Structure-Preserving Transformations of Epistemic Models

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Abstract. The prevailing approaches to modelling interactive uncertainty with epistemic models in economics are state-based and type-based. We explicitly formulate two general procedures that transform state models into type models and vice versa. Both transformation procedures preserve the belief hierarchies as well as the common prior assumption. By means of counterexamples it is shown that the two procedures are not inverse to each other. However, if attention is restricted to maximally reduced epistemic models, then isomorphisms can be constructed and an inverse relationship emerges.

Keywords: belief hierarchies; common prior assumption; epistemic game theory; interactive epistemology; isomorphism; epistemic models; games; maximal reduction; possible worlds; states; transformation procedures; types.

1 Introduction

In game theory it is fundamental to model interactive beliefs to capture the players' reasoning about each other. It is assumed in full generality that a player holds beliefs about his opponents' choices, about his opponents' beliefs about their opponents' choices, about his opponents' beliefs about their opponents' beliefs about their opponents' choices, etc.

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Such infinite doxastic sequences can be formally expressed by the notion of a belief hierarchy.

Initially proposed in the context of incomplete information by Harsanyi (1967-68), a belief hierarchy of a player – in the case of strategic uncertainty (e.g. Böge and Eisele, 1979; Mertens and Zamir, 1985; Brandenburger and Dekel, 1993) – specifies a probability measure about the basic space of uncertainty i.e. the opponents' choice combinations (first-order belief), a probability measure about the opponents' choice combinations and the opponents' first-order beliefs (second-order belief), a probability measure about the opponents' choice combinations, the opponents' first-order beliefs, and the opponents' second-order beliefs (third-order belief), etc. Thus, a k-order belief fixes a belief about the basic space of uncertainty and about each of the lower-order beliefs of the opponents. A player's belief hierarchy can be seen as the formalization of his entire interactive thinking about the game. Different patterns of reasoning (e.g. common belief in rationality) can then be modelled as conditions imposed on a player's belief hierarchy.

Unfortunately, belief hierarchies are cumbersome objects due to their infinite nature. However, there exist finite encodings of belief hierarchies that render them more tractable. The standard way to represent belief hierarchies in a compact and convenient way is due to Harsanyi's (1967-68) seminal idea of types. Accordingly, a type induces a probability measure on the opponents' combinations of choices and types. Any belief of higher order can then be derived. An alternative implicit description of belief hierarchies is based on the idea of states or possible words due to Kripke (1963) and Aumann (1974). Any belief of higher order can be inferred from a player's belief at a given possible world about the worlds in combination with the players' choices and beliefs at worlds. The relation between the so-called *type-based* and *state-based* approaches to modelling belief hierarchies have been investigated by Brandenburger and Dekel (1993) as well as by Tan and Werlang (1992). They essentially show that hypotheses involving common knowledge are preserved across these two epistemic frameworks.

We compare the type-based and state-based approaches to formalizing interactive thinking from a broader perspective and provide two general transformation procedures between type and state models. Belief hierarchies as well as the common prior assumption are preserved by these procedures. In this sense the two different epistemic approaches are equivalent. We then explore whether the two procedures constitute operational inverses to each other by means of an isomorphism. It turns out that they do not do so unless attention is restricted to maximally reduced models which exclude the existence of "superfluous" worlds and types, respectively. This insight emphasizes that type and state models actually exhibit some foundational differences despite their equivalence in terms of preserving belief hierarchies and the common prior assumption. The underyling conceptual reason lies in the distinct degrees of granularity: while the type-based approach only respresents the players' interactive thinking the state-based approach additionally also fixes their choices.

We proceed as follows. Section 2 lays out the formal framework and notation. In particular, type-based and state-based approaches to interactive epistemology are presented. In Section 3 we provide a transformation procedure (Definition 5) to convert state models into type models. Belief hierarchies (Theorem 1) as well as the common prior assumption (Theorem 2) are preserved. Then, in Section 4 our point of departure are type models and we propose a second transformation procedure (Definition 6) to turn them into state models. Again, preservation holds with regards to belief hierarchies (Theorem

3) as well as the common prior assumption (Theorem 4). While the general conclusions of Theorems 1 and 3 about the structural conservation of belief hierarchies are likely to be implicitly known in the game theory community, our purpose is, first, to render these foundational insights explicit in an accessible way, and second, to provide concrete tools to switch back and forth between state and type models. Section 5 explores structural identities within a given epistemic framework. It turns out that the two transformation procedures are not inverse to each other (Examples 1 and 2). By restricting to maximally reduced models inverse relationships between the two operations then ensue (Theorems 5 and 6). Finally, some concluding remarks are offered in Section 6.

2 Preliminaries

A game is modelled as a tuple $\Gamma = \langle I, (C_i, U_i)_{i \in I} \rangle$, where I is a finite set of players, C_i denotes player i's finite choice set, and $U_i : \times_{j \in I} C_j \to \mathbb{R}$ constitutes player i's utility function, which assigns a real number $U_i(c)$ to every choice combination $c \in \times_{j \in I} C_j$. In terms of notation, given a collection $\{S_n : n \in N\}$ of sets and probability measures $p_n \in \Delta(S_n)$ for all $n \in N$, the set S_{-n} refers to the product set $\times_{m \in N \setminus \{n\}} S_m$ and the probability measure p_{-n} refers to the product measure $\Pi_{m \in N \setminus \{n\}} p_m \in \Delta(S_{-n})$ on S_{-n}. Given a probabily measure $p \in \Delta_{n \in N}(\times S_n)$ on a product set, for the sake of simplicity any marginal is also denoted by p if the intended usage is clear from the context.

Belief hierarchies can be inductively formalized as sequences of probability measures. In the context of games, construct for every player $i \in I$ a sequence $(X_i^n)_{n \in \mathbb{N}}$ of spaces, where

$$X_i^1 := C_{-i},$$
$$X_i^2 := X_i^1 \times \big(\times_{j \in I \setminus \{i\}} \Delta(X_j^1) \big),$$
$$\vdots$$
$$X_i^k := X_i^{k-1} \times \big(\times_{j \in I \setminus \{i\}} \Delta(X_j^{k-1}) \big),$$
$$\vdots$$

and a belief hierarchy of player i is then defined as a sequence $\eta_i := (\eta_i^n)_{n \in \mathbb{N}} \in \times_{n \in \mathbb{N}} \big(\Delta(X_i^n) \big)$ of probability measures. For every level $k \in \mathbb{N}$, the probability measure $\eta_i^k \in \Delta(X_i^k)$ is called i's k-th order belief. Note that

$$X_i^k = C_{-i} \times \big(\times_{j \in I \setminus \{i\}} \Delta(X_j^1) \big) \times \big(\times_{j \in I \setminus \{i\}} \Delta(X_j^2) \big) \times \ldots \times \big(\times_{j \in I \setminus \{i\}} \Delta(X_j^{k-1}) \big)$$

holds for all $k \in \mathbb{N}$.

The standard implicit representation of belief hierarchies in terms of types is due to Harsanyi (1967-68). According to this epistemic approach the game-theoretic framework – given by Γ – is enriched by a type-based structure.

