_{-i},t_{-i})v_i(\hat{s}_i,s_{-i})\\
&= \sum_{(s_{-i},t_{-i})\in S_{-i}(h)\times T_{-i}} b_i^k(t_i)(s_{-i},t_{-i})v_i(s_i',s_{-i})\\
&\quad + \sum_{(s_{-i},t_{-i})\in(S_{-i}\setminus S_{-i}(h))\times T_{-i}} b_i^k(t_i)(s_{-i},t_{-i})v_i(s_i,s_{-i})\\
&= b_i^k(t_i)(S_{-i}(h)\times T_{-i})\\
&\quad \times \sum_{(s_{-i},t_{-i})\in S_{-i}(h)\times T_{-i}} \beta_i(f_i(t_i),h)(s_{-i},f_{-i}(t_{-i}))u_i(z(s_i',s_{-i}))\\
&\qquad + \sum_{(s_{-i},t_{-i})\in(S_{-i}\setminus S_{-i}(h))\times T_{-i}} b_i^k(t_i)(s_{-i},t_{-i})v_i(s_i,s_{-i})\\
&= b_i^k(t_i)(S_{-i}(h)\times T_{-i})u_i(s_i',\beta_i(f_i(t_i),h))\\
&\quad + \sum_{(s_{-i},t_{-i})\in(S_{-i}\setminus S_{-i}(h))\times T_{-i}} b_i^k(t_i)(s_{-i},t_{-i})v_i(s_i,s_{-i})\\
&> b_i^k(t_i)(S_{-i}(h)\times T_{-i})u_i(s_i,\beta_i(f_i(t_i),h))\\
&\quad + \sum_{(s_{-i},t_{-i})\in(S_{-i}\setminus S_{-i}(h))\times T_{-i}} b_i^k(t_i)(s_{-i},t_{-i})v_i(s_i,s_{-i})\\
&= \sum_{(s_{-i},t_{-i})\in S_{-i}(h)\times T_{-i}} b_i^k(t_i)(s_{-i},t_{-i})v_i(s_i,s_{-i})\\
&\quad + \sum_{(s_{-i},t_{-i})\in(S_{-i}\setminus S_{-i}(h))\times T_{-i}} b_i^k(t_i)(s_{-i},t_{-i})v_i(s_i,s_{-i})\\
&= v_i^k(s_i,b_i(t_i)).
\end{aligned}
$$

where (1a) and (1b) have been used in the third equality, and the inequality is obtained using ($*$) and the fact that $b_i^k(t_i)(S_{-i}(h)\times T_{-i})>0$. Hence we have the result we wanted to prove. ■

Proof (Lemma 2)**.** First we prove that respect of preferences implies belief in future rationality.

Let $h\in H_i$. Suppose $f_i(t_i)$ does not believe at h in player j's future rationality. Then

$$\beta_i(f_i(t_i),h)(s_j,f_j(t_j))>0$$

for some $s_j\in S_j(h')$ that is a suboptimal strategy for $f_j(t_j)$ at some h' that weakly follows h.

By Lemma 1 there exists $\hat{s}_j\in S_j(h)\cap S_j(h')$ such that t_j prefers $\hat{s}_j$ to s_j. By the hypothesis, t_i respects j's preferences, so it must deem $(\hat{s}_j,t_j)$ infinitely more likely than (s_j,t_j). Since $\hat{s}_j\in S_j(h)$ then

$$\beta_i(f_i(t_i),h)(s_j,f_j(t_j))=0$$

by construction of the conditional belief at h. But this is a contradiction. Therefore, $f_i(t_i)$ believes at h in player j's future rationality for all $h\in H_i$.

Now we prove with a similar argument that respect of preferences implies belief in restricted past rationality.

