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Expected utility as an expression of linear preference intensity

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Abstract

Following Gilboa and Schmeidler (Games Econ Behav 44:184–194, 2003) we consider a scenario where the decision maker holds, for every possible probabilistic belief about the states, a preference relation over his choices. For this setting, Gilboa and Schmeidler have offered conditions that allow for an expected utility representation. Their central condition is the *diversity axiom* which states that for every strict ordering of at most four choices there should be a belief at which it obtains. It turns out that this axiom excludes many natural cases, even when there are no weakly dominated choices. We replace the diversity axiom by two new axioms—*three choice* and *four choice linear preference intensity,* which reflect the assumption that the preference intensity between two choices varies linearly with the belief. It is shown that in the absence of weakly dominated choices, the resulting set of axioms characterizes precisely those scenarios that admit an expected utility representation. In particular, our set of axioms covers a significantly broader class of scenarios than the Gilboa-Schmeidler axioms.

Keywords Expected utility · Decision problems · Games · Conditional preference relation · Preference intensity · Weak dominance

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1 Introduction

Gilboa and Schmeidler (2003) have argued that the von Neumann–Morgenstern foundation for expected utility may be inappropriate for analyzing games. If the player in a game holds a probabilistic belief about the opponents' pure strategies (from now on called *choices*), then each of his own choices may be interpreted as a lottery over outcomes, in line with the von Neumann–Morgenstern framework. However, von Neumann and Morgenstern assume that the decision maker ranks *all* pairs of lotteries over outcomes, whereas many of these pairs cannot be induced by the same belief about the opponents' choices. A second problem is that in a game, a player may care about more than just the physical outcome for himself, as he may engage in various kinds of inter-personal comparisons.

In response, Gilboa and Schmeidler (2003) developed a novel framework where the decision maker (DM) holds a preference relation over the available acts (his choices in the game) for *every* possible probabilistic belief about the states (the opponents' choices in the game). In this paper, we refer to this object as a *conditional preference relation.* By imposing certain axioms on conditional preference relations, Gilboa and Schmeidler single out those that have a *diversified expected utility* representation—a utility matrix where no row is weakly dominated by, or equivalent to, an affine combination of at most three other rows. The key axiom in their characterization is *diversity,* which states that for every strict ordering of at most four acts there must be a belief for which this ordering obtains.

But what can we say about those conditional preference relations that have an expected utility representation, but not a diversified one? This is an important question, because many utility matrices are non-diversified, and many natural conditional preference relations violate the diversity axiom. Indeed, the diversity axiom rules out all cases where some act is weakly dominated by another act, all scenarios with two states and more than two acts, all scenarios with three states and more than three acts, and many other plausible situations as well.

The purpose of this paper is to fill that gap, by providing a list of axioms that, in the absence of weakly dominated acts, is both necessary and sufficient for the conditional preference relation having an expected utility representation—diversified or not. Indeed, our Theorem 3.1 shows that, in the absence of weakly dominated acts, expected utility can be characterized if we replace Gilboa and Schmeidler's diversity axiom by two new axioms: *three choice linear preference intensity* and *four choice linear preference intensity.* Both axioms reveal the idea that the *intensity* with which the decision maker prefers one choice to another changes *linearly* with his belief. More precisely, the first axiom concerns three choices and argues that on two parallel lines of beliefs, the preference intensity between two choices will change at the same rate. This results in a formula that relates the beliefs on these two parallel lines where the decision maker is indifferent between the various pairs of choices. The second axiom concerns four choices, and argues that on a line of beliefs the relative change rates of the preference intensities between the different pairs of choices must be consistent with one another. Also this axiom is expressed in terms of a formula, which relates the beliefs on the line where the decision maker is indifferent between the various pairs of choices. Both axioms may be viewed as testable consequences of the idea that the preference intensity between choices changes linearly with the belief.

If there are no weakly dominated acts, and there is a belief where the decision maker is indifferent between some, but not all, choices (provided there are at least three choices), then it is shown in Proposition 4.1 that the utility differences are unique up to a positive multiplicative constant. In that case, the utility differences between two choices a and b may be viewed as expressing the decision maker's "preference intensity" between a and b. This is similar to the approaches by Anscombe and Aumann (1963) and Wakker (1989), where the axioms of state independence and state independent preference intensity, respectively, guarantee that the utility difference between two consequences is the same at every state, and may be viewed as expressing the "preference intensity" between these consequences.

Our axiomatic characterization can also be extended to the general case that involves weakly dominated acts. In the online appendix of the book (Perea, 2024) it is shown how our axioms can be extended to *signed beliefs* involving negative "probabilities", and how these new axioms jointly characterize expected utility for the general case.

This paper is organized as follows. In Sect. 2 we introduce the notion of a conditional preference relation and some basic regularity axioms which are based on Gilboa and Schmeidler (2003). In Sect. 3 we introduce the new axioms, *three choice* and *four choice linear preference intensity,* and derive the representation theorem for the case when there are no weakly dominated acts. In Sect. 4 we show that under our set of axioms, the utility differences are unique up to a positive multiplicative constant. We conclude with a discussion in Sect. 5. All the proofs, and the necessary mathematical definitions, can be found in the appendix.

2 Conditional preference relations and basic axioms

In this section we formally introduce conditional preference relations as the primitive notion of our model, and subsequently impose some basic regularity axioms on these, based on Gilboa and Schmeidler (2003). In the last part we discuss Gilboa and Schmeidler's *diversity* axiom, and how it rules out many plausible scenarios.

2.1 Conditional preference relations

In line with Gilboa and Schmeidler (2003), the primitive object in this paper is that of a *conditional preference relation* – a mapping that assigns to every probabilistic belief over the states a preference relation over the available choices. In this paper, we also refer to such choices as *acts*. In fact, we will use the terms acts and choices interchangeably. Consider a decision maker (DM) who must choose from a finite set of acts A, and where the finite set of states is given by S. We denote by $\mathcal{B}(A)$ the collection of all binary relations, or preference relations, on A.

A *belief* is a probability distribution p on S, and by $\Delta(S)$ we denote the set of all probability distributions on S.

Definition 2.1 (Conditional preference relation) Consider a finite set of acts A and a finite set of states S. A conditional preference relation relative to (A, S) is a mapping $\succsim : \Delta(S) \to \mathcal{B}(A)$ that assigns to every belief $p \in \Delta(S)$ a preference relation $\succsim_p$ on A.

We write $a \succsim_p b$ to indicate that the DM weakly prefers act a to act b if his belief is p. In line with Gilboa and Schmeidler (2003), a conditional preference relation can be interpreted as follows: Imagine a player (the DM) who is anonymously matched with another player from a given population. Informing the DM about the past choices of the players within that population will then induce a certain belief for the DM. By varying such information in an experimental setup we can, in principle, induce every possible belief for the DM. If subsequently we elicit the DM's preference relation over acts, we would obtain all the ingredients that constitute a conditional preference relation.

For two acts a and b, we write that $a \sim_p b$ if $a \succsim_p b$ and $b \succsim_p a$. Similarly, we write $a \succ_p b$ if $a \succsim_p b$ but not $b \succsim_p a$. For two acts $a, b \in A$ we define the sets of beliefs $P_{a \sim b} := \{p \in \Delta(S) \mid a \sim_p b\}$, $P_{a \succ b} := \{p \in \Delta(S) \mid a \succ_p b\}$ and $P_{a \succsim b} := \{p \in \Delta(S) \mid a \succsim_p b\}$. We say that (a) a *strictly dominates* b under $\succsim$ if $a \succ_p b$ for all $p \in \Delta(S)$; (b) a *weakly dominates* b under $\succsim$ if $a \succsim_p b$ for all $p \in \Delta(S)$, and $a \succ_p b$ for at least one $p \in \Delta(S)$; (c) a is *equivalent* to b under $\succsim$ if $a \sim_p b$ for all $p \in \Delta(S)$.

In the remainder of this paper we will assume that the conditional preference relation does not have equivalent acts. In the discussion section we will briefly explain how our analysis can easily be extended to cover equivalent acts.

An *expected utility representation* can be defined as follows.

Definition 2.2 (Expected-utility representation) A conditional preference relation $\succsim$ has an expected utility representation if there is a utility function $u : A \times S \to \mathbf{R}$ such that for every belief $p \in \Delta(S)$ and every two acts a,b $\in$ A,

$$a \succsim_p b \textit{ if and only if } \sum_{s \in S} p(s) \cdot u(a, s) \geq \sum_{s \in S} p(s) \cdot u(b, s).$$

In this case, we say that the conditional preference relation $\succsim$ is *represented* by the utility function u. For a given vector $v \in \mathbf{R}^S$ we use the notation $u(a, v) := \sum_{s \in S} v(s) \cdot u(a, s)$. Hence, the condition above can be written as $a \succsim_p b$ if and only if $u(a, p) \geq u(b, p)$.

2.2 Regularity axioms

We will start by reviewing some very basic axioms that have already been introduced in Gilboa and Schmeidler (2003), and to which we refer as *regularity axioms.*

Axiom 2.1 (*Completeness and transitivity*) *For every belief p and any two acts $a, b \in A$, either $a \succsim_p b$ or $b \succsim_p a$. Moreover, for every belief p and every three acts $a, b, c \in A$ with $a \succsim_p b$ and $b \succsim_p c$, it holds that $a \succsim_p c$.*

Axiom 2.2 (*Continuity*) *For every two different acts $a, b \in A$ and every two beliefs $p \in P_{a \succ b}$ and $q \in P_{b \succ a}$, there is some $\lambda \in (0, 1)$ such that $(1 - \lambda)p + \lambda q \in P_{a \sim b}$.*

Axiom 2.3 (*Preservation of indifference*) *For every two different acts $a, b \in A$ and every two beliefs $p \in P_{a \sim b}$ and $q \in P_{a \sim b}$, we have that $(1 - \lambda)p + \lambda q \in P_{a \sim b}$ for all $\lambda \in (0, 1)$.*

Axiom 2.4 (*Preservation of strict preference*) *For every two different acts $a, b \in A$ and every two beliefs $p \in P_{a \succsim b}$ and $q \in P_{a \succ b}$, we have that $(1 - \lambda)p + \lambda q \in P_{a \succ b}$ for all $\lambda \in (0, 1)$.*

Completeness and transitivity together resemble the *ranking* axiom in Gilboa and Schmeidler (2003). Our definition of continuity is formally different from Gilboa and Schmeidler (2003) version, but reveals the same idea. When taken together, our axioms of preservation of indifference and preservation of strict preference correspond precisely to Gilboa and Schmeidler (2003) axiom of *combination.*

It can be shown that for the case of two acts, the regularity axioms are both necessary and sufficient for a conditional preference relation having an expected utility representation. A proof can be found in the appendix (see Lemma 6.4).

2.3 Diversity axiom

The central axiom in Gilboa and Schmeidler (2003) foundation for expected utility is *diversity.* It states that for every strict ranking of four choices or less there is at least one belief at which this ranking obtains. In Gilboa and Schmeidler (2003) it is shown that this axiom, together with the regularity axioms above, characterize precisely those conditional preference relations that admit a *diversified* expected utility representation—a utility matrix where no row is weakly dominated by, or equivalent to, an affine combination of at most three other rows.

As it turns out, the diversity axiom rules out many natural scenarios. To start with, it excludes all conditional preference relations where an act is weakly dominated by another act. But even in settings where there is no weak dominance between acts it rules out a series of plausible cases. Consider, for instance, the conditional preference relation in Fig. 1 with three acts and three states.

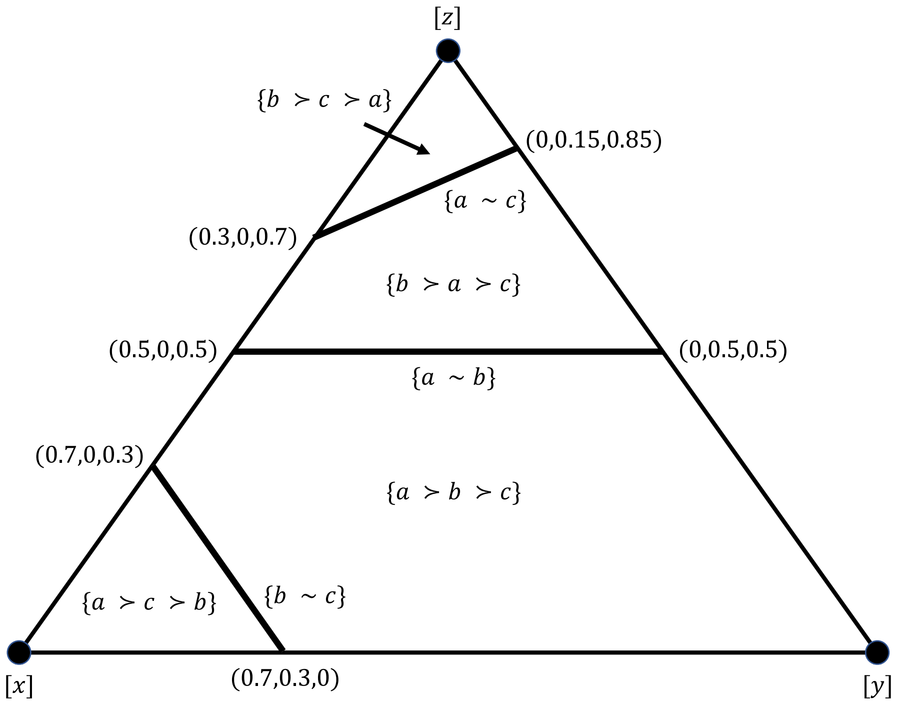

Fig. 1 Violation of diversity with three states and three choices: The area within the triangle represents the set $\Delta(S)$ of all probabilistic beliefs on $S = \{x, y, z\}$, with the probability 1 beliefs $[x]$, $[y]$ and $[z]$ as the extreme points. The two-dimensional plane represents all the vectors in $\mathbf{R}^S$ where the sum of the coordinates is 1, containing the belief simplex $\Delta(S)$ as a subset. The vector (0.7, 0, 0.3) represents the belief that assigns probability 0.7 to x, probability 0 to y and probability 0.3 to z. Similarly for the other beliefs in the picture. Caution: the points in the triangle are probabilistic beliefs, and not lotteries on outcomes as the reader is perhaps used to from the decision theoretic literature

Suppose this conditional preference relation belongs to a two-player game where the decision maker (player 1) can select a choice from $\{a, b, c\}$ and where the opponent (player 2) can choose from $\{x, y, z\}$. It may be verified that no choice of player 1 weakly dominates another choice. Yet, the diversity axiom is violated because there is no belief where player 1 holds the ranking $c \succ b \succ a$, and hence this conditional preference relation is ruled out by the Gilboa-Schmeidler axioms.

Table 1 Expected utility representation for the conditional preference relation in Fig. 1

	x	y	z
a	10	17	0
b	0	7	10
c	3	0	3

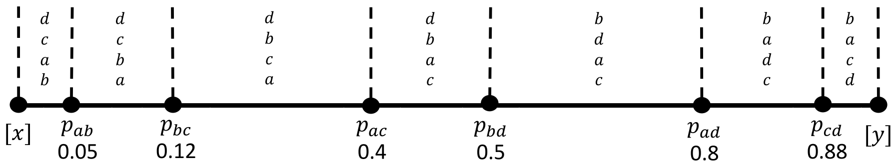

Fig. 2 Violation of diversity with four acts and two states: The numbers below the beliefs indicate the probability assigned to state [y]. For instance, p_{ab} assigns probability 0.05 to state [y] and is the unique belief where the decision maker is indifferent between *a* and *b*. Similarly for the other five beliefs. The vertical position of the four choices in every area indicate the ranking of the four choices held by player 1 in that specific area, with the most preferred choice on top

At the same time this seems a perfectly natural setting for a two-player game. Moreover, it may be verified that the conditional preference relation allows for an expected utility representation, for instance the one in Table 1.

As the conditional preference relation violates the diversity axiom it follows from Gilboa and Schmeidler (2003) that the expected utility representation is necessarily non-diversified.