Definition 1. *Let Γ be a game. A* type model *of Γ is a tuple $\mathcal{T}^\Gamma = \langle (T_i, b_i)_{i \in I} \rangle$, where for every player $i \in I$,*

– *T_i is a finite set of types,*

– $b_i : T_i \to \Delta(C_{-i} \times T_{-i})$ *is* i*'s belief function that assigns to every type* $t_i \in T_i$ *a probability measure* $b_i[t_i]$ *on the set of opponents' choice type combinations.*

A type t_i of some player i naturally induces a belief hierarchy:

$$\eta_i^1[t_i](c_{-i}) := \sum_{t_{-i} \in T_{-i}} b_i[t_i](c_{-i}, t_{-i})$$

for all $c_{-i} \in X_i^1$, as well as

$$\eta_i^k[t_i](c_{-i}, \eta_{-i}^1, \eta_{-i}^2, \dots, \eta_{-i}^{k-1}) := \sum_{t_{-i} \in T_{-i} : \eta_{-i}^l[t_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k-1} b_i[t_i](c_{-i}, t_{-i})$$

for all $(c_{-i}, \eta_{-i}^1, \eta_{-i}^2, \dots, \eta_{-i}^{k-1}) \in X_i^k$ and for all $k \geq 2$, where the sequence $\eta_i[t_i] := (\eta_i^n[t_i])_{n \in \mathbb{N}}$ is called the t_i*-induced belief hierarchy of player* i. The set $H_i[\mathcal{T}^\Gamma] := \{\eta_i \in \times_{n \in \mathbb{N}}\big(\Delta(X_i^n)\big) : \text{there exists } t_i \in T_i \text{ such that } \eta_i[t_i] = \eta_i\}$ is called the $\mathcal{T}^\Gamma$*-induced set of belief hierarchies of player* i.

An alternative way to represent interactive thinking in games is based on the idea of possible worlds – sometimes also called states – due to Kripke (1963) and Aumann (1974). This epistemic approach employs a state-based structure as formal framework added to Γ.

Definition 2. *Let* Γ *be a game. A* state model *of* Γ *is a tuple* $\mathcal{S}^\Gamma = \langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$*, where*

– Ω *is a finite set of all possible worlds,*

and for every player $i \in I$*,*

– $\mathcal{I}_i \subseteq 2^\Omega$ *is a possibility partition of* Ω*,*
– $\sigma_i : \Omega \to C_i$ *is a* $\mathcal{I}_i$*-measurable choice function,*
– $\pi_i \in \Delta(\Omega)$ *is a subjective prior on* Ω *such that* $\pi_i\big(\mathcal{I}_i(\omega)\big) > 0$ *for every world* $\omega \in \Omega$ *with* $\mathcal{I}_i(\omega)$ *denoting the cell of* $\mathcal{I}_i$ *containing* ω*.*

Belief hierarchies also naturally emerge in state models. Given some player i, a possible world ω induces a belief hierarchy as follows:

$$\eta_i^1[\omega](c_{-i}) := \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = c_{-i}} \pi_i\big(\omega' \mid \mathcal{I}_i(\omega)\big)$$

for all $c_{-i} \in X_i^1$, as well as

$$\eta_i^k[\omega](c_{-i}, \eta_{-i}^1, \eta_{-i}^2, \dots, \eta_{-i}^{k-1})$$
$$:= \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = c_{-i}, \eta_{-i}^l[\omega'] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k-1} \pi_i\big(\omega' \mid \mathcal{I}_i(\omega)\big)$$

for all $(c_{-i}, \eta_{-i}^1, \eta_{-i}^2, \dots, \eta_{-i}^{k-1}) \in X_i^k$ and for all $k \geq 2$, where the sequence $\eta_i[\omega] := (\eta_i^n[\omega])_{n \in \mathbb{N}}$ is called the ω*-induced belief hierarchy of player* i. The set $H_i[\mathcal{S}^\Gamma] := \{\eta_i \in$

$\times_{n \in \mathbb{N}}\big(\Delta(X_i^n)\big)$: there exists $\omega \in \Omega$ such that $\eta_i[\omega] = \eta_i\}$ is called the $\mathcal{S}^\Gamma$*-induced set of belief hierarchies of player* i

By the $\mathcal{I}_i$-measurability of σ_i the same choice for player i is assigned throughout an information cell, i.e. $\sigma_i(\omega') = \sigma_i(\omega)$ for all $\omega' \in \mathcal{I}_i(\omega)$. Every information cell $P_i \in \mathcal{I}_i$ thus induces a choice $\sigma_i(P_i) \in C_i$, where $\sigma_i(P_i) := \sigma_i(\omega)$ for all $\omega \in P_i$. Moreover, since the belief hierarchies are constructed on the basis of posterior beliefs, it follows that i's belief hierarchies are also constant throughout his information cells, i.e. $\eta_i[\omega'] = \eta_i[\omega]$ for all $\omega' \in \mathcal{I}_i(\omega)$.

The common prior assumption constitutes a frequently used premise in game theory. Accordingly, all beliefs are derived from a single probability measure. The common prior assumption formalizes the conceptual viewpoint that differences in beliefs are only due to differences in information.

Within the framework of type models the common prior assumption requires the probability measure of every type induced by the belief function to be obtained via Bayesian conditionalization on some common probability measure on all players' choice type combinations.

Definition 3. *Let* Γ *be a game and* $\mathcal{T}^\Gamma$ *a type model of* Γ*. The type model* $\mathcal{T}^\Gamma$ *satisfies the* common prior assumption*, if there exists a probability measure* $\rho \in \Delta\big(\times_{j \in I}(C_j \times T_j)\big)$ *such that for every player* $i \in I$*, and for every type* $t_i \in T_i$ *it is the case that* $\rho(t_i) > 0$ *and*

$$b_i[t_i](c_{-i}, t_{-i}) = \frac{\rho(c_i, c_{-i}, t_i, t_{-i})}{\rho(c_i, t_i)}$$

for all $c_i \in C_i$ *with* $\rho(c_i, t_i) > 0$*, and for all* $(c_{-i}, t_{-i}) \in C_{-i} \times T_{-i}$*. The probability measure* ρ *is called* common prior*.*

The preceding formalization of the common prior assumption is equivalent to the conjunction of Dekel and Siniscalchi's (2015) Definition 12.13 with their Definition 12.15 as well as to Bach and Perea's (2020) Definition 4.

In state models the common prior assumption simply postulates all subjective priors to coincide.

Definition 4. *Let* Γ *be a game and* $\mathcal{S}^\Gamma$ *a state model of* Γ*. The state model* $\mathcal{S}^\Gamma$ *satisfies the* common prior assumption*, if there exists a probability measure* $\pi \in \Delta(\Omega)$ *such that* $\pi_i = \pi$ *for every player* $i \in I$*. The probability measure* π *is called* common prior*.*

3 Transformation of State Models into Type Models

The following transformation procedure converts state models into type models.