Let $h \in H_i$. Suppose $f_i(t_i)$ does not believe at h in player j's restricted past rationality. Then

$$\beta_i(f_i(t_i), h)(s_j, f_j(t_j)) > 0$$

for some $s_j \in S_j(h) \cap S_j(h'')$ that is a suboptimal strategy for $f_j(t_j)$ among strategies in $S_j(h) \cap S_j(h'')$ at h'' that weakly precedes h. By Lemma 1 there exists $\hat{s}_j \in S_j(h) \cap S_j(h'')$ such that t_j prefers $\hat{s}_j$ to s_j. By the hypothesis, t_i respects j's preferences, so it must deem $(\hat{s}_j, t_j)$ infinitely more likely than (s_j, t_j). Since $\hat{s}_j \in S_j(h)$, then by construction of the conditional belief at h

$$\beta_i(f_i(t_i), h)(s_j, f_j(t_j)) = 0$$

which is a contradiction. Therefore $f_i(t_i)$ believes at h in player j's restricted past rationality. ■

We define the set $T^*(t_i)$ as the set of types in t_i's belief hierarchy in the normal form, that is, $T^*(t_i)$ is the smallest set with the property that $t_i \in T^*(t_i)$, and for every $t_j \in T^*(t_i)$, if t_j deems possible t_k, then $t_k \in T^*(t_i)$.

Similarly we define $\hat{T}^*(\hat{t}_i)$ as the set of types in $\hat{t}_i$'s belief hierarchy in the dynamic form. More precisely, $\hat{T}^*(\hat{t}_i)$ is the smallest set such that $\hat{t}_i \in \hat{T}^*(\hat{t}_i)$ and for every $\hat{t}_j \in \hat{T}^*(\hat{t}_i)$, if $\beta_j(\hat{t}_j, h)(s_k, t_k) > 0$ for some $h \in H_j$, then $t_k \in \hat{T}^*(\hat{t}_i)$.

Proof (Lemma 3)**.** Let $t_i \in T_i$ and construct the set $T^*(t_i)$. Since t_i is properly rationalizable, every type in $T^*(t_i)$ is cautious and respects the opponents' preferences.

By construction, every type in $T^*(t_i)$ induces a type in $\hat{T}^*(f_i(t_i))$. It then follows, by Lemma 2, that all types in $\hat{T}^*(f_i(t_i))$ believe in the opponents' future and restricted past rationality.

Then by definition, since all of the types in $\hat{T}^*(f_i(t_i))$ only refer to types in $\hat{T}^*(f_i(t_i))$, all express common belief in future and restricted past rationality.

Hence, in particular, $f_i(t_i)$ expresses common belief in future and restricted past rationality. ■

Proof (Theorem 1)**.** Since s_i is properly rationalizable, there is a type t_i that is properly rationalizable such that s_i is optimal for t_i. By Lemma 3, $f_i(t_i)$ expresses common belief in future and restricted past rationality.

Now we show that s_i is also optimal for type $f_i(t_i)$ at every information set $h \in H_i(s_i)$.

Suppose that s_i is suboptimal for $f_i(t_i)$ at information set h. By Lemma 1, choosing $h' = h$, there is a strategy $\hat{s}_i \in S_i(h)$ such that t_i prefers $\hat{s}_i$ to s_i. Then s_i is not an optimal strategy for t_i, which is a contradiction. ■

8.2 Proofs for Section 6

Before we prove Theorem 2 we require some auxiliary results, and the construction of an epistemic model according to the algorithm, which will have the desired properties.

We state the following result, first proved in Pearce (1984) for games with two players. A general proof can be found in Perea (2012).

Theorem 3 (Pearce's lemma)**.** Consider a reduced decision problem $\Gamma^k(h) = (S_i^k(h), S_{-i}^k(h))$, $A_i \subseteq S_i^k(h)$ and $s_i \in A_i$. Then s_i is optimal among strategies in A_i for some belief $b_i \in \Delta(S_{-i}^k(h))$ if and only if s_i is not strictly dominated on $S_{-i}^k(h)$ by a randomization on A_i.

For $i \in I$, $h \in H_i$ and $k \geq 1$ let $B_{-i}^k(h)$ be the set of opponents' strategy combinations $(s_j)_{j \neq i} \in S_{-i}(h)$ such that there is some type t_i expressing up to k-fold belief in future and restricted past rationality that at h assigns positive probability to s_{-i}.