Consider next the conditional preference relation in Fig. 2 with four acts and two states. Suppose this belongs to a two-player game where the decision maker (player 1) can choose from $\{a, b, c, d\}$, and the opponent (player 2) chooses from $\{x, y\}$. It may be verified that under this conditional preference relation no choice is weakly dominated by another choice. Yet, the conditional preference relation violates the diversity axiom as there is no belief where player 1 holds the ranking $a \succ b \succ c \succ d$. Also this conditional preference relation seems perfectly natural, and it can be shown to have an expected utility representation. See, for instance, the utility matrix in Table 2. Again, in the light of Gilboa and Schmeidler (2003) this expected utility representation is necessarily non-diversified.

The two examples above show that there are natural instances of games where the diversity axiom is violated. In general, it can be shown that every conditional preference relation with two states and more than two acts, and every conditional preference relation with three states and more than three acts, necessarily violates the diversity axiom. Yet, as we will see, our axiomatic treatment will still be able to cover such scenarios, provided there are no weakly dominated acts.

Table 2 Expected utility representation for the conditional preference relation in Fig. 2

	x	*y*
a	6	6
b	5	25
c	8	3
d	30	0

3 New axioms

In this section we present the new axioms: *three choice* and *four choice linear preference intensity*. For the case where no act is weakly dominated by another act, we show that these axioms, together with the regularity axioms from the previous section, jointly characterize those conditional preference relations that admit an expected utility representation.

3.1 Three choice linear preference intensity

If there are more than two acts, the regularity axioms no longer suffice to guarantee an expected utility representation. To see this, consider the conditional preference relation $\succsim$ represented by Fig. 3. It may be verified that $\succsim$ satisfies all the regularity axioms. Yet, there is no expected utility representation for $\succsim$. To see why, suppose there would be a utility function u that represents $\succsim$. Then, the induced expected utilities of a and b must be equal on the line through $P_{a\sim b}$, the expected utilities of b and c must be equal on the line through $P_{b\sim c}$ and the expected utilities of a and c must be equal on the line through $P_{a\sim c}$, also at vectors that lie outside the belief simplex. But then, the expected utilities of a and c must be the same at the vector v where the lines through $P_{a\sim b}$ and $P_{b\sim c}$ intersect, which is impossible since v does not belong to the line through $P_{a\sim c}$. This insight leads us to introduce further axioms

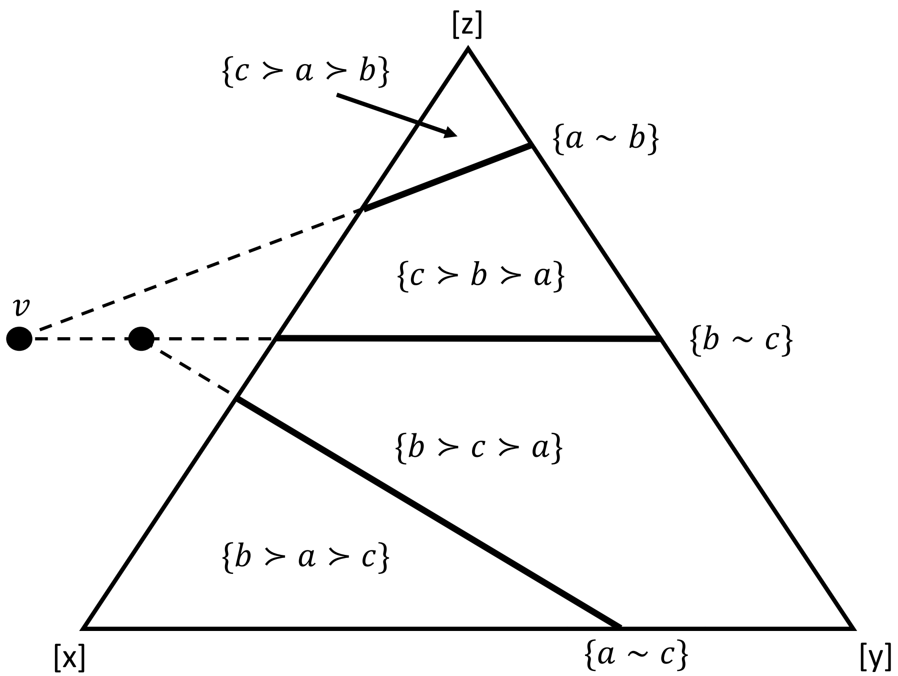

Fig. 3 Why regularity axioms are not sufficient

which *do* guarantee an expected utility representation, at least when no act weakly dominates another act.

Consider three acts $a, b, c,$ and beliefs p_{ab}, p_{ac} and p_{bc} where the DM is indifferent between a and $b,$ between a and $c,$ and between b and c respectively. Suppose that these three beliefs lie on the same line L. Now assume that the DM displays at every belief and for every two choices not only a binary preference between these two choices, but also an "intensity" by which he prefers one choice to the other. Then, as we move from p_{ab} to p_{bc} on this line, the preference intensity between a and b changes from 0 to some value α. Moreover, as we move from p_{ac} to p_{bc} the preference intensity between a and c changes from 0 to the same value α, since at p_{bc} the preference intensities between a and b and between a and c must be the same.

Consider next a line L' that is parallel to L. If we assume that the DM's preference intensities change *linearly* with the belief, then the preference intensity between any given two choices should change at the same speed at the two parallel lines L and L'. Now take three indifference beliefs p'_{ab}, p'_{ac} and p'_{bc} on the line L', and suppose that the distance between p'_{ab} and p'_{bc} on L' is λ times the distance between p_{ab} and p_{bc} on L. Then, as we move from p'_{ab} to p'_{bc} the preference intensity between a and b changes from 0 to $\lambda \cdot \alpha$, since we assume that the preference intensity between a and b changes at the same speed on L and L'. Therefore, as we move from p'_{ac} to p'_{bc}, the preference intensity between a and c changes from 0 to $\lambda \cdot \alpha$ also. But then, the distance between p'_{ac} and p'_{bc} on L' must be λ times the distance between p_{ac} and p_{bc} on L. As such, the ratio of the distances between p_{ac} and p_{bc} and between p_{ab} and p_{bc} must be the same as the ratio of the distances between p'_{ac} and p'_{bc} and between p'_{ab} and p'_{bc}.

For a fixed state s and two beliefs p and q on $L,$ or on L', the difference $p(s) - q(s)$ can be used as a measure for the distance between p and q. Then, by the equal ratio property above we have that

$$\frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)} = \frac{p'_{ac}(s) - p'_{bc}(s)}{p'_{ab}(s) - p'_{bc}(s)},$$

which implies that

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)). \quad (3.1)$$

This equality will be the content of the axiom *three choice linear preference intensity.*

To state this axiom formally, we need the following definitions. A *line of beliefs* is a subset $L \subseteq \Delta(S)$ such that $L = \{(1-\lambda)p + \lambda q \mid \lambda \in [0,1]\}$ for some beliefs $p, q \in \Delta(S)$. Two lines of beliefs L and L' are *parallel* if for every $p, q \in L$ and every $p', q' \in L'$ there is some $\mu \in \mathbf{R}$ with $p - q = \mu(p' - q')$.

Axiom 3.1 (Three choice linear preference intensity) *For every three acts $a, b, c,$ for every line L of beliefs with beliefs p_{ab}, p_{bc}, p_{ac} where the DM is indifferent between the respective acts and which contains a belief where the DM is not indifferent between any of these acts, every line L' parallel to L with beliefs $p'_{ab}, p'_{bc}, p'_{ac}$ where* *the DM is indifferent between the respective acts and which contains a belief where the DM is not indifferent between any of these acts, it holds that*

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) \\ = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s))$$

for every state s.

It may be verified that the conditional preference relation in Fig. 1 satisfies this condition. It turns out that, geometrically, this axiom can be verified in an easy way if the conditional preference relation has no weakly dominated choices. In this case, it is equivalent to checking that every vector (possible outside the belief simplex) which is both on a line through two points in $P_{a\sim b}$ and on a line through two points in $P_{b\sim c}$, must also be on a line through two points in $P_{a\sim c}$. This is the content of Proposition 6.1 in the appendix.[1]

The conditional preference relation in Fig. 3 clearly violates this property, since the vector v in that figure belongs to the line through $P_{a\sim b}$ and the line through $P_{b\sim c}$, but not to the line through $P_{a\sim c}$. Hence, in view of the insight above, it cannot satisfy three choice linear preference intensity.

From the proof of Proposition 6.1 in the appendix it follows that under the regularity axioms, the condition of three choice linear preference intensity holds if we can find *one* line L and *one* parallel line L' for which (3.1) holds. This is important for empirically testing the axiom, because it suffices to select six indifference beliefs $p_{ab}, p_{ac}, p_{bc}, p'_{ab}, p'_{ac}, p'_{bc}$ on two parallel lines to verify the axiom.[2]

3.2 Four choice linear preference intensity

We will now show that in the case of four choices or more, the linearity of preference intensity implies yet another testable condition. Consider four choices $a, b, c, d,$ a line of beliefs $L,$ and beliefs $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}, p_{cd}$ on that line where the DM is indifferent between the respective choices. Similarly to what we have seen above, moving from p_{ab} to p_{bc} changes the preference intensity between a and b from 0 to some value α, whereas moving from p_{ac} to p_{bc} changes the preference intensity between a and c from 0 to α as well. Now suppose that the DM's preference intensity between two choices changes linearly with the belief. Then, the ratio

[1] To be precise, the verbal characterization of three choice linear preference intensity in the text also relies on Lemma 6.1 in the appendix, which shows that under preservation of indifference every vector in the linear span of $P_{a\sim b}$ can be written as the linear combination of only *two* points in $P_{a\sim b}$. Consequently, every vector in the linear span of $P_{a\sim b}$ where the sum of the components is 1 lies on a line through two points in $P_{a\sim b}$.

[2] On a given line of beliefs, the belief is characterized by one parameter only. To elicit the indifference belief p_{ab} on that line one could offer the subject a discretized version of the line, and ask for every associated belief whether the subject prefers a to b, b to a, or whether he is indifferent. Alternatively, one could offer the subject a slider with which he can increase or decrease the parameter of the belief, and ask to point at the belief where he is indifferent.

of the speeds with which the preference intensity between b and a and the preference intensity between c and a change is equal to the ratio of the distance between p_{ac} and p_{bc} and the distance between p_{ab} and p_{bc}. Let us denote this ratio of speeds by r_{bc}.

A similar property holds for the triple of choices a, c, d and the triple of choices a, b, d, resulting in ratios of speeds r_{cd} and r_{bd}. Note that, by construction, $r_{bd} = r_{bc} \cdot r_{cd}$. Since we have seen above that the ratios of speed are equal to the ratios of the distances between the respective indifference beliefs, it follows that

$$\frac{dist(p_{ad}, p_{bd})}{dist(p_{ab}, p_{bd})} = \frac{dist(p_{ac}, p_{bc})}{dist(p_{ab}, p_{bc})} \cdot \frac{dist(p_{ad}, p_{cd})}{dist(p_{ac}, p_{cd})},$$

where $dist$ stands for distance. Again, for a given state s and any two beliefs p, q on the line L we can use $p(s) - q(s)$ as a measure for the distance between p and q. The equality above can then be translated into

$$\frac{p_{ad}(s) - p_{bd}(s)}{p_{ab}(s) - p_{bd}(s)} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)} \cdot \frac{p_{ad}(s) - p_{cd}(s)}{p_{ac}(s) - p_{cd}(s)}. \tag{3.2}$$

By cross-multiplication, we thus obtain the following condition.

Axiom 3.2 (*Four choice linear preference intensity*) *For every four choices a, b, c, d, and for every line L of beliefs with beliefs $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}, p_{cd}$ where the DM is indifferent between the respective choices, and such that L contains a belief where the DM is not indifferent between any of these choices, it holds that*

$$\begin{aligned}(p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{cd}(s)) \cdot (p_{ad}(s) - p_{bd}(s)) \\ = (p_{ab}(s) - p_{bd}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) \cdot (p_{ad}(s) - p_{cd}(s)).\end{aligned}$$

for every state s.

It may be verified that the conditional preference relation in Fig. 2 satisfies the four choice linear preference intensity condition. From the proof of Theorem 3.1 in the appendix it follows that under the regularity axioms and three choice linear preference intensity, the condition of four choice linear preference intensity is satisfied if there is *one* line L for which (3.2) holds. Again, this is important if we want to test this condition empirically, since we need only check condition (3.2) for six indifference beliefs $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}, p_{cd}$ in total.[3]

3.3 Representation theorem

If there are no weakly dominated acts, then the axioms we have gathered so far are not only necessary, but also sufficient, for an expected utility representation. We thus obtain the following representation result.

[3] The six indifference beliefs on the line can be elicited according to Footnote 2.

Theorem 3.1 (Expected utility representation) *Consider a finite set of acts A, a finite set of states S, and a conditional preference relation $\succsim$ on (A, S) such that no act weakly dominates another act and no two acts are equivalent under $\succsim$. Then, $\succsim$ has an expected utility representation, if and only if, it satisfies completeness, transitivity, continuity, preservation of indifference, preservation of strict preference, three choice linear preference intensity and four choice linear preference intensity.*

Note that this theorem also covers conditional preference relations that violate the diversity axiom in Gilboa and Schmeidler (2003). For such scenarios, our result still provides necessary and sufficient conditions that lead to a (necessarily non-diversified) expected utility representation.

4 Unique relative utility differences

So far we have identified a system of axioms that is both necessary and sufficient for an expected utility representation, provided there is no weak dominance between acts. But how unique is this representation? As we will see below, the expected utility differences are "typically" unique up to a positive multiplicative constant.

Proposition 4.1 (Unique relative utility differences) *Consider a finite set of acts A, a finite set of states S, and a conditional preference relation $\succsim$ on (A, S), such that it admits an expected utility representation, no act weakly dominates another act, no two acts are equivalent under $\succsim$, and in the case of at least three acts there is a belief where the DM is indifferent between some, but not all, acts. Then, for every two utility functions u, v that represent $\succsim$ there is some $\alpha > 0$ such that $v(a, s) - v(b, s) = \alpha \cdot (u(a, s) - u(b, s))$ for all $a, b \in A$ and all $s \in S$.*

Under the conditions of the proposition, there would be exactly $|S| + 1$ degrees of freedom for choosing a representing utility function: $|S|$ degrees because we can choose the utilities for one of the choices freely at each of the $|S|$ states, and another degree of freedom because the utility differences at each of the states may be multiplied by the same positive number without changing the induced conditional preference relation.

Moreover, under these conditions the utility difference $u(a, p) - u(b, p)$ at a belief p, which is unique up to a positive multiplicative constant, may be viewed as expressing the "preference intensity" between a and b at p. The conditions above thus guarantee that the relative preference intensities are unique. As an example, suppose that $0 < u(a, x) - u(b, x) = 2 \cdot (u(b, y) - u(a, y))$. Then, the DM will be indifferent between a and b at the belief $1/3[x] + 2/3[y]$,[4] which seems to reflect that the intensity by which the DM prefers a to b at x is twice the intensity by which he prefers b to a at y. This indeed corresponds to the fact that the utility difference

[4] Here, $[x]$ denotes the denegerate belief that assigns probability 1 to the state $[x]$. Similarly for $[y]$.

between a and b at x is twice as large as at y, in absolute terms. However, we will not enter the debates on whether such utility differences, or preference intensities, can be interpreted as reflecting neo-classical cardinal utility (see, for instance, Baccelli and Mongin (2016); Baumol (1958) and Moscati (2018)).

The above interpretation of the utility differences may no longer hold, however, if the conditions in the proposition above are not satisfied. Suppose there are three acts a, b and c, two states x and y, and let $\succsim$ be such that $a \succ_p b \succ_p c$ if $p(x) > 1/2$, $a \sim_p b \sim_p c$ if $p(x) = 1/2$, and $c \succ_p b \succ_p a$ if $p(x) < 1/2$. Hence, the three indifference sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are all equal to $\{1/2[x] + 1/2[y]\}$, and thus there is no belief where the DM is indifferent between some, but not all, acts. Note that the utility functions u, v given by $u(a,x) = 3, u(b,x) = 2, u(c,x) = 0, u(a,y) = -3, u(b,y) = -2, u(c,y) = 0$ and $v(a,x) = 3, v(b,x) = 1, v(c,x) = 0, v(a,y) = -3, v(b,y) = -1, v(c,y) = 0$ both represent $\succsim$. Yet, the utility differences in u and v differ by more than just a multiplicative constant. The reason is that in this case, $\succsim$ does not provide us with sufficiently many data to derive the DM's preference intensity over the three acts at the various beliefs.