Definition 5. *Let* Γ *be a game, and* $\mathcal{S}^\Gamma$ *a state model of* Γ*. The tuple*

$$\langle (T_i, b_i)_{i \in I} \rangle$$

forms the $\mathcal{S}^\Gamma$-generated type model of Γ*, where for every player* $i \in I$*,*

– $T_i := \{t_i^{P_i} : P_i \in \mathcal{I}_i\}$ *is* i*'s set of types,*

– $b_i : T_i \to \Delta(C_{-i} \times T_{-i})$ *is i's belief function with*

$$b_i[t_i^{P_i}](c_{-i}, t_{-i}^{P_{-i}}) := \sum_{\omega \in P_i : \sigma_{-i}(\omega) = c_{-i}, \mathcal{I}_{-i}(\omega) = P_{-i}} \pi_i(\{\omega\} \mid P_i),$$

for all $(c_{-i}, t_{-i}^{P_{-i}}) \in C_{-i} \times T_{-i}$ *and for all* $t_i^{P_i} \in T_i$.

In a nutshell, information cells are transformed into types and the types' beliefs are then given by the subjective priors conditionalized on the corresponding information cells. Note that the type model generated by a given state model actually is unique.

It turns out that the transformation procedure laid out in Definition 5 preserves the induced belief hierarchies of state models.

Theorem 1. *Let Γ be a game, $\mathcal{S}^\Gamma$ a state model of Γ with $\mathcal{S}^\Gamma$-generated type model $\langle (T_i, b_i)_{i \in I} \rangle$ of Γ, $i \in I$ some player, and $\omega \in \Omega$ some world. Then,*

$$\eta_i[\omega] = \eta_i[t_i^{\mathcal{I}_i(\omega)}].$$

Proof. It is shown inductively that $\eta_i^k[\omega] = \eta_i^k[t_i^{\mathcal{I}_i(\omega)}]$ holds for all $k \geq 1$. It then directly follows that $\eta_i[\omega] = (\eta_i^n[\omega])_{n \in \mathbb{N}} = \big(\eta_i^n[t_i^{\mathcal{I}_i(\omega)}]\big)_{n \in \mathbb{N}} = \eta_i[t_i^{\mathcal{I}_i(\omega)}]$.

First of all, observe that

$$\begin{aligned}
&\eta_i^1[\omega](c_{-i}) \\
&= \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = c_{-i}} \pi_i\big(\omega' \mid \mathcal{I}_i(\omega)\big) \\
&= \sum_{t_{-i}^{P_{-i}} \in T_{-i}} \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = c_{-i}, \mathcal{I}_{-i}(\omega') = P_{-i}} \pi_i\big(\omega' \mid \mathcal{I}_i(\omega)\big) \\
&= \sum_{t_{-i}^{P_{-i}} \in T_{-i}} b_i[t_i^{\mathcal{I}_i(\omega)}](c_{-i}, t_{-i}^{P_{-i}}) \\
&= \eta_i^1[t_i^{\mathcal{I}_i(\omega)}](c_{-i})
\end{aligned}$$

for all $c_{-i} \in C_{-i}$.

Now, suppose that $\eta_i^k[\omega] = \eta_i^k[t_i^{\mathcal{I}_i(\omega)}]$ holds up to some $k > 1$. It then follows that

$$\begin{aligned}
&\eta_i^{k+1}[\omega](c_{-i}, \eta_{-i}^1, \ldots, \eta_i^k) \\
&= \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = c_{-i}, \eta_{-i}^l[\omega'] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \pi_i\big(\omega' \mid \mathcal{I}_i(\omega)\big) \\
&= \sum_{t_{-i}^{P_{-i}} \in T_{-i} : \eta_{-i}^l[t_{-i}^{P_{-i}}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = c_{-i}, \mathcal{I}_{-i}(\omega') = P_{-i}} \pi_i\big(\omega' \mid \mathcal{I}_i(\omega)\big) \\
&= \sum_{t_{-i}^{P_{-i}} \in T_{-i} : \eta_{-i}^l[t_i] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} b_i[t_i^{\mathcal{I}_i(\omega)}](c_{-i}, t_{-i}^{P_{-i}}) \\
&= \eta_i^{k+1}[t_i^{\mathcal{I}_i(\omega)}](c_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^k)
\end{aligned}$$

for all $(c_{-i}, \eta_{-i}^1, \ldots, \eta_{-i}^k) \in X_i^{k+1}$. ■

Also, the common prior assumption is maintained from state to type models.

Theorem 2. *Let Γ be a game, and $\mathcal{S}^\Gamma$ a state model of Γ satisfying the common prior assumption. Then, the $\mathcal{S}^\Gamma$-generated type model $\langle (T_i, b_i)_{i\in I}\rangle$ of Γ satisfies the common prior assumption.*

Proof. Define a probability measure $\rho \in \Delta\big(\times_{i\in I} (C_i \times T_i)\big)$ in the $\mathcal{S}^\Gamma$-generated type model $\langle (T_i, b_i)_{i\in I}\rangle$ such that for all $(c_i, t_i^{P_i})_{i\in I} \in \times_{i\in I}(C_i \times T_i)$

$$\rho\big((c_i, t_i^{P_i})_{i\in I}\big) := \begin{cases} \pi(\cap_{i\in I} P_i), & \text{if } \sigma_i(P_i) = c_i \text{ for all } i \in I, \\ 0, & \text{otherwise.} \end{cases}$$

First of all it is established that $\rho(t_i^{P_i}) > 0$ holds for all $t_i^{P_i} \in T_i$ and for all $i \in I$. Let $t_i^{P_i} \in T_i$ and observe that $\rho(t_i^{P_i}) = \sum_{t_{-i}^{P_{-i}} \in T_{-i}} \sum_{(c_j)_{j\in I} \in \times_{j\in I} C_j} \rho\big((c_j, t_j^{P_j})_{j\in I}\big) = \sum_{P_{-i}\in \mathcal{I}_{-i}} \pi(\cap_{j\in I} P_j) = \pi(P_i)$ and since $\pi(P_i) > 0$ it thus follows that $\rho(t_i^{P_i}) > 0$ holds.

Next it is shown that for all $i \in I$ and for all $t_i^{P_i} \in T_i$, the equation

$$b_i[t_i^{P_i}](c_{-i}, t_{-i}^{P_{-i}}) = \frac{\rho(c_i, c_{-i}, t_i^{P_i}, t_{-i}^{P_{-i}})}{\rho(c_i, t_i^{P_i})}$$

holds for all $c_i \in C_i$ with $\rho(c_i, t_i^{P_i}) > 0$, and for all $(c_{-i}, t_{-i}^{P_{-i}}) \in C_{-i} \times T_{-i}$ Note that $\rho(c_i, t_i^{P_i}) = \sum_{t_{-i}^{P_{-i}} \in T_{-i}} \sum_{c_{-i}\in C_{-i}} \rho\big((c_j, t_j^{P_j})_{j\in I}\big) = \sum_{\omega\in\Omega:\sigma_i(\omega)=c_i, \mathcal{I}_i(\omega)=P_i} \pi\big(\cap_{j\in I} \mathcal{I}_j(\omega)\big) = \pi(P_i) > 0$ holds, if and only if, $\sigma_i(P_i) = c_i$. Thus, the following equation

$$b_i[t_i^{P_i}](c_{-i}, t_{-i}^{P_i}) = \frac{\rho\big(\sigma_i(P_i), c_{-i}, t_i^{P_i}, t_{-i}^{P_i}\big)}{\rho\big(\sigma_i(P_i), t_i^{P_i}\big)}$$

has to be validated for all $(c_{-i}, t_{-i}^{P_{-i}}) \in C_{-i} \times T_{-i}$ and for all $t_i^{P_i} \in T_i$.