Lemma 4. For every player $i \in I$, every information set $h \in H_i$ and every $k \geq 1$ we have that $B_{-i}^k(h) \subseteq S_{-i}^k(h)$.

Proof. We prove this statement by induction on k.

Let $k = 1$. Consider a player $i \in I$, an information set $h \in H_i$ and let $(s_j)_{j \neq 1} \in B_{-i}^1(h)$. Then there is a type t_i expressing up to 1-fold belief in future and restricted past rationality such that t_i assigns positive probability to $(s_j)_{j \neq i}$ at h.

Now consider an opponent $j \neq i$. Since t_i believes in j's future and restricted past rationality, then for every $h' \in H_j(s_j)$ weakly following h we can find a conditional belief $\beta_j(t_j, h')$ for which s_j is optimal among strategies in $S_j(h')$, and for every $h'' \in H_j(s_j)$ weakly preceding h we can find a conditional belief $\beta_j(t_j, h'')$ for which s_j is optimal among strategies in $S_j(h) \cap S_j(h'')$.

Then by Pearce's lemma, for every $h' \in H_j(s_j)$ weakly following h, s_j is not strictly dominated on $S_{-j}^0(h')$ by a randomization on $S_j(h')$ and for every $h'' \in H_j(s_j)$ weakly preceding h, s_j is not strictly dominated on $S_{-j}^0(h'')$ by a randomization on $S_j(h) \cap S_j(h'')$. Therefore $s_{-i} \in S_{-i}^1(h)$. Hence $B_{-i}^1(h) \subseteq S_{-i}^1(h)$, and this is true for all players $i \in I$ and every information set $h \in H_i$.

Now we proceed with the induction step. Fix $k \geq 2$ and assume that for every player $i \in I$ and every information set $h \in H_i$, $B_{-i}^{k-1}(h) \subseteq S_{-i}^{k-1}(h)$.

Consider a player i, and let $(s_j)_{j \neq i} \in B_{-i}^k(h)$. Then there is a type t_i that expresses up to k-fold belief in future and restricted past rationality such that t_i assigns positive probability to $(s_j)_{j \neq i}$ at h.

Take an opponent $j \neq i$. Then there must be some type t_j expressing up to $(k-1)$-fold belief in future and restricted past rationality such that s_j is optimal for t_j at every $h' \in H_j(s_j)$ weakly following h among strategies in $S_j(h')$, and at every $h'' \in H_j(s_j)$ weakly preceding h among strategies in $S_j(h) \cap S_j(h'')$.

By the induction assumption, since t_j assigns at every $h' \in H_j$ positive probability only to opponents' strategies in $B_{-j}^{k-1}(h')$, then t_j must assign, at every $h' \in H_j$ positive probability only to opponents' strategies in $S_{-j}^{k-1}(h')$. Then s_j is optimal at every $h' \in H_j(s_j)$ weakly following h among strategies in $S_j(h')$ for some conditional belief $\beta_j(t_j, h')$ on $S_{-j}^{k-1}(h')$, and at every $h'' \in H_j(s_j)$ weakly preceding h among strategies in $S_j(h) \cap S_j(h'')$ for some conditional belief $\beta_j(t_j, h'')$ on $S_{-j}^{k-1}(h'')$. Therefore by Pearce's lemma, at every $h' \in H_j(s_j)$

weakly following h, s_j is not strictly dominated on $S_{-j}^{k-1}(h')$ by a randomization on $S_j(h')$, and at every $h'' \in H_j(s_j)$ weakly preceding h, s_j is not strictly dominated on $S_{-j}^{k-1}(h'')$ by a randomization on $S_j(h) \cap S_j(h'')$. Hence, $s_{-i} \in S_{-i}^k(h)$. Then $B_{-i}^k(h) \subseteq S_{-i}^k(h)$ and this is true for every player $i \in I$ and every information set $h \in H_i$. ■

For $i \in I$ and $k \geq 1$ let BR_i^k be the set of strategies for player i that are optimal for some type that expresses up to k-fold belief in future and restricted past rationality. We also define $S_i^k = \{s_i \in S_i \mid s_i \in S_i^k(h) \text{ for all } h \in H_i(s_i)\}$.