5 Discussion

(a) Comparison with Savage One important difference with the framework of Savage (1954) is that we view the DM's belief as a primitive notion, which then induces a preference relation over acts. This is precisely how a conditional preference relation is defined: It takes the belief as an input, and delivers the preferences over acts as an output. One of the beautiful features of the Savage framework is that the DM's belief can be derived from his preferences over acts. That is, Savage views the DM's preferences over acts as the primitive notion, which then induces his belief. There has been a long-standing debate about which of the two, belief or preferences, should be taken as the primitive object, and we do not want to enter this debate here.

Another difference with Savage lies in the role of the utility function. In our model, the utility function generates the DM's preferences over acts for *all* possible beliefs over the states. As the Savage axiom system leads to a unique probabilistic belief over states, the utility function in the Savage framework can only be viewed in combination with this specific belief.

A final difference we would like to stress concerns the uniqueness of the utility representation. Recall from Proposition 4.1 that in the absence of weakly dominated acts there are $|S| + 1$ degrees of freedom for the utility function in our framework, provided there is a belief where the DM is indifferent between some, but not all, acts in the case of at least three acts. Unless all acts are equivalent, this is also the smallest number of degrees of freedom possible. There may be more degrees of freedom, up to $|A| \cdot |S|$, which would be the case if every act strictly dominates, or is strictly dominated by, another act.

In the Savage framework, on the other hand, the utility representation is always unique up to a positive affine transformation, leaving only two degrees of freedom. The reason is that a DM in the Savage framework holds preferences over *all possible* mappings from states to consequences, providing us with "more data" that restrict the possible utilities compared to a DM in our framework. However, the two degrees of freedom in Savage's framework are only possible because Savage's axiom of small event continuity implies that there are infinitely many states. We assume only finitely many states, but our "richness of data" comes from the fact that a conditional preference relation specifies a preference relation for infinitely many beliefs (if there are at least two states). Most comments here also apply to the framework in Anscombe and Aumann (1963).

(b) Related foundations for expected utility in decision problems and games The foundation for expected utility that is closest to ours is by Gilboa and Schmeidler (2003). As we have already discussed, their axiom system singles out those conditional preference relations that can be represented by a *diversified* utility function, and the crucial axiom in their analysis is *diversity.* The diversity axiom by Gilboa and Schmeidler may be viewed as a "richness" condition on the set of states, and seems plausible if the number of states is very large, or even (countably or uncountably) infinite, as is allowed by the Gilboa-Schmeidler framework. In contrast, we mainly concentrate on settings like finite games where, tyically, the number of states is relatively small. In such scenarios, the diversity axiom seems overly restrictive. Our axiom system, in turn, imposes no such richness condition on the set of states, and puts no restrictions on the utility matrix that can be used to represent the conditional preference relation.

Jagau (2022) shows that the regularity axioms, together with the axioms of *constant preference intensity* and *transitive preference sensitivity,* are necessary and sufficient for an expected utility representation if there are no weakly dominated acts. Constant preference intensity and transitive preference sensitivity are strongly based on our axioms of three choice linear preference intensity and four choice linear preference intensity, respectively.

Perea (2020) proves that the regularity axioms, together with the axiom *existence of a uniform preference increase,* are both necessary and sufficient for an expected utility representation. The existence of a uniform preference increase states that from the conditional preference relation at hand, one should be able to increase the preference intensity between a fixed choice and each of the other choices by a uniform amount.

Luce and Raiffa (1957)'s formulation of a decision problem under uncertainty is rather similar to ours, in that they view the DM's sets of actions and states as primitive notions. On top of this, they assume a consequence mapping, assigning to every act and state the consequence that results. Battigalli et al. (2017) show how the Anscombe-Aumann model can be reconciled with the Luce-Raiffa framework, by letting the DM hold preferences over mixed actions in the Luce-Raiffa model, and proposing an axiomatic characterization of expected utility within this setup.

Fishburn (1976) and Fishburn and Roberts (1978) concentrate on games, and assume that every player holds a preference relation over the combinations of randomized choices—or mixed strategies—of all the players. Combinations of mixed strategies may be viewed as lotteries with objective probabilities on the set of possible (pure) choice combinations in the game. By imposing certain axioms on these preference relations over mixed strategy combinations, they are able to identify those that admit an expected utility representation. It may thus be viewed as a generalization of von Neumann and Morgenstern (1947) axiomatic characterization of expected utility for lotteries. The crucial difference with our approach is that we do not consider randomizations over choices, and that we use conditional preference relations as the primitive, rather than preferences over lotteries with objective probabilities.

Still, probabilities enter our analysis, but at a different level: Within a game-theoretic setting we assume that players choose pure strategies, but hold *probabilistic beliefs* about the opponents' pure strategy combinations. For two players such beliefs are identical to mixed strategies of the opponent,[5] but their interpretation is different: Whereas a mixed strategy represents a conscious randomization over pure strategies, a probabilistic belief represents the player's uncertainty about the other player's *pure* strategy choice.

In Aumann and Drèze (2002), a game is modelled as a mapping that assigns to every choice combination by the players a lottery over consequences for each of the players. The DM (a player in the game) is then assumed to hold a preference relation on the probability distributions over such mappings. Aumann and Drèze (2005) take a different approach, by supposing that the DM in a game holds a preference relation on lotteries which are defined over his own choices and over the possible consequences in the game. In both papers, it is shown that certain axioms on the preference relation lead to an expected utility representation that involves a unique, or essentially unique, probabilistic belief for the DM about the opponents' choice combinations. In that sense, these results are similar to Savage (1954).

Mariotti (1995) points out that a DM in Savage (1954) is required to hold preferences over acts that do not belong to his actual decision problem, and finds this problematic. Mariotti (1995) goes even further, and shows that certain game-theoretic principles are inconsistent with the axioms of completeness and monotonicity in Savage's framework, thus establishing a degree of "incompatibility" between games on the one hand and the framework of Savage on the other hand.

(c) Comparison with case-based decision theory Case-based decision theory, as originally formulated in Gilboa and Schmeidler (1995), assumes that the DM evaluates an act based on how this act performed in previous decision problems. More precisely, assume that C represents the collection of decision problems, or cases, the DM faced in the past, and that $s(c)$ measures the similarity of decision problem c to the present decision problem. Then, the desirability of an act a in the present decision problem is measured by $\sum_{c \in C} s(c) \cdot u(a, c)$, where $u(a, c)$ is the utility that selecting act a generated in decision problem c.

Our framework can be embedded into case-based decision theory as follows: If a conditional preference relation is represented by a utility function u, then the desirability of an act a in the present decision problem, for a given $p \in \Delta(S)$, is given by

[5] For more than two players, a probabilistic belief about the opponents' pure strategy combinations may differ, mathematically, from a profile of mixed strategies, as the beliefs concerning the pure strategies of two different opponents may be correlated.

$\sum_{s\in S} p(s) \cdot u(a, s)$. Now suppose that the states s represent decision problems that the DM faced in the past, and that $p(s)$ measures the similarity of problem s to the decision problem he is facing now. Then, the measure for the desirability of act a resembles exactly that in Gilboa and Schmeidler (1995).

Alternatively, one could still interpret p as a probabilistic belief over states, and identify every state s with the degenerate belief $[s]$ that assigns probability 1 to s. Suppose that, for some reason, the DM has had each of these degenerate beliefs $[s]$ in the past, and remembers the utility $u(a, s)$ that each act a generated under that belief. Then, every belief $[s]$ can be viewed as a case in the Gilboa-Schmeidler framework. If the DM's actual belief is p, then the belief probability $p(s)$ can be viewed as the similarity of the actual belief p to the past belief $[s]$. Also in this scenario, the measure for the desirability of act a in the actual problem, with the actual belief p, coincides with that of the Gilboa-Schmeidler framework.

(d) Utility differences as preference intensities In Proposition 4.1 we have shown that under certain conditions, the utility differences are unique up to a positive multiplicative constant. In that case, the expected utility difference between two acts a and b at a state s may be interpreted as the "preference intensity" between a and b at the state s. This is similar to how utility differences are interpreted in Anscombe and Aumann (1963) and Wakker (1989). The *state independence* axiom in Anscombe and Aumann (1963) states that the preference relation over objective lotteries on consequences must be independent of the state. This implies, in turn, that the utility differences between two consequences must be the same at every state, and these may be viewed as expressing the "preference intensity" between the two consequences.

The key condition in Wakker (1989) axiom system is *state independent preference intensity*. The main idea is that the "preference intensity" between two consequences c_1 and c_2 at a state s can be measured by taking two acts, where one is strictly preferred to the other, and replacing the two acts at state s by c_1 and c_2, respectively, such that the DM becomes indifferent between the two new acts. State independent preference intensity requires that if the preference intensities between c_1 and c_2 and between c_3 and c_4 coincide at one state, then they must coincide at all states. In that case, the utility difference between two consequences will always be the same at all states, and may thus be viewed as expressing the "preference intensity" between the two consequences.[6]

(e) Linear preference intensity The axiomatic characterization in this paper shows that expected utility may be viewed as an expression of linear preference intensity. Indeed, some of the regularity axioms for two choices, and the axioms of three and four choice linear preference intensity for more than two choices, represent consequences of scenarios where the preference intensity between two choices changes linearly with the belief. But how natural is this idea of linear preference intensity? From a behavioral and empirical point of view, one could conduct behavioral

[6] Also in vNM-settings, expected utility differences are often interpreted as representing preference intensities. See, for instance, Börgers and Postl (2009), which focusus on voting scenarios between two parties.

experiments to test these axioms. On a more theoretical basis, the idea states that (i) the change in preference intensity should only depend on the *change* in belief, not on the particular initial and final belief, thereby revealing a specific type of invariance, and (ii) for a given direction of belief change, the change in preference intensity must be proportional to the size of the belief change. Conceptually, it thus represents the simplest possible way in which the preference intensity can vary with the belief. A problem, of course, is that preference intensity cannot be measured directly, but our axioms represent verifiable properties that logically follow from the assumption of linear preference intensity.

(f) Belief revision A conditional preference relation does not only specify the DM's preferences over acts for a given belief, but also describes how these preferences would change if he were to *revise* his belief in the light of new information. In a dynamic game it may happen, for instance, that some opponent's strategy is ruled out by some new information, forcing the DM to change his belief in response. And such information events may even take place sequentially, such that more and more opponents' strategies can be ruled out. The notion of a conditional preference relation is thus able to describe how the DM's preferences would change as a result of belief revision during the course of a dynamic game.

(g) Game theory with conditional preference relations In principle we could build a theory of games based on conditional preference relations, which may or may not satisfy our system of axioms. In a game, the DM would be a player i, his set of acts A_i would be the set of pure strategies in the game, and the states would be the set $S_i = \times_{j \neq i} A_j$ of opponents' pure strategy profiles. Fix a conditional preference relation $\succsim^i$ for every player i. A *Nash equilibrium* (Nash, 1950, 1951) could be defined as a tuple of probability distributions $(\sigma_i)_{i \in I}$, with $\sigma_i \in \Delta(A_i)$ for every player i, such that $\sigma_i(a_i) > 0$ only if a_i is optimal for the induced preference relation $\succsim^i_{\sigma_{-i}}$. Here, σ_{-i} denotes the product of the probability distributions σ_j for $j \neq i$, which is a probability distribution over A_{-i} and hence a belief for player i. With this definition, a Nash equilibrium is thus interpreted as a tuple of beliefs about the opponents' pure strategies, as in Aumann and Brandenburger (1995).

Similarly, *correlated rationalizability* (Brandenburger and Dekel, 1987; Bernheim, 1984; Pearce, 1984) could be defined by the recursive procedure where $A_i^0 := A_i$ for all players i, and

$$A_i^k := \{a_i \in A_i^{k-1} | a_i \text{ optimal for } \succsim^i_{p_i} \text{ for some } p_i \in \Delta(A_{-i}^{k-1})\}$$

for every $k \geq 1$.

In fact, many concepts for static games, including the various equilibrium and rationalizability concepts, could be generalized in terms of conditional preference relations. In particular, every conditional preference relation in a static game setting induces for every player i a best response correspondence $R_i : \Delta(\times_{j \neq i} A_j) \twoheadrightarrow A_i$, given by

$$R_i(p_i) := \{a_i \in A_i | a_i \text{ optimal for } \succsim^i_{p_i}\}$$

for every $p_i \in \Delta(\times_{j \neq i} A_j)$, generalizing the standard best response correspondences based on expected utility.

The analysis could also be extended to dynamic games, since we have seen in (f) that conditional preference relations are able to model how belief revision about the opponents' strategies changes a player's preference relation over his own strategies. As such, conditional preference relations can be used to provide a generalized version of *sequential equilibrium* (Kreps and Wilson, 1982) or of dynamic game rationalizability concepts like *strong rationalizability* (Pearce, 1984; Battigalli, 1997) (also known as *extensive-form rationalizability*), *backward dominance* (Perea, 2014) and *backwards rationalizability* (Perea, 2014; Penta, 2015; Catonini and Penta, 2022).

(h) Equivalent acts. In this paper we have restricted attention to scenarios where no two acts are equivalent. In fact, our entire analysis can easily be extended to the case where equivalent acts are allowed. Suppose we start with a set of acts A where some acts are equivalent. Then, we can partition A into equivalence classes $\{A_1, A_2, \ldots, A_K\}$ with representative acts $a_1, a_2, \ldots, a_K$, and subsequently restrict the conditional preference relation $\gtrsim$ to the set $A' = \{a_1, a_2, \ldots, a_K\}$, resulting in a new conditional preference relation $\gtrsim'$. Then, Theorem 3.1 can be generalized as follows: The conditional preference relation $\gtrsim$ has an expected utility representation, if and only if, $\gtrsim'$ satisfies the conditions in Theorem 3.1. The proof is easy: If $\gtrsim'$ satisfies the conditions in the theorem, then by the same theorem it is represented by a utility function u. Extend u to a utility function v on $A \times S$ by setting $v(a, s) := u(a_k, s)$ for all acts $a \in A$ and all $s \in S$, where $a \in A_k$. Clearly, v will then represent $\gtrsim$. In the same way, the other results in this paper can also be extended to cases that allow for equivalent acts.

Appendix

Mathematical Definitions

In this section we introduce the mathematical definitions and notation needed for this paper, mainly from linear algebra. For a finite set X, we denote by $\mathbf{R}^X$ the set of all functions $v : X \to \mathbf{R}$. Scalar multiplication and addition on $\mathbf{R}^X$ are defined in the usual way: For a function $v \in \mathbf{R}^X$ and a number $\lambda \in \mathbf{R}$, the function $\lambda \cdot v$ is given by $(\lambda \cdot v)(x) = \lambda \cdot v(x)$ for all $x \in X$. Similarly, for functions $v, w \in \mathbf{R}^X$, the sum $v + w$ is given by $(v + w)(x) = v(x) + w(x)$ for all $x \in X$. The set $\mathbf{R}^X$ together with these two operations constitutes a *linear space,* and elements in $\mathbf{R}^X$ are called *vectors*. By $\underline{0}$ we denote the vector in $\mathbf{R}^X$ where $\underline{0}(x) = 0$ for all $x \in X$.

A subset $V \subseteq \mathbf{R}^X$ is called a *linear subspace* of $\mathbf{R}^X$ if for every $v, w \in V$ and every $\alpha, \beta \in \mathbf{R}$, we have that $\alpha v + \beta w \in V$. For a subset $V \subseteq \mathbf{R}^X$, we denote by

$$span(V) := \left\{ \sum_{k=1}^{K} \alpha_k v_k \mid K \geq 1,\ \alpha_k \in \mathbf{R} \text{ and } v_k \in V \text{ for all } k \in \{1, \ldots, K\} \right\}$$

the set of all (finite) *linear combinations* of elements in V, and call it the (*linear*) *span* of V. Here, $\sum_{k=1}^{K} \alpha_k v_k$ is called a *linear combination* of the vectors $v_1, \ldots, v_K$. A linear combination $v = \lambda_1 v_1 + \cdots + \lambda_K v_K$, where $v_1, \ldots, v_K \in \mathbf{R}^X$ and $\lambda_1, \ldots, \lambda_K \in \mathbf{R}$, is called a *convex combination* if $\lambda_1, \ldots, \lambda_K \geq 0$ and $\lambda_1 + \cdots + \lambda_K = 1$.