Consider some $P_i \in \mathcal{I}_i$ and distinguish two cases (I) and (II).

Case (I). Suppose that $P_i \cap (\cap_{j\in I\setminus\{i\}} P_j) \neq \emptyset$ and $c_j = \sigma_j(P_j)$ for all $j \in I \setminus \{i\}$. Then,

$$\begin{aligned} b_i[t_i^{P_i}](c_{-i}, t_{-i}^{P_{-i}}) &= b_i[t_i^{P_i}](\sigma_{-i}(P_{-i}), t_{-i}^{P_{-i}}) \\ &= \sum_{\omega\in P_i:\sigma_{-i}(\omega)=\sigma_{-i}(P_{-i}), \mathcal{I}_{-i}(\omega)=P_{-i}} \pi(\omega \mid P_i) \\ &= \sum_{\omega\in P_i:\omega\in P_j \text{ for all } j\in I\setminus\{i\}} \pi(\omega \mid P_i) \\ &= \frac{\pi(\cap_{k\in I} P_k)}{\pi(P_i)} \\ &= \frac{\pi(\cap_{k\in I} P_k)}{\sum_{\hat{P}_j\in\mathcal{I}_j \text{ for all } j\in I\setminus\{i\}\}} \pi\big(P_i \cap (\cap_{j\in I\setminus\{i\}} \hat{P}_j)\big)} \\ &= \frac{\rho\big(\sigma_i(P_i), t_i^{P_i}, \sigma_{-i}(P_{-i}), t_{-i}^{P_{-i}}\big)}{\sum_{\hat{P}_{-i}\in\mathcal{I}_{-i}} \rho\big(\sigma_i(P_i), t_i^{P_i}, \sigma_{-i}(\hat{P}_{-i}), t_{-i}^{\hat{P}_{-i}}\big)} \end{aligned}$$

$$= \frac{\rho\big(\sigma_i(P_i), t_i^{P_i}, \sigma_{-i}(P_{-i}), t_{-i}^{P_{-i}}\big)}{\sum_{(c_{-i}, t_{-i}^{\hat{P}_{-i}}) \in C_{-i} \times T_{-i}} \rho\big(\sigma_i(P_i), t_i^{P_i}, c_{-i}, t_{-i}^{\hat{P}_{-i}}\big)}$$

$$= \frac{\rho\big(\sigma_i(P_i), t_i^{P_i}, c_{-i}, t_{-i}^{P_{-i}}\big)}{\rho\big(\sigma_i(P_i), t_i^{P_i}\big)}$$

for all $(c_{-i}, t_{-i}^{P_{-i}}) \in C_{-i} \times T_{-i}$.

Case (II). Suppose that $P_i \cap (\cap_{j \in I \setminus \{i\}} P_j) = \emptyset$ or $c_j \neq \sigma_j(P_j)$ for some $j \in I \setminus \{i\}$. Then, $\rho\big(\sigma_i(P_i), t_i^{P_i}, c_{-i}, t_{-i}^{P_{-i}}\big) = 0$ holds by definition of ρ as well as $b_i[t_i^{P_i}](c_{-i}, t_{-i}^{P_{-i}}) = \sum_{\omega \in P_i : \sigma_{-i}(\omega) = c_{-i}, \mathcal{I}_{-i}(\omega) = P_{-i}} \frac{\pi(\{\omega\} \cap P_i)}{\pi_i(P_i)} = \sum_{\omega \in P_i : \sigma_{-i}(\omega) = c_{-i}, \mathcal{I}_{-i}(\omega) = P_{-i}} \pi(\{\omega\} \mid P_i) = 0$. It directly follows that

$$b_i[t_i^{P_i}](c_{-i}, t_{-i}^{P_{-i}}) = \frac{\rho\big(\sigma_i(P_i), t_i^{P_i}, c_{-i}, t_{-i}^{P_{-i}}\big)}{\rho\big(\sigma_i(P_i), t_i^{P_i}\big)}$$

for all $(c_{-i}, t_{-i}^{P_{-i}}) \in C_{-i} \times T_{-i}$.

Therefore, the $\mathcal{S}^{\Gamma}$-generated type model $\langle (T_i, b_i)_{i \in I} \rangle$ satisfies the common prior assumption. ■

4 Transformation of Type Models into State Models

Taking type models as input the following transformation procedure defines corresponding state models.

Definition 6. *Let Γ be a game, and $\mathcal{T}^{\Gamma}$ be a type model of Γ. The tuple*

$$\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$$

forms a $\mathcal{T}^{\Gamma}$-generated state model of Γ, where

- $\Omega := \{\omega^{(c_i, t_i)_{i \in I}} : c_i \in C_i, t_i \in T_i$ *for all* $i \in I\}$ *is the set of all possible worlds,*

and for every player $i \in I$,

- $\mathcal{I}_i \subseteq 2^{\Omega}$ *is i's possibility partition with*

$$\mathcal{I}_i(\omega^{(c_j, t_j)_{j \in I}}) := \{\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \in \Omega : c'_{-i} \in C_{-i}, t'_{-i} \in T_{-i}\}$$

for all $\omega^{(c_j, t_j)_{j \in I}} \in \Omega$,
- $\sigma_i : \Omega \to C_i$ *is i's choice function with*

$$\sigma_i(\omega^{(c_i, t_i, c_{-i}, t_{-i})}) := c_i$$

for all $\omega^{(c_j, t_j)_{j \in I}} \in \Omega$,
- $\pi_i \in \Delta(\Omega)$ *is i's subjective prior with*

$$\pi_i\big(\omega^{(c_i, t_i, c_{-i}, t_{-i})} \mid \mathcal{I}_i(\omega^{(c_i, t_i, c_{-i}, t_{-i})})\big) := b_i[t_i](c_{-i}, t_{-i})$$

for all $\omega^{(c_i, t_i, c_{-i}, t_{-i})_{j \in I}} \in \Omega$.

The transformation procedure generates a possible world for every combination of choices and types of the players. An information cell is associated with a choice type pair of a player and contains all worlds where the choices and types are varied for the opponents. The choice functions pick the choices of the players in line with the corresponding worlds. Finally, the subjective priors are indirectly fixed via their induced posteriors. The belief of a given player i about a world conditional on his information is defined as his belief of the corresponding type about the opponents' choice type combinations attached to the world. Only varying i's choices thus results in the same belief. Observe that for every cell the conditional probabilty measures on the set of possible worlds do indeed sum up to one and are well-defined.

A state model constructed by the transformation procedure based on the type model $\mathcal{T}^{\Gamma}$ is generally not unique, as the subjective priors can be varied. The possible multiplicity of generated state models ensues because of their richer structure compared to type models. While type models only specify posterior beliefs, state models fix prior beliefs and choices on top of (implicit) posterior beliefs. In terms of interactive thinking this additional information is superfluous, and results in some ambiguity when deducing a state model from a type model which constitutes a sparser formal representation of interactive thinking. The ensueing freedom in constructing a $\mathcal{T}^{\Gamma}$-generated state model manifests itself in specifying the subjective priors. Only the engendered posterior beliefs are required to coincide with the corresponding types' beliefs in $\mathcal{T}^{\Gamma}$.