Lemma 5. For every player $i \in I$ and every $k \geq 1$, $BR_i^k \subseteq S_i^{k+1}$.

Proof. Fix $i \in I$ and $k \geq 1$. Let $s_i \in BR_i^k$, then there is a type t_i that expresses up to k-fold belief in future and restricted past rationality such that s_i is optimal for t_i at every $h \in H_i(s_i)$. By definition, at every $h \in H_i(s_i)$, t_i assigns positive probability to s_{-i} only if $s_{-i} \in B_{-i}^k(h)$. By Lemma 4, at every $h \in H_i(s_i)$, t_i assigns positive probability to s_{-i} only if $s_{-i} \in S_{-i}^k(h)$. Therefore, s_i is optimal at $h \in H_i(s_i)$ for some conditional belief $\beta_i(t_i, h)$ on $S_{-i}^k(h)$. Hence by Pearce's lemma, s_i is not strictly dominated at $h \in H_i(s_i)$ on $S_{-i}^k(h)$ by a randomization on $S_i(h)$. This implies that s_i survives step $k+1$ of the algorithm, that is, $s_i \in S_i^{k+1}$. Then $BR_i^k \subseteq S_i^{k+1}$ and this holds for every player $i \in I$ and every $k \geq 1$. ■

For every $i \in I$, $h \in H$ and $k \geq 1$ we define $R_i^k(h)$ as the set of strategies $s_i \in S_i(h)$ such that s_i is not strictly dominated on $S_{-i}^{k-1}(h')$ at every $h' \in H_i(s_i)$ weakly following h among strategies in $S_i(h')$, and s_i is not strictly dominated on $S_{-i}^{k-1}(h'')$ at every $h'' \in H_i(s_i)$ weakly preceding h among strategies in $S_i(h) \cap S_i(h'')$. Notice that $R_i^k(h) \subseteq S_i^k(h)$ for all $i \in I$, $h \in H_i$ and $k \geq 1$.

Suppose that the algorithm ends after K steps, that is $S_i^{K+1}(h) = S_i^K(h)$ and $S_{-i}^{K+1}(h) = S_{-i}^K(h)$ for every player $i \in I$ and every information set $h \in H_i$. In order to prove that $S_i^{k+1} \subseteq BR_i^k$ we construct an epistemic model with the following characteristics:

1. For every information set h, every player i and every strategy $s_i \in R_i^1(h)$ there is a type $t_i^{s_i,h}$ such that s_i is optimal for $t_i^{s_i,h}$ at every $h' \in H_i(s_i)$ weakly following h among strategies in $S_i(h')$ and at every $h'' \in H_i(s_i)$ weakly preceding h among strategies in $S_i(h) \cap S_i(h'')$.

2. For every $k \geq 2$, if $s_i \in R_i^k(h)$ then the associated type $t_i^{s_i,h}$ expresses up to $(k-1)$-fold belief in future and restricted past rationality.

3. If $s_i \in R_i^K(h)$ then the associated type $t_i^{s_i,h}$ expresses common belief in future and restricted past rationality.

Construction of the epistemic model

We start with the construction of beliefs for the model. For $i \in I$ take an information set $h \in H$ and let $D_i^k(h) = R_i^k(h) \setminus R_i^{k+1}(h)$ for all $k \geq 1$.

Consider $k \in \{1, 2, \dots, K-1\}$ and $s_i \in D_i^k(h)$. By definition and Pearce's lemma, for every $h' \in H_i(s_i)$ weakly following h there is a conditional belief $\beta_i^{s_i,h}(h')$ on $S_{-i}^{k-1}(h')$ such that s_i is optimal for $\beta_i^{s_i,h}(h')$ among strategies in $S_i(h')$, and for every $h'' \in H_i(s_i)$ weakly preceding h there is a conditional belief $\beta_i^{s_i,h}(h'')$ on $S_{-i}^{k-1}(h'')$ such that s_i is optimal for $\beta_i^{s_i,h}(h'')$ among strategies in $S_i(h) \cap S_i(h'')$. For every other $h''' \in H_i$, define $\beta_i^{s_i,h}(h''')$ on $S_{-i}^{k-1}(h''')$ arbitrarily.