The set *span*(V) is always a linear subspace, and if V itself is a linear subspace then $span(V) = V$. Vectors $v_1, \ldots, v_K \in \mathbf{R}^X$ are called *linearly independent* if none of the vectors is a linear combination of the other vectors. The set of vectors $\{v_1, \ldots, v_K\}$ is a *basis* for V if $v_1, \ldots, v_K$ are linearly independent, and $span(\{v_1, \ldots, v_K\}) = V$. Every basis for V has the same number of vectors, and this number is called the *dimension* of V, denoted by $\dim(V)$. If $V = \{\underline{0}\}$, then $\dim(V) = 0$.

A *probability distribution* on X is a vector $p \in \mathbf{R}^X$ such that $\sum_{x \in X} p(x) = 1$ and $p(x) \geq 0$ for all $x \in X$. The set of probability distributions on X is denoted by $\Delta(X)$. For a given element $x \in X$, we denote by $[x]$ the probability distribution in $\Delta(X)$ where $[x](x) = 1$ and $[x](y) = 0$ for all $y \in X \backslash \{x\}$. A probability distribution p has *full support* if $p(x) > 0$ for all $x \in X$.

For every two vectors $v, w \in \mathbf{R}^X$, the *vector product* is given by $v \cdot w := \sum_{x \in X} v(x) w(x)$. A *hyperplane* is a set of the form $H = \{v \in \mathbf{R}^X \mid v \cdot w = c\}$, where $w \in \mathbf{R}^X \backslash \{\underline{0}\}$ and $c \in \mathbf{R}$. If $c = 0$ then H is a linear subspace of dimension $|X| - 1$, where $|X|$ denotes the number of elements in X.

Proof of Theorem 3.1

In this subsection we will prove Theorem 3.1. Before doing so, we first derive some preparatory results. The first characterizes the span of the set of beliefs where the DM is indifferent between a and b.

Lemma 6.1 (Span of an indifference set) *Consider a conditional preference relation $\gtrsim$ that satisfies preservation of indifference, and two choices a and b. Then,*

$$span(P_{a \sim b}) = \{\lambda_1 p_1 + \lambda_2 p_2 | p_1, p_2 \in P_{a \sim b} \text{ and } \lambda_1, \lambda_2 \in \mathbf{R}\}.$$

Proof Let

$$A := \{\lambda_1 p_1 + \lambda_2 p_2 | p_1, p_2 \in P_{a \sim b} \text{ and } \lambda_1, \lambda_2 \in \mathbf{R}\}.$$

We will show that $span(P_{a \sim b}) = A$. Clearly, $A \subseteq span(P_{a \sim b})$. Hence, it remains to show that $span(P_{a \sim b}) \subseteq A$. Take some $p \in span(P_{a \sim b})$. Then, there are some beliefs $p_1, \ldots, p_k, p_{k+1}, \ldots, p_{k+m} \in P_{a \sim b}$ and numbers $\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_{k+m} > 0$ such that

$$p = \lambda_1 p_1 + \cdots + \lambda_k p_k - \lambda_{k+1} p_{k+1} - \cdots - \lambda_{k+m} p_{k+m}. \tag{6.1}$$

Let $\alpha_1 := \lambda_1 + \cdots + \lambda_k$ and $\alpha_2 := \lambda_{k+1} + \cdots + \lambda_{k+m}$. If $\alpha_1 > 0$ and $\alpha_2 > 0$, then define the vectors

$$q_1 := \frac{\lambda_1}{\alpha_1}p_1 + \cdots + \frac{\lambda_k}{\alpha_1}p_k \text{ and } q_2 := \frac{\lambda_{k+1}}{\alpha_2}p_{k+1} + \cdots + \frac{\lambda_{k+m}}{\alpha_2}p_{k+m}.$$

It may be verified that q_1 and q_2 are convex combinations of beliefs in $P_{a\sim b}$. Hence, by repeatedly using preservation of indifference, it follows that $q_1, q_2 \in P_{a\sim b}$. By (6.1) we have that $p = \alpha_1 q_1 - \alpha_2 q_2$, and thus $p \in A$.

If $\alpha_1 > 0$ and $\alpha_2 = 0$, then we have that $p = \alpha_1 q_1 + 0 \cdot q_1$, which is in A. The case when $\alpha_1 = 0$ and $\alpha_2 > 0$ is similar. Finally, when $\alpha_1 = 0$ and $\alpha_2 = 0$, then $p = 0 \cdot p_1 + 0 \cdot p_2$ for two arbitrary beliefs $p_1, p_2 \in P_{a\sim b}$, and hence $p \in A$.

In general, we thus see that every $p \in span(P_{a\sim b})$ is also in A, and thus $span(P_{a\sim b}) \subseteq A$. Together with the observation above that $A \subseteq span(P_{a\sim b})$, we conclude that $span(P_{a\sim b}) = A$. This completes the proof. □

The second preparatory result contains some further properties of the set of beliefs where the DM is indifferent between a and b, gathered in Lemma 6.2. In this lemma, we denote by $S_{a\sim b}$ the set of states s where $a \sim_{[s]} b$. Moreover, we say that there are preference reversals between a and b if there are beliefs $p, q \in \Delta(S)$ such that $a \succ_p b$ and $b \succ_q a$.

Lemma 6.2 (Linear structure of indifference sets) *Suppose there are two choices, a and b, and n states. Consider a conditional preference relation* $\succsim$ *that satisfies the regularity axioms. Then, the following properties hold: (a)* $P_{a\sim b} = span(P_{a\sim b}) \cap \Delta(S)$; *(b) if* $\succsim$ *has preference reversals between a and b, then* $span(P_{a\sim b})$ *is a hyperplane with dimension* $n-1$, *and there is a full support belief* $p \in P_{a\sim b}$ *with* $p(s) > 0$ *for all* $s \in S$;*(c) if a weakly dominates b under* $\succsim$ *then* $P_{a\sim b} = \{p \in \Delta(S) \mid \sum_{s\in S_{a\sim b}} p(s) = 1\}$.

Proof **(a)** Clearly, $P_{a\sim b} \subseteq span(P_{a\sim b}) \cap \Delta(S)$. It remains to show that $span(P_{a\sim b}) \cap \Delta(S) \subseteq P_{a\sim b}$. Take some $p \in span(P_{a\sim b}) \cap \Delta(S)$. Then, by Lemma 6.1, there are beliefs $p_1, p_2 \in P_{a\sim b}$ and numbers λ_1, λ_2 such that $p = \lambda_1 p_1 + \lambda_2 p_2$. Since $p \in \Delta(S)$, we must have that $\sum_{s\in S} p(s) = 1$. Moreover, as p_1, p_2 are beliefs, it holds that $\sum_{s\in S} p_1(s) = \sum_{s\in S} p_2(s) = 1$. But then, it must be that $\lambda_1 + \lambda_2 = 1$.

Suppose first that $\lambda_1 = 0$. Then, $\lambda_2 = 1$, and hence $p = p_2$, which is in $P_{a\sim b}$. The case where $\lambda_2 = 0$ is similar. Assume next that $\lambda_1, \lambda_2 > 0$. As $\lambda_1 + \lambda_2 = 1$, it follows that p is a convex combination of p_1 and p_2, which are both in $P_{a\sim b}$. By preservation of indifference, it follows that $p \in P_{a\sim b}$.

Suppose now that $\lambda_1 > 0$ and $\lambda_2 < 0$. Since $\lambda_1 + \lambda_2 = 1$, it must be that $\lambda_1 > 1$. Hence, we have that

$$p_1 = \frac{1}{\lambda_1}p - \frac{\lambda_2}{\lambda_1}p_2 = \frac{1}{\lambda_1}p + \left(1 - \frac{1}{\lambda_1}\right)p_2 \tag{6.2}$$

since $\lambda_2 = 1 - \lambda_1$. As $\lambda_1 > 1$, it follows that p_1 is a convex combination of p and p_2, where p_1 and p_2 are both in $P_{a\sim b}$.

We will show that p must be in $P_{a\sim b}$. Suppose, on the contrary, that $p \notin P_{a\sim b}$. Assume, without loss of generality, that $p \in P_{a\succ b}$. Then, it follows from (6.2) and preservation of strict preference that $p_1 \in P_{a \succ b}$, which is a contradiction. Hence, $p \in P_{a \sim b}$. The case where $\lambda_1 < 0$ and $\lambda_2 > 0$ is similar. In general, we conclude that every $p \in span(P_{a \sim b}) \cap \Delta(S)$ is also in $P_{a \sim b}$. Hence, $span(P_{a \sim b}) \cap \Delta(S) \subseteq P_{a \sim b}$. As we have already seen that $P_{a \sim b} \subseteq span(P_{a \sim b}) \cap \Delta(S)$, we have that $P_{a \sim b} = span(P_{a \sim b}) \cap \Delta(S)$.

(b) Suppose that $\succsim$ has preference reversals on $\{a, b\}$. Then, there must be a state x where $a \succ_x b$, and another state y where $b \succ_y a$. Here, we write $a \succ_x b$ as a shortcut for $a \succ_{[x]} b$. By continuity, there must be a belief $p_2 = (1 - \lambda_2)[x] + \lambda_2[y]$ on the line segment between $[x]$ and $[y]$ where $a \sim_{p_2} b$. Now, let the remaining states be numbered $s_3, \dots, s_n$ such that

$$
\begin{aligned}
&a \succ_{s_k} b \text{ for all } k \in \{3, \dots, m\},\\
&b \succ_{s_k} a \text{ for all } k \in \{m+1, \dots, m+l\}, \text{ and}\\
&a \sim_{s_k} b \text{ for all } k \in \{m+l+1, \dots, n\}.
\end{aligned}
$$

We choose (i) for every $k \in \{3, \dots, m\}$ a belief $p_k = (1 - \lambda_k)[s_k] + \lambda_k[y]$ with $a \sim_{p_k} b$, (ii) for every $k \in \{m+1, \dots, m+l\}$ a belief $p_k = (1 - \lambda_k)[s_k] + \lambda_k[x]$ with $a \sim_{p_k} b$, and (iii) for every $k \in \{m+l+1, \dots, n\}$ the belief $p_k = [s_k]$ with $a \sim_{p_k} b$.

We will now show that $p_2, \dots, p_n$ are linearly independent. Take some numbers $\alpha_2, \dots, \alpha_n$ such that $\sum_{k=2}^{n} \alpha_k \cdot p_k = \underline{0}$. By construction, this sum is equal to

$$
\begin{aligned}
&\alpha_2((1 - \lambda_2)[x] + \lambda_2[y]) + \sum_{k=3}^{m} \alpha_k((1 - \lambda_k)[s_k] + \lambda_k[y])\\
&\quad + \sum_{k=m+1}^{m+l} \alpha_k((1 - \lambda_k)[s_k] + \lambda_k[x]) + \sum_{k=m+l+1}^{n} \alpha_k[s_k]\\
&= \left(\alpha_2(1 - \lambda_2) + \sum_{k=m+1}^{m+l} \alpha_k \lambda_k\right)[x] + \left(\alpha_2 \lambda_2 + \sum_{k=3}^{m} \alpha_k \lambda_k\right)[y]\\
&\quad + \sum_{k=3}^{m+l} \alpha_k(1 - \lambda_k)[s_k] + \sum_{k=m+l+1}^{n} \alpha_k[s_k] = \underline{0}.
\end{aligned}
$$

As the vectors $[x], [y], [s_3], \dots, [s_n]$ are linearly independent, and $0 < \lambda_k < 1$ for all $k \in \{2, \dots, m+l\}$, it follows that $\alpha_k = 0$ for all $k \in \{3, \dots, n\}$. This, in turn, implies that also $\alpha_2 = 0$. Hence, the indifference beliefs $p_2, \dots, p_n \in P_{a \sim b}$ are linearly independent.

As a consequence, the dimension of $span(P_{a \sim b})$ is at least $n - 1$. The dimension of $span(P_{a \sim b})$ cannot be n, since otherwise we would have that $span(P_{a \sim b}) = \mathbf{R}^S$, and hence, by (a), $P_{a \sim b} = \mathbf{R}^S \cap \Delta(S) = \Delta(S)$. This would contradict the assumption that there are preference reversals between a and b. We thus conclude that the dimension of $span(P_{a \sim b})$ must be $n - 1$, and therefore $span(P_{a \sim b})$ is a hyperplane.

To show that $P_{a \sim b}$ contains a belief p with $p(s) > 0$ for every state s, consider the vector $p := \frac{1}{n-1} p_2 + \dots + \frac{1}{n-1} p_n$. It may be verified that p is a belief. Moreover, by construction of the beliefs $p_2, \dots, p_n$, we have that $p(s) > 0$ for all states s.

(c) Let $A = \{p \in \Delta(S) \mid \sum_{s \in S_{a \sim b}} p(s) = 1\}$. To show that $P_{a \sim b} \subseteq A$, take some $p \in P_{a \sim b}$. Assume, contrary to what we want to show, that $p \notin A$. Then, $p(s) > 0$ for some $s \in S_{a \succ b}$, where $S_{a \succ b}$ is the set of states t with $a \succ_t b$. As $p = \sum_{s \in S_{a \sim b}} p(s) \cdot [s] + \sum_{s \in S_{a \succ b}} p(s) \cdot [s]$ it follows by preservation of indifference and preservation of strict preference that $p \in P_{a \succ b}$. This is a contradiction to the assumption that $p \in P_{a \sim b}$. We thus conclude that $p \in A$. Hence, $P_{a \sim b} \subseteq A$. The inclusion $A \subseteq P_{a \sim b}$ follows directly by preservation of indifference. We thus see that $P_{a \sim b} = A$. This completes the proof. □

The third preparatory result provides sufficient conditions for an expected utility representation between two choices.

Lemma 6.3 (Sufficient conditions for expected utility representation) *Consider a conditional preference relation $\succsim$ that satisfies the regularity axioms, two choices a and b, and a utility function u. Suppose that $\succsim$ has preference reversals between a and b, and that there are n states. If there is a belief p^* with $a \succ_{p^*} b$ and $u(a, p^*) > u(b, p^*)$, and $n-1$ linearly independent vectors $v_1, \dots, v_{n-1}$ in $span(P_{a \sim b})$ with $u(a, v_k) = u(b, v_k)$ for all $k \in \{1, \dots, n-1\}$, then u represents $\succsim$ on $\{a, b\}$.*

Proof Let $P_{u(a)=u(b)}$ be the set of beliefs p with $u(a,p) = u(b,p)$, and similarly for $P_{u(a)>u(b)}$. To show that u represents $\succsim$ on $\{a, b\}$, it is thus sufficient to show that $P_{a \sim b} = P_{u(a)=u(b)}$ and $P_{a \succ b} = P_{u(a)>u(b)}$.

We start by showing that $P_{a \sim b} = P_{u(a)=u(b)}$. Consider the set $V_{u(a)=u(b)} := \{v \in \mathbf{R}^S \mid u(a, v) = u(b, v)\}$. It may be verified that $V_{u(a)=u(b)}$ is a linear space. Moreover, $P_{u(a)=u(b)} = V_{u(a)=u(b)} \cap \Delta(S)$. We now show that $span(P_{a \sim b}) = V_{u(a)=u(b)}$. We first prove that $span(P_{a \sim b}) \subseteq V_{u(a)=u(b)}$. In Lemma 6.2 (b) we have seen that $span(P_{a \sim b})$ has dimension $n-1$. Since the vectors $v_1, \dots, v_{n-1}$ in $span(P_{a \sim b})$ are linearly independent, we conclude that $\{v_1, \dots, v_{n-1}\}$ is a basis of $span(P_{a \sim b})$. Take some $v \in span(P_{a \sim b})$. Then, we can write $v = \lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1}$ for some numbers $\lambda_1, \dots, \lambda_{n-1}$. Since $v_k \in V_{u(a)=u(b)}$ for all $k \in \{1, \dots, n-1\}$ and $V_{u(a)=u(b)}$ is a linear subspace, it follows that $v \in V_{u(a)=u(b)}$. Thus, $span(P_{a \sim b}) \subseteq V_{u(a)=u(b)}$.