The transformation procedure yields the same induced belief hierarchies in the type model of departure and its corresponding state models.

Theorem 3. *Let Γ be a game, and $\mathcal{T}^{\Gamma}$ a type model of Γ with some $\mathcal{T}^{\Gamma}$-generated state model $\mathcal{S}^{\Gamma}$ of Γ, $i \in I$ some player, and $t_i \in T_i$ some type of player i. Then,*

$$\eta_i[t_i] = \eta_i[\omega^{(c_i, t_i, c_{-i}, t_{-i})}]$$

for all $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$.

Proof. It is shown inductively that $\eta_i^k[t_i] = \eta_i^k[\omega^{(c_i, t_i, c_{-i}, t_{-i})}]$ holds for all $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$, and for all $k \geq 1$. It then directly follows that $\eta_i[t_i] = (\eta_i^n[t_i])_{n \in \mathbb{N}} = \big(\eta_i^n[\omega^{(c_i, t_i, c_{-i}, t_{-i})}]\big)_{n \in \mathbb{N}} = \eta_i[\omega^{(c_i, t_i, c_{-i}, t_{-i})}]$ for all $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$.

First of all, let $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$ and observe that

$$\begin{aligned}
&\eta_i^1[t_i](c'_{-i}) \\
&= \sum_{t'_{-i} \in T_{-i}} b_i[t_i](c'_{-i}, t'_{-i}) \\
&= \sum_{t'_{-i} \in T_{-i}} \pi_i\big(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \mid \mathcal{I}_i(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})})\big) \\
&= \sum_{t'_{-i} \in T_{-i}} \pi_i\big(\omega^{(c_i, t_i, c'_{-i}, t'_{-i})} \mid \mathcal{I}_i(\omega^{(c_i, t_i, c_{-i}, t_{-i})})\big) \\
&= \sum_{\omega \in \mathcal{I}_i(\omega^{(c_i, t_i, c_{-i}, t_{-i})}) : \sigma_{-i}(\omega) = c'_{-i}} \pi_i\big(\omega \mid \mathcal{I}_i(\omega^{(c_i, t_i, c_{-i}, t_{-i})})\big)
\end{aligned}$$

$$= \eta_i^1[\omega^{(c_i,t_i,c_{-i},t_{-i})}](c'_{-i})$$

holds for all $c'_{-i} \in C_{-i}$.

Now, suppose that $\eta_i^k[t_i] = \eta_i^k[\omega^{(c_i,t_i,c_{-i},t_{-i})}]$ holds for all $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$ up to some $k > 1$. Let $(c_i, c_{-i}, t_{-i}) \in C_i \times C_{-i} \times T_{-i}$ and observe that

$$\eta_i^{k+1}[t_i](c'_{-i}, \eta_{-i}^1, \dots, \eta_{-i}^k)$$

$$= \sum_{t'_{-i} \in T_{-i} : \eta_{-i}^l[t'_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} b_i[t_i](c'_{-i}, t'_{-i})$$

$$= \sum_{t'_{-i} \in T_{-i} : \eta_{-i}^l[t'_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \pi_i\big(\omega^{(c_i,t_i,c'_{-i},t'_{-i})} \mid \mathcal{I}_i(\omega^{(c_i,t_i,c'_{-i},t'_{-i})})\big)$$

$$= \sum_{t'_{-i} \in T_{-i} : \eta_{-i}^l[t'_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \pi_i\big(\omega^{(c_i,t_i,c'_{-i},t'_{-i})} \mid \mathcal{I}_i(\omega^{(c_i,t_i,c_{-i},t_{-i})})\big)$$

for all $(c'_{-i}, \eta_{-i}^1, \dots, \eta_{-i}^k) \in X_i^{k+1}$.

By the inductive assumption, it is the case $\eta_j^l[t_j] = \eta_j^l[\omega^{(c_j,t_j,c_{-j},t_{-j})}]$ for all $j \in I \setminus \{i\}$, for all $t_j \in T_j$, for all $1 \leq l \leq k$, and for all $c_j, c_{-j}, t_{-j} \in C_j \times C_{-j} \times T_{-j}$. Therefore,

$$= \sum_{t'_{-i} \in T_{-i} : \eta_{-i}^l[t'_{-i}] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k} \pi_i\big(\omega^{(c_i,t_i,c'_{-i},t'_{-i})} \mid \mathcal{I}_i(\omega^{(c_i,t_i,c_{-i},t_{-i})})\big)$$

$$= \sum_{\omega' \in \mathcal{I}_i(\omega^{(c_i,t_i,c_{-i},t_{-i})}) : \eta_{-i}^l[\omega'] = \eta_{-i}^l \text{ for all } 1 \leq l \leq k, \sigma_{-i}(\omega') = c'_{-i}} \pi_i\big(\omega' \mid \mathcal{I}_i(\omega^{(c_i,t_i,c_{-i},t_{-i})})\big)$$

$$= \eta_i^{k+1}[\omega^{(c_i,t_i,c_{-i},t_{-i})}](c'_{-i}, \eta_{-i}^1, \dots, \eta_{-i}^k)$$

for all $(c'_{-i}, \eta_{-i}^1, \dots, \eta_{-i}^k) \in X_i^{k+1}$. ■

The common prior assumption is preserved from type to state models, too.

Theorem 4. *Let Γ be a game, and $\mathcal{T}^\Gamma$ a type model of Γ satisfying the common prior assumption. Then, there exists a $\mathcal{T}^\Gamma$-generated state model $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ of Γ that satisfies the common prior assumption.*

Proof. Define a state model $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ of Γ with the objects $\Omega, (\mathcal{I}_i, \sigma_i)_{i \in I}$ as in Definition 6, as well as with a probability measure $\pi \in \Delta(\Omega)$ such that $\pi(\omega^{(c_i,t_i)_{i \in I}}) := \rho\big((c_i, t_i)_{i \in I}\big)$ for all $\omega^{(c_i,t_i)_{i \in I}} \in \Omega$ and $\pi_i = \pi$ for all $i \in I$. By construction $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ thus satisfies the common prior assumption. Since

$$\pi_i\big(\omega^{(c_i,t_i,c'_{-i},t'_{-i})} \mid \mathcal{I}_i(\omega^{(c_j,t_j)_{j \in I}})\big) = \frac{\pi(\omega^{(c_i,t_i,c'_{-i},t'_{-i})})}{\pi\big(\mathcal{I}_i(\omega^{(c_j,t_j)_{j \in I}})\big)} = \frac{\rho(c_i, t_i, c'_{-i}, t'_{-i})}{\rho(c_i, t_i)} = b_i[t_i](c'_{-i}, t'_{-i})$$

holds for all $(c'_{-i}, t'_{-i}) \in C_{-i} \times T_{-i}$, for all $\omega^{(c_j,t_j)_{j \in I}} \in \Omega$ and for all $i \in I$, the state model $\langle (\Omega, (\mathcal{I}_i, \pi_i, \sigma_i)_{i \in I}) \rangle$ also forms a $\mathcal{T}^\Gamma$-generated state model of Γ. ■

5 Isomorphism

The transformation procedure in Definition 5 converts state models into type models, while the one in Definition 6 moulds state moldels from type models. In terms of structural equivalence of epistemic models the question whether these two transformation procedures are inverse to each other naturally emerges. We explore the relationship between the two transformation procedures by means of isomorphism. Intuitively, two epistemic models are isomorphic if they formalize the same interactive thinking. In our context two issues need to be addressed. Firstly, it has to be determined whether a type model is isomorphic to the type model generated via Definition 5 by the state model which itself is generated via Definition 6 by the type model of departure. Secondly, it needs to be established whether a state model is isomorphic to the state model generated via Definition 6 by the type model which itself is generated via Definition 5 by the state model of departure.