Consider $s_i \in R_i^K(h)$. Then $s_i \in R_i^{K+1}(h)$ as well. By definition of $R_i^{K+1}(h)$, for every $h' \in H_i(s_i)$ weakly following h there is a conditional belief $\beta_i^{s_i,h}(h')$ on $S_{-i}^K(h')$ such that s_i is optimal for $\beta_i^{s_i,h}(h')$ among strategies in $S_i(h')$, and for every $h'' \in H_i(s_i)$ weakly preceding h there is a conditional belief $\beta_i^{s_i,h}(h'')$ on $S_{-i}^K(h'')$ such that s_i is optimal for $\beta_i^{s_i,h}(h'')$ among strategies in $S_i(h) \cap S_i(h'')$. For every other $h''' \in H_i$, define $\beta_i^{s_i,h}(h''')$ on $S_{-i}^K(h''')$ arbitrarily.

Now we proceed with the construction of types for the epistemic model. For player $i \in I$ we define the set of types $T_i = \{t_i^{s_i,h} \mid h \in H \text{ and } s_i \in R_i^1(h)\}$. For every player $i \in I$, every information set $h \in H$ and every $k \in \{1, \dots, K\}$ let $T_i^k(h) = \{t_i^{s_i,h} \mid s_i \in R_i^k(h)\}$. Since $R_i^K(h) \subseteq R_i^{K-1}(h) \subseteq \cdots \subseteq R_i^2(h) \subseteq R_i^1(h)$, then $T_i^K(h) \subseteq T_i^{K-1}(h) \subseteq \cdots \subseteq T_i^2(h) \subseteq T_i^1(h)$ for every player $i \in I$ and every information set $h \in H$.

For every player $i \in I$ and every information set $h \in H$ we now construct the beliefs for each type in $T_i^1(h)$.

Consider $t_i^{s_i,h}$ with $s_i \in D_i^1(h)$, that is, $t_i^{s_i,h} \in T_i^1(h) \setminus T_i^2(h)$. We define the conditional belief vector $\beta_i(t_i^{s_i,h})$ in the following way: For each $j \neq i$ take an arbitrary type $\hat{t}_j$ and consider an information set $h' \in H_i$. Let

$$\beta_i(t_i^{s_i,h}, h')((s_j, t_j)_{j \neq i}) = \begin{cases} \beta_i^{s_i,h}(h')((s_j)_{j \neq i}) & \text{if } t_j = \hat{t}_j \text{ for every } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

Then at every $h' \in H_i$, type $t_i^{s_i,h}$ holds the same belief about the opponents' strategy choices as $\beta_i^{s_i,h}$. By construction of the beliefs, s_i is optimal for $\beta_i^{s_i,h}(h')$ at every $h' \in H_i(s_i)$ weakly following h among strategies in $S_i(h')$ and s_i is optimal for $\beta_i^{s_i,h}(h'')$ at every $h'' \in H_i(s_i)$ weakly preceding h among strategies in $S_i(h) \cap S_i(h'')$.

Therefore s_i is optimal for type $t_i^{s_i,h}$ at every $h' \in H_i(s_i)$ weakly following h among strategies in $S_i(h')$ and at every $h'' \in H_i(s_i)$ weakly preceding h among strategies in $S_i(h) \cap S_i(h'')$.

Now consider $t_i^{s_i,h}$ with $s_i \in D_i^k(h)$ for some $k \in \{2, 3, \dots, K-1\}$. Hence $t_i^{s_i,h} \in T_i^k(h) \setminus T_i^{k+1}(h)$. We define the conditional belief vector $\beta_i(t_i^{s_i,h})$ as follows: For every information set $h' \in H_i$ let $\beta_i(t_i^{s_i,h}, h')$ be the conditional belief at h' about the opponents' strategy-type pairs given by:

$$\beta_i(t_i^{s_i,h}, h')((s_j, t_j)_{j \neq i}) = \begin{cases} \beta_i^{s_i,h}(h')((s_j)_{j \neq i}) & \text{if } t_j = t_j^{s_j,h'} \text{ for every } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