We now show that $V_{u(a)=u(b)} = span(P_{a \sim b})$. Since $V_{u(a)=u(b)}$ is a linear subspace of $\mathbf{R}^S$, its dimension can be at most n. Moreover, as $span(P_{a \sim b}) \subseteq V_{u(a)=u(b)}$ and $span(P_{a \sim b})$ has dimension $n-1$, the dimension of $V_{u(a)=u(b)}$ is at least $n-1$. Suppose, contrary to what we want to prove, that $V_{u(a)=u(b)} \neq span(P_{a \sim b})$. Then, the dimension of $V_{u(a)=u(b)}$ must be n, and hence $V_{u(a)=u(b)} = \mathbf{R}^S$. However, this is a contradiction since $u(a, p^*) > u(b, p^*)$, and hence $p^* \notin V_{u(a)=u(b)}$. We thus conclude that $V_{u(a)=u(b)} = span(P_{a \sim b})$. Since $P_{u(a)=u(b)} = V_{u(a)=u(b)} \cap \Delta(S)$ and, by Lemma 6.2 (a), $P_{a \sim b} = span(P_{a \sim b}) \cap \Delta(S)$, we conclude that $P_{a \sim b} = P_{u(a)=u(b)}$.

We next prove that $P_{a \succ b} = P_{u(a)>u(b)}$. Let p^* be the belief where $a \succ_{p^*} b$ and $u(a, p^*) > u(b, p^*)$. Consider the set

$$A := \{p \in \Delta(S) | \text{ there is no } \lambda \in [0,1] \text{ with } (1-\lambda)p + \lambda p^* \in P_{a \sim b}\}.$$

We show that $P_{a \succ b} = A$. To prove that $P_{a \succ b} \subseteq A$, take some $p \in P_{a \succ b}$. Since $p^* \in P_{a \succ b}$ it follows by preservation of strict preference that $(1-\lambda)p + \lambda p^* \in P_{a \succ b}$ for every $\lambda \in [0,1]$, and hence $p \in A$. Thus, $P_{a \succ b} \subseteq A$.

To show that $A \subseteq P_{a \succ b}$, take some $p \in A$. Suppose that $p \notin P_{a \succ b}$. Since $p \in A$, we must have that $p \notin P_{a \sim b}$, and hence $p \in P_{b \succ a}$. By continuity, there must then be some $\lambda \in (0,1)$ with $(1-\lambda)p + \lambda p^* \in P_{a \sim b}$. This, however, contradicts the assumption that $p \in A$. Hence, $p \in P_{a \succ b}$, which yields $A \subseteq P_{a \succ b}$. Altogether, we conclude that $P_{a \succ b} = A$.

We next show that $P_{u(a)>u(b)} = A$. Since $P_{a \sim b} = P_{u(a)=u(b)}$, it follows that

$$A = \{p \in \Delta(S) | \text{ there is no } \lambda \in [0,1] \text{ with } (1-\lambda)p + \lambda p^* \in P_{u(a)=u(b)}\}.$$

As $p^* \in P_{u(a)>u(b)}$ by construction, it can be shown in a similar same way as above that $P_{u(a)>u(b)} = A$. As such, $P_{a \succ b} = A = P_{u(a)>u(b)}$.

Since $P_{a \sim b} = P_{u(a)=u(b)}$ and $P_{a \succ b} = P_{u(a)>u(b)}$, the utility function u represents $\succsim$ on $\{a, b\}$. This completes the proof. □

The following result contains an axiomatic characterization of expected utility for the case of two choices.

Lemma 6.4 (Characterization of expected utility for two choices) *Consider a set A consisting of two acts, a finite set of states S, and a conditional preference relation $\succsim$ on (A, S). Then, $\succsim$ has an expected utility representation, if and only if, it satisfies completeness, transitivity, continuity, preservation of indifference and preservation of strict preference.*

Proof of Lemma 6.4 Suppose first that $\succsim$ has an expected utility representation u. Then, it can easily be verified that $\succsim$ satisfies the regularity axioms. We leave this to the reader.

Assume next that $\succsim$ satisfies the regularity axioms. We will show that $\succsim$ has an expected utility representation. We distinguish three cases: (a) there are preference reversals between a and b, (b) a weakly dominates b, and (c) b weakly dominates a. For the remainder of this proof, we assume that the number of states is n.

(a) Suppose that there are preference reversals between a and b. Since we know from Lemma 6.2 (b) that $span(P_{a \sim b})$ has dimension $n-1$, there are $n-1$ linearly independent beliefs $p_1, \ldots, p_{n-1} \in P_{a \sim b}$. Moreover, there must be some state x with $a \succ_{[x]} b$. As $[x] \notin P_{a \sim b}$, we know from Lemma 6.2 (a) that $[x] \notin span(P_{a \sim b})$, and hence the beliefs $p_1, \ldots, p_{n-1}, [x]$ are linearly independent. Fix some number $\alpha < u(a,x)$, and find the unique utilities $\{u(b,s) \mid s \in S\}$ such that $u(b,x) = \alpha < u(a,x)$ and $u(b,p_k) = u(a,p_k)$ for all $k \in \{1, \ldots, n-1\}$. By Lemma 6.3 it then follows that u represents $\succsim$.

(b) Suppose that a weakly dominates b. Choose a utility function u such that, for every state s, we have $u(a,s) > u(b,s)$ when $[s] \in P_{a \succ b}$, and $u(a,s) = u(b,s)$ when $[s] \in P_{a \sim b}$. It then follows by Lemma 6.2 (c) that $P_{a \sim b} = P_{u(a)=u(b)}$. Since every belief p is either in $P_{a \sim b}$ or $P_{a \succ b}$, it follows that $P_{a \succ b} = P_{u(a)>u(b)}$. We thus conclude that the utility function u represents $\succsim$.

(c) This proof is similar to that for (b). The proof is hereby complete. □

The following result guarantees the existence of a line of beliefs with certain properties.

Lemma 6.5 (Line containing three indifference beliefs) *Consider a conditional preference relation* $\succsim$ *that has preference reversals for all pairs of choices, and satisfies the regularity axioms. Then, for every three choices* a, b, c, *there is a line of beliefs that contains full support beliefs* p_{ab}, p_{ac}, p_{bc} *where the DM is indifferent between the respective choices, and that contains a belief where the DM is not indifferent between any of these three choices.*

Proof Suppose first that there is a full support belief $p \in P_{a\sim b} \cap P_{b\sim c}$. Then, by transitivity, $p \in P_{a\sim c}$. We can then choose a line of beliefs through p that contains a belief where the DM is not indifferent between any of the three choices. Such a line will satisfy the statement in the lemma.

Assume next that there is no full support belief in $P_{a\sim b} \cap P_{b\sim c}$. By transitivity, there will be no full support belief in $P_{a\sim b} \cap P_{a\sim c}$ or $P_{b\sim c} \cap P_{a\sim c}$ either. Let $\Delta^+(S)$ be the set of full support beliefs. Then, the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ will be pairwise disjoint on $\Delta^+(S)$. As, by Lemma 6.2 (a), these indifference sets are the intersections of hyperplanes with $\Delta(S)$, it must be that one of these indifference sets is "in between" the other two. Suppose, without loss of generality, that $P_{b\sim c}$ is in between $P_{a\sim b}$ and $P_{a\sim c}$. By Lemma 6.2 (b), there is a full support belief $p_{ab} \in P_{a\sim b}$ and a full support belief $p_{ac} \in P_{a\sim c}$. Let l be the line of beliefs that goes through p_{ab} and p_{ac}. As the set $P_{b\sim c}$ is in between $P_{a\sim b}$ and $P_{a\sim c}$, there must be a belief $p_{bc} \in P_{b\sim c}$ on the line l between p_{ab} and p_{ac}. Moreover, p_{bc} is a full support belief, since p_{ab} and p_{ac} are full support beliefs. Finally, the full support beliefs p_{ab} and p_{ac} can be chosen such that l contains a belief where the DM is not indifferent between any of the three choices. The line l thus satisfies the requirements of the lemma. This completes the proof.

□

The following result provides a geometric characterization of three choice linear preference intensity.

Proposition 6.1 (Characterization of three choice linear preference intensity) *Consider a conditional preference relation* $\succsim$ *that has no weakly dominated choices and that satisfies the regularity axioms. Then,* $\succsim$ *satisfies three choice linear preference intensity, if and only if, for every three choices* a, b, c *it holds that* $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$.

Proof Consider a conditional preference relation $\succsim$ that has no weakly dominated choices and satisfies the regularity axioms. Since we exclude equivalent choices, it must be that $\succsim$ has preference reversals between every pair of choices.

(a) Assume first that $\succsim$ satisfies three choice linear preference intensity. Consider three choices a, b and c. We must show that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$.

Take some $q \in span(P_{a\sim b}) \cap span(P_{b\sim c})$. We distinguish two cases: (1) $\sum_{s\in S} q(s) \neq 0$, and (2) $\sum_{s\in S} q(s) = 0$.

Case 1. Assume that $\sum_{s\in S} q(s) \neq 0$. Then, there is some number $\lambda \neq 0$ such that $\hat{q} := \lambda q$ satisfies $\sum_{s\in S} \hat{q}(s) = 1$. Moreover, $\hat{q} \in span(P_{a\sim b}) \cap span(P_{b\sim c})$ also. By Lemma 6.5 there is a line l that contains full support beliefs $p_{ab} \in P_{a\sim b}, p_{bc} \in P_{b\sim c}$ and $p_{ac} \in P_{a\sim c}$. Then, there is some $\varepsilon \in (0, 1)$ small enough such that (i) the vectors $p'_{ab} := (1-\varepsilon)p_{ab} + \varepsilon\hat{q}$, $p'_{bc} := (1-\varepsilon)p_{bc} + \varepsilon\hat{q}$ and $p' := (1-\varepsilon)p_{ac} + \varepsilon\hat{q}$ are all in $\Delta(S)$, and (ii) the line l' through p'_{ab} and p'_{bc} contains a belief $p'_{ac} \in P_{a\sim c}$. Since $p'_{ab} - p'_{bc} = (1-\varepsilon)\cdot(p_{ab} - p_{bc})$, we conclude that the lines l and l' are parallel. Moreover, the lines l and l' can be chosen such that they contain beliefs where the DM is not indifferent between any of the three choices. Hence, by preservation of strict preference, p'_{ac} is the unique belief in $P_{a\sim c}$ on the line l'. Also, the lines l and l' can be chosen such that the probability of no state is constant on l or l'.

Recall that $\hat{q} \in span(P_{a\sim b})$. Thus, we conclude that $p'_{ab} \in span(P_{a\sim b}) \cap \Delta(S)$. By Lemma 6.2 (a) it follows that $p'_{ab} \in P_{a\sim b}$. As $\hat{q} \in span(P_{b\sim c})$ it can be shown in a similar way that $p'_{bc} \in P_{b\sim c}$.

Recall that $p' := (1-\varepsilon)p_{ac} + \varepsilon\hat{q}$. We will now show that $p'_{ac} = p'$. Suppose first that $p_{ab} = p_{bc}$. Then, by transitivity, $p_{ac} = p_{ab} = p_{bc}$. Moreover, by definition of p'_{ab} and p'_{bc} it follows that $p'_{ab} = p'_{bc}$, and hence by transitivity we must have that $p'_{ac} = p'_{ab} = p'_{bc}$. Thus, $p' = (1-\varepsilon)p_{ac} + \varepsilon\hat{q} = (1-\varepsilon)p_{ab} + \varepsilon\hat{q} = p'_{ab} = p'_{ac}$.

Suppose now that $p_{ab} \neq p_{bc}$. Then, by transitivity, the beliefs p_{ab}, p_{bc} and p_{ac} are pairwise different. By definition of p'_{ab} and p'_{bc}, we then have that $p'_{ab} \neq p'_{bc}$. Hence, by transitivity, the beliefs p'_{ab}, p'_{bc} and p'_{ac} are pairwise different. By three choice linear preference intensity, we have for every state s that

$$(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) = (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)). \quad (6.3)$$

Note that, by definition, $(p'_{ab}(s) - p'_{bc}(s)) = (1-\varepsilon)(p_{ab}(s) - p_{bc}(s))$. Since the beliefs p_{ab}, p_{bc} and p_{ac} are pairwise different, the beliefs p'_{ab}, p'_{bc} and p'_{ac} are pairwise different, and no state has constant probability on the lines l and l', it follows together with (6.3) that $(p'_{ac}(s) - p'_{bc}(s)) = (1-\varepsilon)(p_{ac}(s) - p_{bc}(s))$, and thus

$$p'_{ac}(s) = (1-\varepsilon)(p_{ac}(s) - p_{bc}(s)) + p'_{bc}(s) = (1-\varepsilon)p_{ac}(s) + \varepsilon\hat{q}(s) = p'(s).$$

As this holds for every state s, we conclude that $p'_{ac} = p'$. Thus, the belief $p' = (1-\varepsilon)p_{ac} + \varepsilon\hat{q}$ is in $P_{a\sim c}$. As such, $\hat{q} = \frac{1}{\varepsilon}p' + (1 - \frac{1}{\varepsilon})p_{ac} \in span(P_{a\sim c})$, which implies that $q \in span(P_{a\sim c})$ also.

Case 2. Assume next that $\sum_{s\in S} q(s) = 0$. Let $V_0 := \{v \in \mathbf{R}^S \mid \sum_{s\in S} v(s) = 0\}$. We distinguish two subcases: (2.1) $span(P_{a\sim b}) \cap span(P_{b\sim c}) \nsubseteq V_0$, and (2.2) $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq V_0$.

Case 2.1. Assume that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \nsubseteq V_0$. Then, there is some $r \in span(P_{a\sim b}) \cap span(P_{b\sim c})$ with $\sum_{s\in S} r(s) \neq 0$. Hence, we know by Case 1 that $r \in span(P_{a\sim c})$. Moreover, as $q, r \in span(P_{a\sim b}) \cap span(P_{b\sim c})$, we conclude that $q - r \in span(P_{a\sim b}) \cap span(P_{b\sim c})$ also, with $\sum_{s\in S}(q-r)(s) \neq 0$. Hence, by Case 1, also $q - r \in span(P_{a\sim c})$. As $q = r + (q-r)$, and both r and $q - r$ are in $span(P_{a\sim c})$, it follows that $q \in span(P_{a\sim c})$.

Case 2.2. Suppose that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq V_0$. It can be shown that $span(P_{a\sim b}) \cap span(P_{b\sim c}) = span(P_{a\sim b}) \cap V_0$. To see this, note first that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim b}) \cap V_0$, since $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq V_0$. Moreover, we also know that $span(P_{a\sim b}) \neq span(P_{b\sim c})$, since otherwise $span(P_{a\sim b}) \cap span(P_{b\sim c})$ would contain beliefs in $P_{a\sim b}$ which would clearly not be in V_0. Since, by Lemma 6.2 (b), $span(P_{a\sim b})$ and $span(P_{b\sim c})$ are linear subspaces of dimension $n-1$, it follows that $span(P_{a\sim b}) \cap span(P_{b\sim c})$ is a linear subspace of dimension $n-2$. Now, consider the linear subspace $span(P_{a\sim b}) \cap V_0$. Clearly, $span(P_{a\sim b}) \neq V_0$, since $span(P_{a\sim b})$ contains beliefs in $P_{a\sim b}$ which are not in V_0. Since $span(P_{a\sim b})$ and V_0 are linear subspaces of dimension $n-1$, it follows that $span(P_{a\sim b}) \cap V_0$ is a linear subspace of dimension $n-2$. Since $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim b}) \cap V_0$ and both linear subspaces have the same dimension, $n-2$, both spaces must be equal. Hence, $span(P_{a\sim b}) \cap span(P_{b\sim c}) = span(P_{a\sim b}) \cap V_0$.

Moreover, it must be that $span(P_{a\sim b}) \cap span(P_{a\sim c}) \subseteq V_0$ also. To see this, assume on the contrary that $span(P_{a\sim b}) \cap span(P_{a\sim c}) \nsubseteq V_0$. Then, it would follow from Case 2.1 that $span(P_{a\sim b}) \cap span(P_{a\sim c}) \subseteq span(P_{b\sim c})$, and thus $span(P_{a\sim b}) \cap span(P_{a\sim c}) \subseteq span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq V_0$. This would be a contradiction. Hence, we conclude that $span(P_{a\sim b}) \cap span(P_{a\sim c}) \subseteq V_0$. It can then be shown, in the same way as above, that $span(P_{a\sim b}) \cap span(P_{a\sim c}) = span(P_{a\sim b}) \cap V_0$.