For the epistemic framework of type models the notion of isomorphism can be spelled out as follows.

Definition 7. *Let Γ be a game, and $\langle (T_i, b_i)_{i \in I} \rangle$ as well as $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$ be type models of Γ. The type models $\langle (T_i, b_i)_{i \in I} \rangle$ and $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$ are* isomorphic*, if for all $i \in I$ there exists a bijection $f_i : T_i \to \tilde{T}_i$ such that*

$$\tilde{b}_i[f_i(t_i)]\big(c_{-i}, f_{-i}(t_{-i})\big) = b_i[t_i](c_{-i}, t_{-i})$$

for all $(c_{-i}, t_{-i}) \in C_{-i} \times T_{-i}$ and for all $t_i \in T_i$.

Intuitively, in two isomorphic type models the same belief hierarchies are present – in fact only their labels differ – and thus the described interactive thinking is alike. The bijection in Definition 7 is essentially equivalent to the notion of type isomorphism due to Heifetz and Samet (1998, Definition 3.2).

Take some type model $\mathcal{T}^{\Gamma} = \langle (T_i, b_i)_{i \in I} \rangle$ as input and construct a type model $\hat{\mathcal{T}}^{\Gamma} = \langle (\hat{T}_i, \hat{b}_i)_{i \in I} \rangle$ as output by first applying Definition 6 to $\mathcal{T}^{\Gamma}$ and then Definition 5 to the $\mathcal{T}^{\Gamma}$-generated state model. It turns out that the isomorphic relationship does actually not always hold between such input and output type models. To see this consider the following example.

Example 1. Let Γ be a game with $I = \{i, j\}$, $C_i = \{a\}$ as well as $C_j = \{b, c\}$, and $\mathcal{T}^{\Gamma}$ a type model of Γ with

- $T_i = \{t_i\}$,
- $T_j = \{t_j\}$,
- $b_i[t_i](b, t_j) = \frac{1}{2}$ and $b_i[t_i](c, t_j) = \frac{1}{2}$,
- $b_j[t_j](a, t_i) = 1$.

Then, $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ with

- $\Omega = \{\omega^{(a,t_i,b,t_j)}, \omega^{(a,t_i,c,t_j)}\}$,
- $\mathcal{I}_i = \{\Omega\}$,
- $\mathcal{I}_j = \big\{\{\omega^{(a,t_i,b,t_j)}\}, \{\omega^{(a,t_i,c,t_j)}\}\big\}$,
- $\pi_i(\omega) = \pi_j(\omega) = \frac{1}{2}$ for all $\omega \in \Omega$,

- $\sigma_i(\omega^{(a,t_i,b,t_j)}) = \sigma_i(\omega^{(a,t_i,c,t_j)}) = a$,
- $\sigma_j(\omega^{(a,t_i,b,t_j)}) = b$ and $\sigma_j(\omega^{(a,t_i,c,t_j)}) = c$,

forms a $\mathcal{T}^\Gamma$-induced state model of Γ. The $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$-induced type model of Γ is given by $\hat{\mathcal{T}^\Gamma} = \langle (\hat{T}_i, \hat{b}_i)_{i \in I} \rangle$ with

- $\hat{T}_i = \{t_i^\Omega\}$ and $T_j = \{t_j^{\{\omega^{(a,t_i,b,t_j)}\}}, t_j^{\{\omega^{(a,t_i,c,t_j)}\}}\}$,
- $\hat{b}_i[t_i^\Omega](b, t_j^{\{\omega^{(a,t_i,b,t_j)}\}}) = \sum_{\omega \in \Omega : \sigma_j(\omega) = b, \mathcal{I}_j(\omega) = \{\omega^{(a,t_i,b,t_j)}\}} \pi_i(\omega \mid \{\Omega\}) = \frac{1}{2}$ and
  $\hat{b}_i[t_i^\Omega](c, t_j^{\{\omega^{(a,t_i,c,t_j)}\}}) = \sum_{\omega \in \Omega : \sigma_j(\omega) = c, \mathcal{I}_j(\omega) = \{\omega^{(a,t_i,c,t_j)}\}} \pi_i(\omega \mid \{\Omega\}) = \frac{1}{2}$
- $\hat{b}_j[t_j^{\{\omega^{(a,t_i,b,t_j)}\}}](a, t_i^\Omega) = \sum_{\omega \in \Omega : \sigma_i(\omega) = a, \mathcal{I}_i = \{\Omega\}} \pi_j(\omega \mid \{\omega^{(a,t_i,b,t_j)}\}) = 1$,
- $\hat{b}_j[t_j^{\{\omega^{(a,t_i,c,t_j)}\}}](a, t_i^\Omega) = \sum_{\omega \in \Omega : \sigma_i(\omega) = a, \mathcal{I}_i = \{\Omega\}} \pi_j(\omega \mid \{\omega^{(a,t_i,c,t_j)}\}) = 1$.

Since $\mid T_j \mid < \mid \hat{T}_j \mid$, there does not exist a bijection $f_j : T_j \rightarrowtail\!\!\!\!\rightarrow \hat{T}_j$ and consequently $\mathcal{T}^\Gamma$ and $\hat{\mathcal{T}^\Gamma}$ are not isomorphic. ♣

In the preceding example the input type model only contains one type for player j, yet there are two cells for him in the generated state model, which in turn imply two corresponding types in its induced type model. It thus becomes impossible to construct a bijection between the two type models. However, one of the two types in the output type model is superfluous in the sense of interactive thinking, as it encodes precisely the same belief hierarchy as the other type.

To remove any superflous ingredients from type models we now introduce the idea of reduction.

Definition 8. *Let Γ be a game, and $\langle (T_i, b_i)_{i \in I} \rangle$ as well as $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$ type models of Γ.*

(a) The type model is $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$ is a reduction *of the type model $\langle (T_i, b_i)_{i \in I} \rangle$, if for every player $i \in I$ there exists a reduction function $r_i : T_i \to \tilde{T}_i$ such that r_i is surjective and*

$$\tilde{b}_i\left(r_i(t_i)\right)\left((c_j, \tilde{t}_j)_{j \in I \setminus \{i\}}\right) = b_i(t_i)\left(\left(\{c_j\} \times r_j^{-1}(\tilde{t}_j)\right)_{j \in I \setminus \{i\}}\right) \tag{1}$$

for all $(c_j, \tilde{t}_j)_{j \in I \setminus \{i\}} \in \times_{j \in I \setminus \{i\}} (C_j, \tilde{T}_j)$ and for all $t_i \in T_i$.