By construction of the beliefs, strategy s_i is optimal for $\beta_i^{s_i,h}(h')$ at every $h' \in H_i(s_i)$ weakly following h among strategies in $S_i(h')$ and s_i is optimal for $\beta_i^{s_i,h}(h'')$ at every $h'' \in H_i(s_i)$ weakly preceding h among strategies in $S_i(h) \cap S_i(h'')$. Therefore s_i is optimal for type $t_i^{s_i,h}$ at every $h' \in H_i(s_i)$ weakly following h among strategies in $S_i(h')$ and at every $h'' \in H_i(s_i)$ weakly preceding h among strategies in $S_i(h) \cap S_i(h'')$.

Since at every $h' \in H_i$ the belief $\beta_i^{s_i,h}(h') \in \Delta(S_{-i}^{k-1}(h'))$ and $S_{-i}^{k-1}(h') = \times_{j \neq i} R_j^{k-1}(h')$, then $\beta_i^{s_i,h}(h')$ assigns positive probability only to opponents' strategies in $R_j^{k-1}(h')$. Hence type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' types $t_j^{s_j,h'}$ where $s_j \in R_j^{k-1}(h')$. That is, type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' types in $T_j^{k-1}(h')$.

Finally, consider types $t_i^{s_i,h}$ with $s_i \in R_i^K(h)$, that is, $t_i^{s_i,h} \in T_i^K(h)$. We define the conditional belief vector $\beta_i(t_i^{s_i,h})$ as follows: For every $h' \in H_i$ let $\beta_i(t_i^{s_i,h}, h')$ be the conditional belief at h' about the opponents' strategy-type pairs given by:

$$\beta_i(t_i^{s_i,h}, h')((s_j, t_j)_{j \neq i}) = \begin{cases} \beta_i^{s_i,h}(h')((s_j)_{j \neq i}) & \text{if } t_j = t_j^{s_j,h'} \text{ for every } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

This way, for every $h' \in H_i$, type $t_i^{s_i,h}$ holds the same belief about the opponents' strategy choices as $\beta_i^{s_i,h}$. By construction, strategy s_i is optimal for $\beta_i^{s_i,h}(h')$ at every $h' \in H_i(s_i)$ weakly following h among strategies in $S_i(h')$ and s_i is optimal for $\beta_i^{s_i,h}(h'')$ at every $h'' \in H_i(s_i)$ weakly preceding h among strategies in $S_i(h) \cap S_i(h'')$. Therefore s_i is optimal for type $t_i^{s_i,h}$ at every $h' \in H_i(s_i)$ weakly following h among strategies in $S_i(h')$ and at every $h'' \in H_i(s_i)$ weakly preceding h among strategies in $S_i(h) \cap S_i(h'')$.

We also have that for every $h' \in H_i$ the belief $\beta_i^{s_i,h}(h') \in \Delta(S_{-i}^K(h'))$ and $S_{-i}^K(h') = \times_{j \neq i} R_j^K(h')$. So $\beta_i^{s_i,h}(h')$ assigns positive probability only to opponents' strategies in $R_j^K(h')$. Therefore type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' types $t_j^{s_j,h'}$ where $s_j \in R_j^K(h')$. Then type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' types in $T_j^K(h')$. ♦

Now we proceed to prove some properties of this epistemic model.

Lemma 6. For the epistemic model constructed above, every type $t_i \in T_i^k(h)$ expresses up to $(k-1)$-fold belief in future and restricted past rationality.

Proof. We prove the result by induction on k.

Let $k = 2$, and consider a player $i \in I$ and an information set $h \in H$. Take $t_i \in T_i^2(h)$, then $t_i = t_i^{s_i,h}$ for some $s_i \in R_i^2(h)$. By construction, type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' strategy-type pairs $(s_j, t_j^{s_j,h'})$ where $s_j \in R_j^1(h')$ and $t_j^{s_j,h'} \in T_j^1(h')$.