By combining the latter two equalities, we get

$$span(P_{a\sim b}) \cap span(P_{b\sim c}) = span(P_{a\sim b}) \cap V_0 = span(P_{a\sim b}) \cap span(P_{a\sim c}),$$

which implies that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$. As $q \in span(P_{a\sim b}) \cap span(P_{b\sim c})$, it follows that $q \in span(P_{a\sim c})$. This completes the proof of (a).

(b) Suppose now that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$ for all three choices a, b, c. We must show that $\succsim$ satisfies three choice linear preference intensity. Consider two parallel lines of beliefs l, l' that (i) contain beliefs where the DM is not indifferent between any two choices from $\{a, b, c\}$, (ii) where l contains indifference beliefs $p_{ab} \in P_{a\sim b}, p_{bc} \in P_{b\sim c}$ and $p_{ac} \in P_{a\sim c}$, and (iii) l' contains indifference beliefs $p'_{ab} \in P_{a\sim b}, p'_{bc} \in P_{b\sim c}$ and $p'_{ac} \in P_{a\sim c}$. Let l_{ab} be the line through p_{ab} and p'_{ab}, let l_{bc} be the line through p_{bc} and p'_{bc}, and l_{ac} the line through p_{ac} and p'_{ac}. Note that all these lines belong to the same two-dimensional plane: the plane that goes through l and l'.

Assume first that the lines l_{ab}, l_{bc} and l_{ac} are all parallel. Then, there is a vector q such that $p'_{ab} = p_{ab} + q$, $p'_{bc} = p_{bc} + q$ and $p'_{ac} = p_{ac} + q$. As a consequence, for every state s,

$$\begin{aligned}(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) =& (p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) \\ =& (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)).\end{aligned}$$

Hence, the formula for three choice linear preference intensity is satisfied.

Assume next that the lines l_{ab}, l_{bc} and l_{ac} are not all parallel. Without loss of generality, we suppose that l_{ab} and l_{bc} are not parallel. Since these two lines lie in the same two-dimensional plane, they must intersect at a unique vector q. Since q lies on l_{ab}, which goes through p_{ab} and p'_{ab} in $P_{a\sim b}$, we conclude that $q \in span(P_{a\sim b})$. Similarly, as q lies on l_{bc}, it follows that $q \in span(P_{b\sim c})$. Since we assume that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$, we conclude that $q \in span(P_{a\sim c})$ too.

Let V be the two-dimensional plane that goes through the lines l and l'. Since, by condition (i) above, l and l' contain beliefs where the DM is not indifferent between a and c, it follows that $span(P_{a\sim c}) \cap V = l_{ac}$. As $q \in span(P_{a\sim c}) \cap V$, we conclude that q lies on the line l_{ac}.

As q lies on l_{ab}, l_{bc} and l_{ac}, the beliefs p_{ab}, p_{bc}, p_{ac} lie on l, the beliefs p'_{ab}, p'_{bc} and p'_{ac} lie on l', and the lines l and l' are parallel, there is a unique number λ such that $p'_{ab} = (1-\lambda)q + \lambda p_{ab}$, $p'_{bc} = (1-\lambda)q + \lambda p_{bc}$ and $p'_{ac} = (1-\lambda)q + \lambda p_{ac}$. Hence, for every state s we have that

$$\begin{aligned}(p_{ab}(s) - p_{bc}(s)) \cdot (p'_{ac}(s) - p'_{bc}(s)) =& \lambda \cdot (p_{ab}(s) - p_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)) \\ =& (p'_{ab}(s) - p'_{bc}(s)) \cdot (p_{ac}(s) - p_{bc}(s)).\end{aligned}$$

Thus, the formula for three choice linear preference intensity is satisfied. We therefore conclude that $\gtrsim$ satisfies three choice linear preference intensity. This completes the proof. □

In our last preparatory result, we characterize the span of an indifference set $P_{a\sim b}$ in case of an expected utility representation.

Lemma 6.6 (Span of indifference set under utility representation) *Consider a conditional preference relation $\gtrsim$ with an expected utility representation u. Suppose there are preferene reversals between choices a and b. Then,*

$$span(P_{a\sim b}) = \{q \in \mathbf{R}^S | u(a,q) = u(b,q)\}.$$

Proof Let $A := \{q \in \mathbf{R}^S \,|\, u(a,q) = u(b,q)\}$. We first show that $span(P_{a\sim b}) \subseteq A$. Take some $q \in span(P_{a\sim b})$. Then, by Lemma 6.1, there are $p_1, p_2 \in P_{a\sim b}$ and numbers λ_1, λ_2 such that $q = \lambda_1 p_1 + \lambda_2 p_2$. As $u(a,p_1) = u(b,p_1)$ and $u(a,p_2) = u(b,p_2)$, it follows that $u(a,q) = u(b,q)$, and hence $q \in A$. Thus, $span(P_{a\sim b}) \subseteq A$. By Lemma 6.2 (b) we know that $span(P_{a\sim b})$ has dimension $n-1$. Since A is a linear subspace with dimension $n-1$ also, and $span(P_{a\sim b}) \subseteq A$, it must be that $span(P_{a\sim b}) = A$. This completes the proof. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. (a) Suppose first that $\gtrsim$ has an expected utility representation u. From Lemma 6.4, we know that $\gtrsim$ satisfies the regularity axioms.

To show three choice linear preference intensity it suffices, in view of Proposition 6.1, to show that $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$ for all three choices a, b, c. Take some $q \in span(P_{a\sim b}) \cap span(P_{b\sim c})$. Then, by Lemma 6.1, there are $p^1_{ab}, p^2_{ab} \in P_{a\sim b}$, $p^1_{bc}, p^2_{bc} \in P_{b\sim c}$ and numbers $\lambda_1, \lambda_2, \mu_1, \mu_2$ such that

$q = \lambda_1 p^1_{ab} + \lambda_2 p^2_{ab} = \mu_1 p^1_{bc} + \mu_2 p^2_{bc}$. As $u(a, p^1_{ab}) = u(b, p^1_{ab})$ and $u(a, p^2_{ab}) = u(b, p^2_{ab})$, it follows that $u(a, q) = u(b, q)$. In a similar fashion, it follows that $u(b, q) = u(c, q)$, and hence $u(a, q) = u(c, q)$. By Lemma 6.6 it thus follows that $q \in span(P_{a\sim c})$. Hence, $span(P_{a\sim b}) \cap span(P_{b\sim c}) \subseteq span(P_{a\sim c})$, which implies by Proposition 6.1 that $\succsim$ satisfies three choice linear preference intensity.

We finally show four choice linear preference intensity. Consider a line of beliefs l, and four choices a, b, c, d such that there is a belief on the line where the DM is not indifferent between any pair of choices in $\{a, b, c, d\}$. Moreover, let $p_{ab}, p_{ac}, p_{ad}, p_{bc}, p_{bd}$ and p_{cd} be corresponding indifference beliefs on this line. Consider some state s. If the probability of s is constant on the line l, then the formula for four choice linear preference intensity holds trivially.

We therefore assume from now on that the probability of s is not constant on l, so that every belief on l is uniquely given by the probability it assigns to s. Suppose that $p_{ab} = p_{ac}$. Then, by transitivity, it must be that $p_{ab} = p_{ac} = p_{bc}$, and the formula for four choice linear preference intensity would hold trivially. Similarly, the formula would trivially hold if $p_{ab} = p_{ad}$ or $p_{ac} = p_{ad}$.

We now assume that p_{ab}, p_{ac}, p_{ad} are pairwise different. Then, by transitivity, p_{bc} is different from p_{ab} and p_{ac}, the belief p_{bd} is different from p_{ab} and p_{ad}, and the belief p_{cd} is different from p_{ac} and p_{ad}.

Consider two arbitrary, but different, beliefs p_1, p_2 on l, and define

$$\Delta(u(a) - u(b)) := (u(a, p_1) - u(b, p_1)) - (u(a, p_2) - u(b, p_2)).$$

As there is a belief on the line where the DM is indifferent between a and b, and another belief on the line where the DM is not, we must have that $\Delta(u(a) - u(b)) \neq 0$. In a similar way, we define $\Delta(u(a) - u(c))$ and $\Delta(u(a) - u(d))$.

By applying the arguments from Section 4.3 to expected utility differences, instead of preference intensity, it follows that

$$\frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(c))} = \frac{p_{ac}(s) - p_{bc}(s)}{p_{ab}(s) - p_{bc}(s)}. \tag{6.4}$$

Recall that also $\Delta(u(a) - u(c)) \neq 0$. Moreover, since $p_{ab} \neq p_{bc}$ and the belief on the line is uniquely given by its probability on s, we have that $p_{ab}(s) \neq p_{bc}(s)$. Thus, the two ratios above are well-defined. In a similar fashion, it follows that

$$\frac{\Delta(u(a) - u(c))}{\Delta(u(a) - u(d))} = \frac{p_{ad}(s) - p_{cd}(s)}{p_{ac}(s) - p_{cd}(s)} \text{ and } \frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(d))} = \frac{p_{ad}(s) - p_{bd}(s)}{p_{ab}(s) - p_{bd}(s)}. \tag{6.5}$$

As, by definition,

$$\frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(d))} = \frac{\Delta(u(a) - u(b))}{\Delta(u(a) - u(c))} \cdot \frac{\Delta(u(a) - u(c))}{\Delta(u(a) - u(d))},$$

it follows by (6.4) and (6.5) that the formula for four choice linear preference intensity obtains. Thus, $\succsim$ satisfies four choice linear preference intensity.

(b) Suppose that $\succsim$ satisfies the regularity axioms, three choice linear preference intensity and four choice linear preference intensity. If there are only two choices, then we know from Lemma 6.4 that there is an expected utility representation. We therefore assume, from now on, that there are at least three choices.

To show that $\succsim$ has an expected utility representation, we distinguish two cases: (1) $P_{a\sim b} = P_{c\sim d}$ for every two pairs of choices $\{a, b\}$ and $\{c, d\}$, and (2) $P_{a\sim b} \neq P_{c\sim d}$ for some pairs of choices $\{a, b\}$ and $\{c, d\}$.

Case 1. Suppose that $P_{a\sim b} = P_{c\sim d}$ for every two pairs of choices $\{a, b\}$ and $\{c, d\}$. Let $A := P_{a\sim b}$ for some pair of choices $\{a, b\}$. Note that $A \neq \Delta(S)$, as we assume that no two choices are equivalent under $\succsim$. Since we also assume that no choice weakly dominates another choice, there will be preference reversals between all pairs of choices. Let x be a state where $[x] \notin A$. Hence, $[x] \notin P_{a\sim b}$ for every two choices a and b. By transitivity, we can order the choices $c_1, c_2, \dots, c_K$ such that

$$c_1 \succ_{[x]} c_2 \succ_{[x]} c_3 \succ_{[x]} \cdots \succ_{[x]} c_K.$$

Choose numbers $v_1, \dots, v_K$ with $v_1 > v_2 > \cdots > v_K$.

For choice c_1, set $u(c_1, x) = v_1$, and set the utilities $u(c_1, s)$ for states $s \neq x$ arbitrarily.

By Lemma 6.2 (b) we know that $span(A)$ has dimension $n - 1$, where n is the number of states. Let $\{p_1, \dots, p_{n-1}\}$ be a basis for $span(A)$. As $[x] \notin span(A)$, we know that $\{p_1, \dots, p_{n-1}, [x]\}$ is a basis for $\mathbf{R}^S$. For every choice c_k with $k \geq 2$ find the unique utilities $u(c_k, s)$ such that

$$u(c_k, p_1) = u(c_1, p_1), \dots, u(c_k, p_{n-1}) = u(c_1, p_{n-1}) \text{ and } u(c_k, x) = v_k. \quad (6.6)$$

We will show that the utility function u represents $\succsim$.

Take two choices a, b with $a \succ_{[x]} b$. Then, by construction of the utility function, we have that $u(a, p_k) = u(b, p_k)$ for all $k \in \{1, \dots, n-1\}$, and $u(a, x) > u(b, x)$. As $\{p_1, \dots, p_{n-1}\}$ is a basis for $span(P_{a\sim b})$, we know that $p_1, \dots, p_{n-1}$ are linearly independent. It thus follows by Lemma 6.3 that u represents $\succsim$ on the pair of choices $\{a, b\}$. As this holds for every pair of choices $\{a, b\}$, we conclude that u represents $\succsim$.

Case 2. Suppose that $P_{a\sim b} \neq P_{c\sim d}$ for some pairs of choices $\{a, b\}$ and $\{c, d\}$. Then, there must be some choices a, b, c such that $P_{a\sim c} \neq P_{b\sim c}$. To see this, suppose on the contrary that $P_{a\sim c} = P_{b\sim c}$ for all three choices a, b, c. Then, take two arbitrary pairs of choices $\{a, b\}$ and $\{c, d\}$ where $\{a, b\} \cap \{c, d\} = \emptyset$. By assumption we would then have that $P_{a\sim b} = P_{b\sim c} = P_{c\sim d}$, and hence $P_{a\sim b} = P_{c\sim d}$ for all pairs $\{a, b\}$ and $\{c, d\}$. This would be a contradiction. Hence, $P_{a\sim c} \neq P_{b\sim c}$ for some choices a, b, c.

Now take some choice d different from a, b and c, if it exists. Then, either $P_{a\sim d} \neq P_{b\sim d}$ or $P_{a\sim d} \neq P_{c\sim d}$. To see this, suppose on the contrary that $P_{a\sim d} = P_{b\sim d} = P_{c\sim d}$. Define $A := P_{a\sim d} = P_{b\sim d} = P_{c\sim d}$. Since, by transitivity, $P_{a\sim d} \cap P_{b\sim d} \subseteq P_{a\sim b}$ and $P_{b\sim d} \cap P_{c\sim d} \subseteq P_{b\sim c}$, it follows that $A \subseteq P_{a\sim b} \cap P_{b\sim c}$. Thus, $span(A) \subseteq span(P_{a\sim b}) \cap span(P_{b\sim c})$. However, since $P_{a\sim c} \neq P_{b\sim c}$ we have, by transitivity, that $P_{a\sim b} \neq P_{b\sim c}$. As, by Lemma 6.2 (b), both $span(P_{a\sim b})$ and $span(P_{b\sim c})$ have dimension $n - 1$, it must be that $span(P_{a\sim b}) \cap span(P_{b\sim c})$ has dimension $n - 2$. However, A has dimension $n - 1$, and hence it cannot be that

$span(A) \subseteq span(P_{a\sim b}) \cap span(P_{b\sim c})$. We thus obtain a contradiction, and conclude that either $P_{a\sim d} \neq P_{b\sim d}$ or $P_{a\sim d} \neq P_{c\sim d}$.

Based on the two insights above, we can order the choices $c_1, c_2, \ldots, c_K$ such that $P_{c_3\sim c_1} \neq P_{c_3\sim c_2}$, and for every $k \geq 4$ either $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$ or $P_{c_k\sim c_1} \neq P_{c_k\sim c_3}$. Let the utilities for c_1 and c_2 be given as in the proof of Lemma 6.4. For the other choices, we define their utilities according to the following procedure:

Utilities for c_3: By Lemma 6.2 (b), there are $n-1$ linearly independent beliefs $p_1, \ldots, p_{n-1} \in P_{c_3\sim c_1}$. Choose a belief $p_n \in P_{c_3\sim c_2} \backslash P_{c_3\sim c_1}$. Note that this is possible since $P_{c_3\sim c_1} \neq P_{c_3\sim c_2}$, and because of Lemma 6.2 (a) and (b). By Lemma 6.2 (a), $p_n \notin span(P_{c_3\sim c_1})$, and hence $p_1, \ldots, p_{n-1}, p_n$ are linearly independent. Find the unique utilities $\{u(c_3, s) \mid s \in S\}$ such that

$$u(c_3, p_m) = u(c_1, p_m) \text{ for all } m \in \{1, \ldots, n-1\}, \text{ and } u(c_3, p_n) = u(c_2, p_n). \tag{6.7}$$

Utilities for $c_4, \ldots, c_K$. For every $k \geq 4$, inductively define the utilities for c_k as follows. From above, we know that either $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$ or $P_{c_k\sim c_1} \neq P_{c_k\sim c_3}$. Suppose that $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$. Like above, we can choose linearly independent beliefs $p_1, \ldots, p_{n-1}, p_n$ where $p_1, \ldots, p_{n-1} \in P_{c_k\sim c_1}$ and $p_n \in P_{c_k\sim c_2} \backslash P_{c_k\sim c_1}$. Find the unique utilities $\{u(c_k, s) \mid s \in S\}$ such that

$$u(c_k, p_m) = u(c_1, p_m) \text{ for all } m \in \{1, \ldots, n-1\}, \text{ and } u(c_k, p_n) = u(c_2, p_n). \tag{6.8}$$

If $P_{c_k\sim c_1} \neq P_{c_k\sim c_3}$, the utilities can be defined analogously,

We will now show that these utilities represents the conditional preference relation $\gtrsim$.