(b) The type model $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$ is a strict reduction *of the type model $\langle (T_i, b_i)_{i \in I} \rangle$, if $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$ is a reduction of $\langle (T_i, b_i)_{i \in I} \rangle$ and $\mid \tilde{T}_j \mid < \mid T_j \mid$ for some $j \in I$.*

(c) The type model $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$ is a maximal reduction *of the type model $\langle (T_i, b_i)_{i \in I} \rangle$, if $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$ is a reduction of $\langle (T_i, b_i)_{i \in I} \rangle$ and there exists no strict reduction of $\langle (\tilde{T}_i, \tilde{b}_i)_{i \in I} \rangle$.*

Note that the reduction functions r_i for all $i \in I$ correspond to surjective type morphisms of Heifetz and Samet (1998, Definition 3.2).

A couple of preparatory results about reduced type models are established next.

Lemma 1. *Let Γ be a game, $\mathcal{T}^\Gamma$ a type model of Γ, $\tilde{\mathcal{T}^\Gamma}$ a reduction of $\mathcal{T}^\Gamma$ with reduction function $r_j : T_j \to \tilde{T}_j$ for every player $j \in I$, and $i \in I$ some player. Then, $\eta_i[t_i] = \eta_i[r_i(t_i)]$ for all $t_i \in T_i$.*

Proof. It is shown inductively that $\eta_i[t_i]^k = \eta_i[r_i(t_i)]^k$ holds for all $t_i \in T_i$, for all $i \in I$, and for all $k \geq 1$. It then directly follows that $\eta_i[t_i] = (\eta_i^n[t_i])_{n\in\mathbb{N}} = (\eta_i^n[r_i(t_i)])_{n\in\mathbb{N}}\eta_i[r_i(t_i)]$ for all $t_i \in T_i$ and for all $i \in I$.

Let $k = 1$ and consider some player $i \in I$, some type $t_i \in T_i$ of player i, as well as some opponents' choice combination $c_{-i} \in C_{-i}$. By definition,

$$\eta_i^1[t_i](c_{-i}) = \sum_{t_{-i}\in T_{-i}} b_i[t_i](c_{-i}, t_{-i}).$$

Moreover, as

$$\tilde{b}_i[r_i(t_i)](c_{-i}, \tilde{t}_{-i}) = \sum_{t_{-i}\in T_{-i}:r_{-i}(t_{-i})=\tilde{t}_{-i}} b_i[t_i](c_{-i}, t_{-i}),$$

it follows that

$$\eta_i^1[r_i(t_i)](c_{-i}) = \sum_{\tilde{t}_{-i}\in \tilde{T}_{-i}} \tilde{b}_i[r_i(t_i)](c_{-i}, \tilde{t}_{-i})$$

$$= \sum_{\tilde{t}_{-i}\in \tilde{T}_{-i}} \sum_{t_{-i}\in T_{-i}:r_{-i}(t_{-i})=\tilde{t}_{-i}} b_i[t_i](c_{-i}, t_{-i}) = \sum_{t_{-i}\in T_{-i}} b_i[t_i](c_{-i}, t_{-i}) = \eta_i^1[t_i](c_{-i}).$$

Let $k \geq 2$ and assume that $\eta_i[t_i]^l = \eta_i[r_i(t_i)]^l$ holds for all $t_i \in T_i$, for all $i \in I$, and for all $l \leq k-1$. Consider some player $i \in I$, some type $t_i \in T_i$ of player i, and some tuple $(c_{-i}, \eta_{-i}^1, \dots, \eta_{-i}^{k-1}) \in X_{-i}^k$. By definition,

$$\eta_i^k[t_i](c_{-i}, \eta_{-i}^1, \dots, \eta_{-i}^{k-1}) = \sum_{t_{-i}\in T_{-i}:\eta_{-i}^l[t_{-i}]=\eta_{-i}^l \text{ for all } l\leq k-1} b_i[t_i](c_{-i}, t_{-i}).$$

Consequently,

$$\eta_i^k[r_i(t_i)] = \sum_{\tilde{t}_{-i}\in \tilde{T}_{-i}:\eta_{-i}^l[\tilde{t}_{-i}]=\eta_{-i}^l \text{ for all } l\leq k-1} b_i[r_i(t_i)](c_{-i}, \tilde{t}_{-i})$$

$$= \sum_{\tilde{t}_{-i}\in \tilde{T}_{-i}:\eta_{-i}^l[\tilde{t}_{-i}]=\eta_{-i}^l \text{ for all } l\leq k-1} \sum_{t_{-i}\in T_{-i}:r_{-i}(t_{-i})=\tilde{t}_{-i}} b_i[t_i](c_{-i}, t_{-i})$$

$$= \sum_{t_{-i}\in T_{-i}:\eta_{-i}^l[r_{-i}(t_{-i})]=\eta_{-i}^l \text{ for all } l\leq k-1} b_i[t_i](c_{-i}, t_{-i}) = \sum_{t_{-i}\in T_{-i}:\eta_{-i}^l[t_{-i}]=\eta_{-i}^l \text{ for all } l\leq k-1} b_i[t_i](c_{-i}, t_{-i})$$

$$= \eta_i^k[t_i](c_{-i}, \eta_{-i}^1, \dots, \eta_{-i}^{k-1}),$$

where the fourth equality follows from the inductive hypothesis. ■

Thus, type models are structurally equivalent to their reduced counterparts. No essential information is lost and the same interactive reasoning is represented. Lemma 1 follows from Heifetz and Samet (1998, Proposition 5.1). Since their formal framework is slightly different and to keep our paper self-contained, we still provide a direct proof.

Lemma 2. *Let Γ be a game, $\mathcal{T}^\Gamma$ a type model of Γ such that there exists no strict reduction of $\mathcal{T}^\Gamma$, and $i \in I$ some player. Then, $\eta_i[t_i'] \neq \eta_i[t_i'']$ for all $t_i', t_i'' \in T_i$ such that $t_i' \neq t_i''$.*

Proof. By contraposition, suppose that there exist $t_i', t_i'' \in T_i$ such that $t_i' \neq t_i''$ and $\eta_i[t_i'] = \eta_i[t_i'']$. For every player $j \in I$ recall the set $H_j[\mathcal{T}^\Gamma] := \{\eta_j \in \times_{n\in\mathbb{N}}\Delta(X_j^n) :$ There exists $t_j \in T_j$ such that $\eta_j[t_j] = \eta_j\}$ of induced belief hierarchies in the type model $\mathcal{T}^\Gamma$. Construct a type model $\tilde{\mathcal{T}^\Gamma} = \langle(\tilde{T}_j, \tilde{b}_j)_{j\in I}\rangle$ where $\tilde{T}_j := H_j[\mathcal{T}^\Gamma]$ for every player $j \in I$ and

$$\tilde{b}_j[h_j](c_{-j}, h_{-j}) := \sum_{t_{-j}\in T_{-j}:\eta_{-j}[t_{-j}]=h_{-j}} b_j[t_j](c_{-j}, t_{-j}) \tag{2}$$

such that $\eta_j[t_j] = h_j$, for all $(c_{-j}, h_{-j}) \in C_{-j} \times \tilde{T}_{-j}$, for all $h_j \in \tilde{T}_j$, and for all $j \in I$. Observe that the belief functions are well-defined, since every two types $t_j, t_j' \in T_j$ such that $\eta_j[t_j] = \eta_j[t_j']$ satisfy

$$\sum_{t_{-j}\in T_{-j}:\eta_{-j}[t_{-j}]=h_{-j}} b_j[t_j](c_{-j}, t_{-j}) = \sum_{t_{-j}\in T_{-j}:\eta_{-j}[t_{-j}]=h_{-j}} b_j[t_j'](c_{-j}, t_{-j})$$

for all $(c_{-j}, t_{-j}) \in C_{-j} \times T_{-j}$ and for all $h_{-j} \in \tilde{T}_{-j}$.