For every such strategy-type pair $(s_j, t_j^{s_j,h'})$, strategy s_j is optimal for type $t_j^{s_j,h'}$ at every $h'' \in H_j(s_j)$ weakly following h' among strategies in $S_j(h'')$ and at every $h''' \in H_j(s_j)$ weakly preceding h' among strategies in $S_j(h') \cap S_j(h''')$. Therefore type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' strategy-type pairs $(s_j, t_j^{s_j,h'})$ where s_j is optimal for type $t_j^{s_j,h'}$ at every $h'' \in H_j(s_j)$ weakly following h' among strategies in $S_j(h'')$, and at every $h''' \in H_j(s_j)$ weakly preceding h' among strategies in $S_j(h') \cap S_j(h''')$. This means that $t_i^{s_i,h}$ believes in the opponents' future and restricted past rationality. Then $t_i^{s_i,h}$ expresses up to 1-fold belief in future and restricted past rationality.

Now the induction step. Fix $k \geq 3$ and assume that for every player $i \in I$ and every information set $h \in H$, every type $t_i \in T_i^{k-1}(h)$ expresses up to $(k-2)$-fold belief in future and restricted past rationality.

Consider a player $i \in I$ and an information set $h \in H$. Take $t_i \in T_i^k(h)$, which means $t_i = t_i^{s_i,h}$ for some $s_i \in R_i^k(h)$. Type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' strategy-type pairs $(s_j, t_j^{s_j,h'})$ where $s_j \in R_j^{k-1}(h')$ and $t_j^{s_j,h'} \in T_j^{k-1}(h')$. For every such strategy-type pair, s_j is optimal for type $t_j^{s_j,h'}$ at every $h'' \in H_j(s_j)$ weakly following h' among strategies in $S_j(h'')$ and at every $h''' \in H_j(s_j)$ weakly preceding h' among strategies in $S_j(h') \cap S_j(h''')$.

By the induction assumption, since type $t_j^{s_j,h'} \in T_j^{k-1}(h')$ then $t_j^{s_j,h'}$ expresses up to $(k-2)$-fold belief in future and restricted past rationality. Then type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' strategy-type pairs $(s_j, t_j^{s_j,h'})$ where s_j is optimal for type $t_j^{s_j,h'}$ at every $h'' \in H_j(s_j)$ weakly following h' among strategies in $S_j(h'')$, and at every $h''' \in H_j(s_j)$ weakly preceding h' among strategies in $S_j(h') \cap S_j(h''')$, and type $t_j^{s_j,h'} \in T_j^{k-1}(h')$ expresses up to $(k-2)$-fold belief in future and restricted past rationality. Hence, $t_i^{s_i,h}$ expresses up to $(k-1)$-fold belief in future and restricted past rationality. This holds for all players $i \in I$ and all information sets $h \in H$, so every type $t_i \in T_i^k(h)$ expresses up to $(k-1)$-fold belief in future and restricted past rationality. By induction, the result is true for every $k \geq 2$. ■

Lemma 7. Given the epistemic model constructed above, for every $k \geq K-1$, every type $t_i \in T_i^K(h)$ expresses up to k-fold belief in future and restricted past rationality.

Proof. This result is proven by induction on k.

Let $k = K-1$. By Lemma 6 we know that every type $t_i \in T_i^K(h)$ expresses up to $(K-1)$-fold belief in future and restricted past rationality, so the result is true for $k = K-1$.

Now we do the induction step. Fix $k \geq K$ and assume that for every player $i \in I$ and every information set $h \in H$, every type $t_i \in T_i^K(h)$ expresses up to $(k-1)$-fold belief in future and restricted past rationality. Consider a player $i \in I$, an information set $h \in H$ and a type $t_i \in T_i^K(h)$, that is $t_i = t_i^{s_i,h}$

for some $s_i \in R_i^K(h)$. By construction, $t_i^{s_i,h}$ assigns, at every $h' \in H_i$ positive probability only to opponents' strategy-type pairs $(s_j, t_j^{s_j,h'})$ where $s_j \in R_j^K(h')$ and $t_j^{s_j,h'} \in T_j^K(h')$. Then for every such pair $(s_j, t_j^{s_j,h'})$ the strategy s_j is optimal for type $t_j^{s_j,h'}$ at every $h'' \in H_j(s_j)$ weakly following h' among strategies in $S_j(h'')$ and at every $h''' \in H_j(s_j)$ weakly preceding h' among strategies in $S_j(h') \cap S_j(h''')$.