We prove, by induction on k, that u represents $\gtrsim$ on $\{c_1, \ldots, c_k\}$. For $k = 2$ we know this is true, in the light of the proof of Lemma 6.4.

Suppose now that $k \geq 3$, and that u represents $\gtrsim$ on $\{c_1, \ldots, c_{k-1}\}$. We must show that u represents $\gtrsim$ on all pairs $\{c_k, c_m\}$ where $m \in \{1, \ldots, k-1\}$.

We start by showing that u represents $\gtrsim$ on $\{c_k, c_1\}$. Assume, without loss of generality, that $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$. Then, by (6.7) and (6.8) we know that $u(c_k, p_n) = u(c_2, p_n)$. As $p_n \in P_{c_k\sim c_2} \backslash P_{c_k\sim c_1}$, we may assume, without loss of generality, that $p_n \in P_{c_k \succ c_1}$. As $p_n \in P_{c_k\sim c_2}$, it follows that $p_n \in P_{c_2 \succ c_1}$, and hence $u(c_2, p_n) > u(c_1, p_n)$. As, by (6.7) and (6.8), $u(c_k, p_n) = u(c_2, p_n)$, we conclude that $u(c_k, p_n) > u(c_1, p_n)$. Thus, $p_n \in P_{c_k \succ c_1}$ is such that $u(c_k, p_n) > u(c_1, p_n)$. Together with (6.7) and (6.8), we conclude from Lemma 6.3 that u represents $\gtrsim$ on $\{c_k, c_1\}$.

We next show that u represents $\gtrsim$ on $\{c_k, c_2\}$. As $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$, it follows by transitivity that $P_{c_k\sim c_1} \neq P_{c_1\sim c_2}$. Hence, it follows by Lemma 6.2 (a) and (b) that $span(P_{c_k\sim c_1}) \cap span(P_{c_1\sim c_2})$ has dimension $n-2$. Take a basis $\{q_1, \ldots, q_{n-2}\}$ for $span(P_{c_k\sim c_1}) \cap span(P_{c_1\sim c_2})$. By Proposition 6.1 we know that $span(P_{c_k\sim c_1}) \cap span(P_{c_1\sim c_2}) \subseteq span(P_{c_k\sim c_2})$, and hence the vectors $q_1, \ldots, q_{n-2}$ are all in $span(P_{c_k\sim c_2})$. As each of these vectors q_m is in $span(P_{c_k\sim c_1})$, it follows by Lemma 6.1 that q_m can be written as $q_m = \lambda_1 r_1 + \lambda_2 r_2$, where $\lambda_1, \lambda_2 \in \mathbf{R}$ and $r_1, r_2 \in P_{c_k\sim c_1}$. Since u represents $\gtrsim$ on $\{c_k, c_1\}$, we know that $u(c_k, r_1) = u(c_1, r_1)$ and $u(c_k, r_2) = u(c_1, r_2)$, which implies that $u(c_k, q_m) = u(c_1, q_m)$. As q_m is also in $span(P_{c_1 \sim c_2})$, and u represents $\succsim$ on $\{c_1, c_2\}$, it follows in a similar way that $u(c_1, q_m) = u(c_2, q_m)$. Hence, we conclude that

$$u(c_k, q_m) = u(c_2, q_m) \text{ for all } m \in \{1, \ldots, n-2\} \text{ and } u(c_k, p_n) = u(c_2, p_n), \quad (6.9)$$

where the last equality follows from (6.7) and (6.8). Moreover, as $p_n \notin P_{c_k \sim c_1}$, we know that $p_n \notin span(P_{c_k \sim c_1}) \cap span(P_{c_1 \sim c_2})$, and hence the $n-1$ vectors above are linearly independent.

Since $P_{c_k \sim c_1} \neq P_{c_k \sim c_2}$, there is a belief $p \in P_{c_k \sim c_1} \backslash P_{c_k \sim c_2}$. Assume, without loss of generality, that $p \in P_{c_k \succ c_2}$. As $p \in P_{c_k \sim c_1}$, it follows by transitivity that $p \in P_{c_1 \succ c_2}$. As u represents $\succsim$ on $\{c_k, c_1\}$ and $\{c_1, c_2\}$, it follows that $u(c_k, p) = u(c_1, p)$ and $u(c_1, p) > u(c_2, p)$, which implies that $u(c_k, p) > u(c_2, p)$. Hence, there is some belief p with $P_{c_k \succ c_2}$ and $u(c_k, p) > u(c_2, p)$. Together with (6.9) and Lemma 6.3, we conclude that u represents $\succsim$ on $\{c_k, c_2\}$.

We finally show that u represents $\succsim$ on $\{c_k, c_m\}$ for every $m \in \{3, \ldots, k-1\}$. Take some $m \in \{3, \ldots, k-1\}$. Then, necessarily, $k \geq 4$. To abbreviate the notation, we define $span_{ml} := span(P_{c_m \sim c_l})$ for every $m, l \in \{1, \ldots, k\}$. We distinguish two cases: (1) $span_{k1} \cap span_{1\,m} \neq span_{k2} \cap span_{2\,m}$ or $span_{k1} \cap span_{1\,m} \neq span_{k3} \cap span_{3\,m}$, and (2) $span_{k1} \cap span_{1\,m} = span_{k2} \cap span_{2\,m}$ and $span_{k1} \cap span_{1\,m} = span_{k3} \cap span_{3\,m}$.

Case 1. Assume, without loss of generality, that $span_{k1} \cap span_{1\,m} \neq span_{k2} \cap span_{2\,m}$. Since, by Lemma 6.2 (b), the four linear spans have dimension $n-1$, it follows that the two intersections have dimension $n-2$ or $n-1$. Moreover, as the two intersections are different, we conclude that

$$A := span[(span_{k1} \cap span_{1m})) \cup (span_{k2} \cap span_{2m})]$$

has dimension $n-1$ or n. Moreover, we know from Proposition 6.1 that $span_{k1} \cap span_{1\,m}$ and $span_{k2} \cap span_{2\,m}$ are both subsets of $span_{km}$, and hence $A \subseteq span_{km}$ also. As c_k and c_m are not equivalent, A cannot have dimension n, and thus the dimension of A must be $n-1$.

Take a basis $\{q_1, \ldots, q_{n-1}\}$ for A, where every q_l is either in $span_{k1} \cap span_{1m}$ or in $span_{k2} \cap span_{2\,m}$. Suppose that q_l is in $span_{k1} \cap span_{1\,m}$. As u represents $\succsim$ on $\{c_k, c_1\}$ and $\{c_1, c_m\}$, it can be shown in the same way as above that $u(c_k, q_l) = u(c_1, q_l)$ and $u(c_1, q_l) = u(c_m, q_l)$, which implies that $u(c_k, q_l) = u(c_m, c_l)$. If q_l is in $span_{k2} \cap span_{2\,m}$, it can be shown in a similar way that $u(c_k, q_l) = u(c_m, q_l)$ also. We thus see that

$$q_l \in span_{km} \text{ and } u(c_k, q_l) = u(c_m, q_l) \text{ for every } l \in \{1, \ldots, n-1\}. \quad (6.10)$$

Since $P_{c_k \sim c_1} \neq P_{c_k \sim c_2}$, either $P_{c_k \sim c_1} \backslash P_{c_k \sim c_m}$ or $P_{c_k \sim c_2} \backslash P_{c_k \sim c_m}$ must be non-empty. Assume, without loss of generality, that $P_{c_k \sim c_1} \backslash P_{c_k \sim c_m}$ is non-empty. Take some $p \in P_{c_k \sim c_1} \backslash P_{c_k \sim c_m}$. Assume, without loss of generality, that $p \in P_{c_k \succ c_m}$. As $p \in P_{c_k \sim c_1}$, it follows by transitivity that $p \in P_{c_1 \succ c_m}$. Since u represents $\succsim$ on $\{c_k, c_1\}$ and $\{c_1, c_m\}$, we know that $u(c_k, p) = u(c_1, p)$ and $u(c_1, p) > u(c_m, p)$, and thus $u(c_k, p) > u(c_m, p)$. We have thus found a belief $p \in P_{c_k \succ c_m}$ with $u(c_k, p) > u(c_m, p)$. Together with (6.10) and Lemma 6.3 we conclude that u represents $\succsim$ on $\{c_k, c_m\}$.

Case 2. Suppose that $span_{k1} \cap span_{1m} = span_{k2} \cap span_{2m}$ and $span_{k1} \cap span_{1m} = span_{k3} \cap span_{3m}$.

Claim. There are $i, j \in \{1, 2, 3\}$ such that for every triple $a, b, c \in \{c_i, c_j, c_m, c_k\}$ the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different.

Proof of claim. We first show that $P_{c_k\sim c_1} \neq P_{c_1\sim c_m}$. Suppose not. Then, $P_{c_k\sim c_1} = P_{c_1\sim c_m}$ and hence, by transitivity, $P_{c_k\sim c_1} = P_{c_k\sim c_m}$. Thus, $span_{k1} \cap span_{1m} = span_{k1} = span_{km}$. Since $span_{k1} \cap span_{1m} = span_{k2} \cap span_{2m}$, it follows that $span_{k2} \cap span_{2m} = span_{km}$, which can only be if $P_{c_k\sim c_2} = P_{c_2\sim c_m} = P_{c_k\sim c_m}$. As such, $P_{c_k\sim c_1} = P_{c_k\sim c_m} = P_{c_k\sim c_2}$, which contradicts the assumption that $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$. Hence, $P_{c_k\sim c_1} \neq P_{c_1\sim c_m}$. By transitivity, $P_{c_k\sim c_1}, P_{c_1\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different.

As a consequence, $span_{k1} \cap span_{1m}$ has dimension $n-2$. Since $span_{k1} \cap span_{1m} = span_{k2} \cap span_{2m}$ it follows that $span_{k2} \cap span_{2m}$ has dimension $n-2$ also, which can only be if $P_{c_k\sim c_2} \neq P_{c_2\sim c_m}$. Thus, by transitivity, $P_{c_k\sim c_2}, P_{c_2\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different. As $span_{k1} \cap span_{1m} = span_{k3} \cap span_{3m}$, it follows in a similar way that $P_{c_k\sim c_3}, P_{c_3\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different also.

Consider the sets $A = \{c_1, c_2, c_m, c_k\}$, $B = \{c_1, c_3, c_m, c_k\}$ and $C = \{c_2, c_3, c_m, c_k\}$. Suppose, contrary to what we want to show, that in each of these sets there is a triple a, b, c such that $P_{a\sim b} = P_{a\sim c} = P_{b\sim c}$. Since, by assumption, $P_{c_k\sim c_1} \neq P_{c_k\sim c_2}$, and we have seen above that $P_{c_k\sim c_1}, P_{c_1\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different and $P_{c_k\sim c_2}, P_{c_2\sim c_m}$ and $P_{c_k\sim c_m}$ are pairwise different, we must have in set A that $P_{c_1\sim c_2} = P_{c_2\sim c_m} = P_{c_1\sim c_m}$.

By a similar argument, we must have in set B that either $P_{c_1\sim c_3} = P_{c_3\sim c_m} = P_{c_1\sim c_m}$ or $P_{c_1\sim c_3} = P_{c_3\sim c_k} = P_{c_1\sim c_k}$. However, if $P_{c_1\sim c_3} = P_{c_3\sim c_m} = P_{c_1\sim c_m}$ then, by the insight above that $P_{c_1\sim c_2} = P_{c_1\sim c_m}$, it would follow that $P_{c_1\sim c_2} = P_{c_1\sim c_3}$, which is a contradiction to the fact that $P_{c_1\sim c_2} \neq P_{c_1\sim c_3}$. We must thus have that $P_{c_1\sim c_3} = P_{c_3\sim c_k} = P_{c_1\sim c_k}$.

By a similar argument, we must have in set C that either $P_{c_2\sim c_3} = P_{c_3\sim c_m} = P_{c_2\sim c_m}$ or $P_{c_2\sim c_3} = P_{c_3\sim c_k} = P_{c_2\sim c_k}$. If $P_{c_2\sim c_3} = P_{c_3\sim c_m} = P_{c_2\sim c_m}$ then, together with the insight above that $P_{c_1\sim c_2} = P_{c_2\sim c_m}$, it would follow that $P_{c_1\sim c_2} = P_{c_2\sim c_3}$. This would contradict the assumption that $P_{c_1\sim c_2} \neq P_{c_2\sim c_3}$. If $P_{c_2\sim c_3} = P_{c_3\sim c_k} = P_{c_2\sim c_k}$ then, together with the fact above that $P_{c_1\sim c_3} = P_{c_3\sim c_k}$, it would follow that $P_{c_1\sim c_3} = P_{c_2\sim c_3}$. This would contradict the assumption that $P_{c_1\sim c_3} \neq P_{c_2\sim c_3}$. We thus arrive at a general contradiction, and hence there are $i, j \in \{1, 2, 3\}$ such that for every triple $a, b, c \in \{c_i, c_j, c_m, c_k\}$ the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different. This completes the proof of the claim.

According to the claim, we can choose $i, j \in \{1, 2, 3\}$ such that for every triple $a, b, c \in \{c_i, c_j, c_m, c_k\}$ the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different. Define the set of choices $D := \{c_i, c_j, c_m, c_k\}$, and let

$$A := span_{ki} \cap span_{im}.$$

We show that A has dimension $n-2$, that $A \subseteq span(P_{a\sim b})$ for all $a, b \in D$, and that $A = span(P_{a\sim b}) \cap span(P_{c\sim d})$ whenever $P_{a\sim b} \neq P_{c\sim d}$.

Since by the choice of i, j we have that $P_{c_k\sim c_i} \neq P_{c_i\sim c_m}$, it follows that A has dimension $n-2$. Note by Proposition 6.1 that $A \subseteq span_{km}$. Moreover, as we assume in Case 2 that $span_{ki} \cap span_{im} = span_{kj} \cap span_{jm}$, it follows that

$A = span_{ki} \cap span_{im} \cap span_{kj}$, and thus we have by Proposition 6.1 that $A \subseteq span_{ij}$ also. Hence, $A \subseteq span(P_{a\sim b})$ for all $a, b \in D$.

Now, let $P_{a\sim b} \neq P_{c\sim d}$ for some $a, b, c, d \in D$. Then $span(P_{a\sim b}) \cap span(P_{c\sim d})$ has dimension $n-2$. As $A \subseteq span(P_{a\sim b}) \cap span(P_{c\sim d})$ and A has dimension $n-2$ as well, it must be that $A = span(P_{a\sim b}) \cap span(P_{c\sim d})$.

Let $\Delta^+(S) := \{p \in \Delta(S) \mid p(s) > 0 \text{ for all } s \in S\}$ be the set of full support beliefs. We distinguish two cases: (2.1) $A \cap \Delta^+(S)$ is empty, and (2.2) $A \cap \Delta^+(S)$ is non-empty.

Case 2.1. Suppose that $A \cap \Delta^+(S)$ is empty. Recall from Lemma 6.2 (b) that each of the indifference sets $P_{a\sim b}$, where $a, b \in D$, has a full support belief in $\Delta^+(S)$, and thus $P_{a\sim b} \cap \Delta^+(S)$ is non-empty. Moreover, recall from above that $P_{a\sim b} \cap P_{c\sim d} = A$ whenever $P_{a\sim b} \neq P_{c\sim d}$. As $A \cap \Delta^+(S)$ is empty, it follows that $P_{a\sim b} \cap P_{c\sim d} \cap \Delta^+(S)$ is empty whenever $P_{a\sim b} \neq P_{c\sim d}$ and $a, b, c, d \in D$.

Let $\{P_1, \dots, P_R\}$ be the collection of pairwise different indifference sets that remains if from $\{P_{a\sim b} \mid a, b \in D\}$ we remove all duplicate sets. Since i, j have been chosen such that for every triple a, b, c in D the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different, we know that $R \geq 3$.

As $P_{a\sim b} \cap P_{c\sim d} \cap \Delta^+(S)$ is empty whenever $P_{a\sim b} \neq P_{c\sim d}$, it follows that the sets $P_1 \cap \Delta^+(S), \dots, P_R \cap \Delta^+(S)$ are pairwise disjoint. Moreover, we have seen that each of the latter sets are non-empty. Since $span(P_1), \dots, span(P_R)$ are hyperplanes of dimension $n-1$, we can order the sets $P_1, \dots, P_R$ such that $P_2 \cap \Delta^+(S), \dots, P_{R-1} \cap \Delta^+(S)$ are in between $P_1 \cap \Delta^+(S)$ and $P_R \cap \Delta^+(S)$. Take some $p_1 \in P_1 \cap \Delta^+(S)$ and $p_R \in P_R \cap \Delta^+(S)$, and let l be the line through p_1 and p_R. Then, the corresponding line segment from p_1 to p_R is included in $\Delta^+(S)$. As $P_2 \cap \Delta^+(S), \dots, P_{R-1} \cap \Delta^+(S)$ are in between $P_1 \cap \Delta^+(S)$ and $P_R \cap \Delta^+(S)$, the line l contains for every $r \in \{2, \dots, R-1\}$ a unique belief p_r in P_r. In particular, for every pair of choices a, b in D, there is a unique belief $p_{ab} \in P_{a\sim b}$ on the line l, and the line l contains a belief where the DM is not indifferent between any of the choices in D.

Recall that for every triple a, b, c in D the sets $P_{a\sim b}, P_{a\sim c}$ and $P_{b\sim c}$ are pairwise different. As $P_{a\sim b} \cap P_{c\sim d} \cap \Delta^+(S)$ is empty whenever $P_{a\sim b} \neq P_{c\sim d}$, we must have for every triple a, b, c in D that p_{ab}, p_{ac} and p_{bc} are pairwise different.

Let s be a state such that the probability of s is not constant on the line l. By four choice linear preference intensity, we have that

$$\frac{p_{ac}(s) - p_{cd}(s)}{p_{ad}(s) - p_{cd}(s)} = \frac{(p_{ab}(s) - p_{bd}(s))(p_{ac}(s) - p_{bc}(s))}{(p_{ab}(s) - p_{bc}(s))(p_{ad}(s) - p_{bd}(s))}, \tag{6.11}$$

where $a := c_i$, $b := c_j$, $c := c_m$ and $d := c_k$. Note that both fractions are well-defined since $p_{ad} \neq p_{cd}$, $p_{ab} \neq p_{bc}$ and $p_{ad} \neq p_{bd}$. Moreover, as p_{ac}, p_{ad}, p_{cd} are pairwise different, we have that $p_{ac}(s) - p_{cd}(s) \neq p_{ad}(s) - p_{cd}(s)$, and hence the fraction on the lefthand side is not equal to 1. As such, the fraction on the righthand side is not equal to 1 either. Let this fraction on the righthand side be called F. Then, by (6.11), p_{cd} is the unique belief on l where

$$p_{cd}(s) = \frac{F \cdot p_{ad}(s) - p_{ac}(s)}{F - 1}. \tag{6.12}$$

Remember that $A \subseteq span(P_{c\sim d})$, that A has dimension $n-2$, and that $span(P_{c\sim d})$ has dimension $n-1$. Let $\{q_2, \ldots, q_{n-1}\}$ be a basis for A. As $p_{cd} \in P_{c\sim d}$ is not in A, we conclude that $\{p_{cd}, q_2, \ldots, q_{n-1}\}$ is a basis for $span(P_{c\sim d})$.

Now, let $\succsim^u$ be the conditional preference relation generated by the utility function u. We have already seen that u represents $\succsim$ on all pairs of choices in $\{a, b, c, d\}$, except $\{c, d\}$. In particular, we thus know that

$$u(a, p_{ab}) = u(b, p_{ab}), u(a, p_{ac}) = u(c, p_{ac}),$$
$$u(a, p_{ad}) = u(d, p_{ad}), u(b, p_{bc}) = u(c, p_{bc}) \text{ and } u(b, p_{bd}) = u(d, p_{bd}).$$

As we have seen in part (a) of the proof that $\succsim^u$ satisfies four choice linear preference intensity, the unique belief on the line l where the DM is indifferent between c and d under $\succsim^u$ is given by (6.12). Therefore,

$$u(c, p_{cd}) = u(d, p_{cd}). \tag{6.13}$$

Recall that $A = span(P_{d\sim a}) \cap span(P_{a\sim c})$. As u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, it follows that $u(d, v) = u(a, v)$ and $u(a, v) = u(c, v)$ for every $v \in span(P_{d\sim a}) \cap span(P_{a\sim c})$. Therefore, $u(c, v) = u(d, v)$ for every $v \in A$. In particular,

$$u(c, q_k) = u(d, q_k) \text{ for every } k \in \{2, \ldots, n-1\}, \tag{6.14}$$

where $\{q_2, \ldots, q_{n-1}\}$ is a basis for A. Moreover, we have seen that $\{p_{cd}, q_2, \ldots, q_{n-1}\}$ is a basis for $span(P_{c\sim d})$.

Recall that $A = span(P_{c_k\sim c_1}) \cap span(P_{c_1\sim c_m})$ has dimension $n-2$, and thus $P_{c_k\sim c_1} \neq P_{c_1\sim c_m}$. Thus, $P_{d\sim a} \neq P_{a\sim c}$. We can thus choose some $p \in P_{d\sim a} \backslash P_{a\sim c}$. Assume, without loss of generality, that $p \in P_{a\succ c}$. By transitivity, we then have that $p \in P_{d\succ c}$. Since u represents $\succsim$ on $\{d, a\}$ and $\{a, c\}$, we know that $u(d, p) = u(a, p)$ and $u(a, p) > u(c, p)$, and hence

$$u(d, p) > u(c, p) \text{ for some } p \in P_{d\succ c}. \tag{6.15}$$

In view of (6.13), (6.14) and (6.15), it follows by Lemma 6.3 that u represents $\succsim$ on $\{c, d\} = \{c_k, c_m\}$.

Case 2.2. Suppose that $A \cap \Delta^+(S)$ is non-empty. Then, there is some full support belief p^* in A, with $p^*(s) > 0$ for all states s. As we have seen that $A \subseteq span(P_{a\sim b})$ for all $a, b \in D$, it follows that $p^* \in P_{a\sim b}$ for all pairs $a, b \in D$.

Since we have seen that A has dimension $n-2$, the linear subspace A is contained in some hyperplane containing the zero vector. Hence, there is some vector $n^A \in \mathbf{R}^S$ such that

$$n^A \cdot v = 0 \text{ for all } v \in A. \tag{6.16}$$

Moreover, we can choose the vector n^A such that for every pair $a, b \in D$ there is some $p \in P_{a\sim b}$ with $n^A \cdot p \neq 0$.

In that case, there is for every pair $a, b \in D$ some $p \in P_{a\sim b}$ with $n^A \cdot p > 0$. To see this, suppose that a, b are such that $n^A \cdot p \leq 0$ for every $p \in P_{a\sim b}$. As there is some $p \in P_{a\sim b}$ with $n^A \cdot p \neq 0$, there must be some $p \in P_{a\sim b}$ with $n^A \cdot p < 0$. Since $p^* \in \Delta^+(S)$, there is some $\lambda > 1$ close enough to 1 such that

$q := (1-\lambda)p + \lambda p^* \in \Delta(S)$. Note that $p^* \in A \subseteq span(P_{a\sim b})$ and $p \in P_{a\sim b}$, which implies that $q \in span(P_{a\sim b}) \cap \Delta(S) = P_{a\sim b}$. At the same time we know, by (6.16) and the fact that $p^* \in A$, that $n^A \cdot p^* = 0$. Since $n^A \cdot p < 0$ and $\lambda > 1$, it follows that $n^A \cdot q = (1-\lambda) \cdot (n^A \cdot p) + \lambda \cdot (n^A \cdot p^*) > 0$. Thus,

$$\text{for every } a, b \in D \text{ there is some } p \in P_{a\sim b} \text{ with } n^A \cdot p > 0. \tag{6.17}$$

Let $P^+ := \{p \in \Delta(S) \mid n^A \cdot p > 0\}$. Then, in view of (6.17),

$$P_{a\sim b} \cap P^+ \text{ is non-empty for all } a, b \in D. \tag{6.18}$$

Recall that $P_{a\sim b} \cap P_{c\sim d} = A$ for every $a, b, c, d \in D$ with $P_{a\sim b} \neq P_{c\sim d}$. In view of (6.16) and (6.18) we conclude that $P_{a\sim b} \cap P_{c\sim d} \cap P^+$ is empty whenever $P_{a\sim b} \neq P_{c\sim d}$. Hence, $(P_{a\sim b} \cap P^+)$ and $(P_{c\sim d} \cap P^+)$ are disjoint whenever $P_{a\sim b} \neq P_{c\sim d}$. But then, the different sets in $\{P_{a\sim b} \mid a, b \in D\}$ can be numbered $P_1, \ldots, P_R$, with $R \geq 3$, such that $P_2 \cap P^+, \ldots, P_{R-1} \cap P^+$ are in between $P_1 \cap P^+$ and $P_R \cap P^+$. In a similar way as in Case 2.1, it can then be shown that u represents $\succsim$ on $\{c_k, c_m\}$.

We thus conclude that u represents $\succsim$ on $\{c_1, \ldots, c_k\}$. By induction on k, the proof is complete. □

Proof of Proposition 4.1

Proof of Proposition 4.1 Let u, v be two different utility representations for $\succsim$. To prove the statement, we distinguish three cases: (1) there are two choices, (2) there are three choices, and (3) there are at least four choices.

Case 1. Suppose there are two choices, a and b. Since there are preference reversals on $\{a, b\}$, there is some $p^* \in P_{a\succ b}$. Define

$$\alpha := \frac{v(a,p^*) - v(b,p^*)}{u(a,p^*) - u(b,p^*)}. \tag{6.19}$$

We show that

$$v(a,p) - v(b,p) = \alpha \cdot (u(a,p) - u(b,p)) \text{ for all beliefs } p \in \Delta(S). \tag{6.20}$$

As there are preference reversals on $\{a, b\}$, it follows by Lemma 6.2 (b) that there are $n-1$ linearly independent beliefs $p_1, \ldots, p_{n-1}$ in $P_{a\sim b}$. Moreover, by Lemma 6.2 (a) we know that $p^* \notin span(P_{a\sim b})$. Hence, $\{p_1, \ldots, p_{n-1}, p^*\}$ are linearly independent, and thus form a basis for $\mathbf{R}^S$. As, by construction, $v(a,p_k) - v(b,p_k) = \alpha \cdot (u(a,p_k) - u(b,p_k)) = 0$ for all $k \in \{1, \ldots, n-1\}$ and, by (6.19), $v(a,p^*) - v(b,p^*) = \alpha \cdot (u(a,p^*) - u(b,p^*))$, it follows that (6.20) holds for every p in the basis $\{p_1, \ldots, p_{n-1}, p^*\}$. Now, take some arbitrary belief $p \in \Delta(S)$. Then, $p = \lambda_1 p_1 + \cdots + \lambda_{n-1} p_{n-1} + \lambda_n p^*$ for some numbers $\lambda_1, \ldots, \lambda_n$. Thus,

$$
\begin{aligned}
v(a,p) - v(b,p) &= \sum_{k=1}^{n-1} \lambda_k \cdot (v(a,p_k) - v(b,p_k)) + \lambda_n \cdot (v(a,p^*) - v(b,p^*)) \\
&= \alpha \cdot \left(\sum_{k=1}^{n-1} \lambda_k \cdot (u(a,p_k) - u(b,p_k)) + \lambda_n \cdot (u(a,p^*) - u(b,p^*)) \right) \\
&= \alpha \cdot (u(a,p) - u(b,p)),
\end{aligned}
$$

which establishes (6.20).

Case 2. Suppose there are three choices, a, b and c. Since, by assumption, there is a belief where the DM is indifferent between some, but not all, choices, it must be that $P_{c\sim a} \neq P_{c\sim b}$. Let the number α be given by (6.19). We show, for every two choices $d, e \in \{a, b, c\}$, that

$$
v(d,p) - v(e,p) = \alpha \cdot (u(d,p) - u(e,p)) \text{ for all beliefs } p \in \Delta(S). \tag{6.21}
$$

By the proof of Case 1, we know that (6.21) holds for the choices a and b. We now show that (6.21) holds for the choices c and a. Let $p_1, \dots, p_{n-1} \in \Delta(S)$ be a basis for $span(P_{c\sim a})$. Then,

$$
v(c,p_k) - v(a,p_k) = \alpha \cdot (u(c,p_k) - u(a,p_k)) = 0 \text{ for all } k \in \{1, \dots, n-1\}. \tag{6.22}
$$

Since $P_{c\sim a} \neq P_{c\sim b}$, there is a belief $p_n \in P_{c\sim b} \backslash P_{c\sim a}$. By Lemma 6.2 (a) we must then have that $p_n \notin span(P_{c\sim a})$, and hence $\{p_1, \dots, p_{n-1}, p_n\}$ is a basis for $\mathbf{R}^S$. As $p_n \in P_{c\sim b}$, it must be that

$$
v(c,p_n) - v(b,p_n) = \alpha \cdot (u(c,p_n) - u(b,p_n)) = 0. \tag{6.23}
$$

Moreover, we know from Case 1 that

$$
v(b,p_n) - v(a,p_n) = \alpha \cdot (u(b,p_n) - u(a,p_n)). \tag{6.24}
$$

If we combine (6.23) and (6.24), we get

$$
\begin{aligned}
v(c,p_n) - v(a,p_n) &= (v(c,p_n) - v(b,p_n)) + (v(b,p_n) - v(a,p_n)) \\
&= \alpha \cdot (u(c,p_n) - u(b,p_n)) + \alpha \cdot (u(b,p_n) - u(a,p_n)) \\
&= \alpha \cdot (u(c,p_n) - u(a,p_n)).
\end{aligned} \tag{6.25}
$$

From (6.22) and (6.25) we conclude, in a similar way as in the proof of Case 1, that

$$
v(c,p) - v(a,p) = \alpha \cdot (u(c,p) - u(a,p)) \text{ for all beliefs } p.
$$

In a similar fashion we can show (6.21) for the choices c and b.

Case 3. Suppose there are at least four choices. By assumption, there is a belief where the DM is indifferent between some, but not all, choices. That is, there are choices a, b, c, d such that $P_{a\sim b} \neq P_{c\sim d}$. Following the proof of Theorem 3.1, it can then be shown that there are three choices a, b and c with $P_{c\sim a} \neq P_{c\sim b}$. Let the number α be given by (6.19). Then, we know by Case 2 that (6.21) holds for every $d, e \in \{a, b, c\}$.

We now show (6.21) for choices d and a, where d is some arbitrary choice not in $\{a, b, c\}$. From the proof of Theorem 3.1 we know that either $P_{d \sim a} \neq P_{d \sim b}$ or $P_{d \sim a} \neq P_{d \sim c}$. Assume, without loss of generality, that $P_{d \sim a} \neq P_{d \sim b}$. Then it can be shown in a similar way as for Case 2 that (6.21) holds for the choices d and a.

Now, take some choice $d \notin \{a, b, c\}$, and some arbitrary choice $e \notin \{a, d\}$. Since we know that (6.21) holds for the choices d and a, and for the choices e and a, it follows that

$$v(d, p) - v(a, p) = \alpha \cdot (u(d, p) - u(a, p)) \text{ for all beliefs } p$$

and

$$v(a, p) - v(e, p) = \alpha \cdot (u(a, p) - u(e, p)) \text{ for all beliefs } p.$$

This implies that

$$\begin{aligned} v(d, p) - v(e, p) =& (v(d, p) - v(a, p)) + (v(a, p) - v(e, p)) \\ =& \alpha \cdot (u(d, p) - u(a, p)) + \alpha \cdot (u(a, p) - u(e, p)) \\ =& \alpha \cdot (u(d, p) - u(e, p)) \text{ for all beliefs } p. \end{aligned}$$

Hence, (6.21) holds for every two choices d, e. This completes the proof. □

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