By the induction assumption, every type $t_j^{s_j,h'} \in T_j^K(h')$ expresses up to $(k-1)$-fold belief in future and restricted past rationality. Therefore, type $t_i^{s_i,h}$ assigns at every $h' \in H_i$ positive probability only to opponents' strategy-type pairs $(s_j, t_j^{s_j,h'})$ where s_j is optimal for type $t_j^{s_j,h'}$ at every $h'' \in H_j(s_j)$ weakly following h' among strategies in $S_j(h'')$ and at every $h''' \in H_j(s_j)$ weakly preceding h' among strategies in $S_j(h') \cap S_j(h''')$, and type $t_j^{s_j,h'} \in T_j^K(h')$ expresses up to $(k-1)$-fold belief in future and restricted past rationality. Then type $t_i^{s_i,h}$ expresses up to k-fold belief in future and restricted past rationality, and this holds for every player $i \in I$ and every information set $h \in H$. Hence, every type $t_i \in T_i^K(h)$ expresses up to k-fold belief in future and restricted past rationality. By induction, the result holds for every $k \geq K-1$. ■

The next result follows from Lemma 7 and the definition of common belief in future and restricted past rationality.

Corollary 2. Given the epistemic model constructed above, every type $t_i \in T_i^K(h)$ expresses common belief in future and restricted past rationality.

Now we proceed with the proof for Theorem 2.

Proof (Theorem 2)**.** The first part of the theorem can be stated as $BR_i^k = S_i^{k+1}$ for every player i and every k. We show this holds by dividing the proof in two parts.

First we prove that $S_i^{k+1} \subseteq BR_i^k$ for every player i and every k. Consider a player $i \in I$ and $k \geq 1$. Take some $s_i \in S_i^{k+1}$. Then $s_i \in S_i^{k+1}(h)$ for all $h \in H_i(s_i)$. This implies that $s_i \in R_i^{k+1}(\varnothing)$. Hence, type $t_i^{s_i,\varnothing}$ is in $T_i^{k+1}(\varnothing)$, so by Lemma 6, $t_i^{s_i,\varnothing}$ expresses up to k-fold belief in future and restricted past rationality. Moreover, s_i is optimal for $t_i^{s_i,\varnothing}$ at every $h \in H_i(s_i)$ weakly following $\varnothing$ among strategies in $S_i(h)$. Therefore $s_i \in BR_i^k$. So every strategy $s_i \in S_i^{k+1}$ is also in BR_i^k, that is $S_i^{k+1} \subseteq BR_i^k$, and this holds for all players $i \in I$ and $k \geq 1$. Moreover, from Lemma 5 we know that $BR_i^k \subseteq S_i^{k+1}$. Hence $BR_i^k = S_i^{k+1}$.

For the second part of the theorem, consider a strategy s_i that can rationally be chosen by a type that expresses common belief in future and restricted past rationality. Then $s_i \in BR_i^k = S_i^{k+1}$ for all k, so s_i survives the full algorithm. Hence, every strategy s_i that can rationally be chosen by a type that expresses common belief in future and restricted past rationality survives the full algorithm.

Now, take a strategy s_i that survives the full algorithm. Hence, $s_i \in S_i^K(h)$ for all $h \in H_i(s_i)$. Then $s_i \in R_i^K(\varnothing)$, and by Corollary 2 we know type

$t_i^{s_i,\varnothing} \in T_i^K(\varnothing)$ expresses common belief in future and restricted past rationality. Moreover, by the construction of the epistemic model, the strategy s_i is optimal for the type $t_i^{s_i,\varnothing}$ at every $h \in H_i(s_i)$ weakly following $\varnothing$ among strategies in $S_i(h)$. Hence s_i is optimal for a type that expresses common full belief in future and restricted past rationality. Therefore, every strategy s_i that survives the full algorithm is optimal for a type that expresses common belief in future and restricted past rationality. ■

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