For every player $j \in I$ define a surjection $r_j : T_j \to \tilde{T}_j$ such that

$$r_j(t_j) := \eta_j[t_j] \tag{3}$$

for all $t_j \in T_j$. By (2) and (3) it follows that

$$\tilde{b}_j[r_j(t_j)](c_{-j}, \tilde{t}_{-j}) = b_j[t_j]\big(\{c_{-j}\} \times r_{-j}^{-1}(\tilde{t}_{-j})\big)$$

for all $(c_{-j}, \tilde{t}_{-j}) \in C_{-j} \times \tilde{T}_{-j}$, for all $t_j \in T_j$, and for all $j \in I$. Consequently, $\tilde{\mathcal{T}^\Gamma}$ constitutes a reduction of $\mathcal{T}^\Gamma$. Since $\eta_i[t_i'] = \eta_i[t_i'']$, it is the case that $|\tilde{T}_i| = |H_i[\mathcal{T}^\Gamma]| < |T_i|$ for player i. Therefore, $\tilde{\mathcal{T}^\Gamma}$ actually is a strict reduction of $\mathcal{T}^\Gamma$. ■

Accordingly, any two different types in an epistemic model without strict reduction possibilities induce distinct belief hierarchies. In this sense, maximally reduced type models do not carry any superfluous ingredients.

If the input type model and output type model of the successive application of the two transformation procedures are considered in their maximally reduced form, then an isomorphism does emerge between the input and output type models.

Theorem 5. *Let Γ be a game, $\mathcal{T}^\Gamma$ a type model of Γ, and $\hat{\mathcal{T}^\Gamma}$ the type model of Γ generated by a $\mathcal{T}^\Gamma$-generated state model. Then, every maximal reduction of $\mathcal{T}^\Gamma$ is isomorphic to every maximal reduction of $\hat{\mathcal{T}^\Gamma}$.*

Proof. Let $i \in I$ be a player and note that the set $\hat{T}_i$ from $\hat{\mathcal{T}^\Gamma}$ can be expressed as $\{\hat{t}_i^{P_i^{(c_i,t_i)}} : t_i \in T_i, c_i \in C_i\}$, where T_i belongs to $\mathcal{T}^\Gamma$. Construct a correspondence $e_i : T_i \to \hat{T}_i$ such that

$$e_i(t_i) := \{\hat{t}_i^{P_i^{(c_i,t_i)}} : c_i \in C_i\}$$

for all $t_i \in T_i$. Thus, e_i maps i's types from the initial input model to the respective types in the output model. Hence, by construction, $\hat{T}_i = \cup_{t_i\in T_i} e_i(t_i)$. By Theorems 3 and 1 it follows that every $t_i \in T_i$ and every $\hat{t}_i \in \hat{T}_i$ such that $\hat{t}_i \in e_i(t_i)$ induce the same belief hierarchy, i.e. $\eta_i[t_i] = \eta_i[\hat{t}_i]$. Consequently, for every player $i \in I$ there exists a collection of belief hierarchies $H_i \subseteq \times_{n\in\mathbb{N}}\big(\Delta(X_i^n)\big)$ such that

$$H_i[\mathcal{T}^\Gamma] = H_i[\hat{\mathcal{T}^\Gamma}] = H_i. \tag{4}$$

Let $\mathcal{T}^\Gamma_\downarrow = \langle (T_{\downarrow i}, b_{\downarrow i})_{i\in I}\rangle$ be a maximal reduction of $\mathcal{T}^\Gamma$ and $\hat{\mathcal{T}}^\Gamma_\downarrow = \langle (\hat{T}_{\downarrow i}, \hat{b}_{\downarrow i})_{i\in I}\rangle$ a maximal reduction of $\hat{\mathcal{T}^\Gamma}$. By Lemma 1 and (4) it follows that $H_i[\mathcal{T}^\Gamma_\downarrow] = H_i[\mathcal{T}^\Gamma] = H_i$ as well as $H_i[\hat{\mathcal{T}}^\Gamma_\downarrow] = H_i[\hat{\mathcal{T}^\Gamma}] = H_i$ for all $i \in i$. Moreover, Lemma 2 implies that two distinct types in $\mathcal{T}^\Gamma_\downarrow$ induce different belief hierarchies. The same holds for $\hat{\mathcal{T}}^\Gamma_\downarrow$. Consequently, for every player $i \in I$ and for every belief hierarchy $h_i \in H_i$ there exists a unique type $t_i \in T_{\downarrow i} \in T_{\downarrow i}$ and a unique type $\hat{t}_i \in \hat{T}_{\downarrow i}$ such that $\eta_i[t_i] = \eta_i[\hat{t}_i] = h_i$.

It follows that for every player $i \in I$ a bijection $f_i : T_{\downarrow i} \to \hat{T}_{\downarrow i}$ can be defined such that

$$\eta_i[t_i] = \eta_i[f_i(t_i)] \tag{5}$$

for all $t_i \in T_{\downarrow i}$. Besides, (5) implies that $\hat{b}_{\downarrow i}[f_i(t_i)]\big(c_{-i}, f(t_{-i})\big) = b_{\downarrow i}[t_i](c_{-i}, t_{-i})$ for all $(c_{-i}, t_{-i}) \in C_{-i} \times T_{-i}$ and for all $t_i \in T_{\downarrow i}$. Therefore, $T_{\downarrow i}$ and $\hat{T}_{\downarrow i}$ are isomorphic. ■

A type model can thus be said to be structurally equivalent to its two-fold transformed counterpart modulo superfluous ingredients.

An notion of isomorphism can also be laid out for the epistemic framework of state models.

Definition 9. *Let Γ be a game, and $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$ as well as $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i\in I}\rangle$ state models of Γ. The state models $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$ and $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i\in I}\rangle$ are* isomorphic, *if there exists a bijection $f : \Omega \to \tilde{\Omega}$ such that for all $\omega \in \Omega$ and for all $i \in I$ it is the case that*

$$\tilde{\mathcal{I}}_i\big(f(\omega)\big) = \{f(\omega') : \omega' \in \mathcal{I}_i(\omega)\}, \tag{6}$$

$$\tilde{\pi}_i\Big(\{f(\omega)\} \mid \hat{\mathcal{I}}_i\big(f(\omega)\big)\Big) = \pi_i\big(\{\omega\} \mid \mathcal{I}_i(\omega)\big), \tag{7}$$

$$\tilde{\sigma}_i\big(f(\omega)\big) = \sigma_i(\omega). \tag{8}$$

In two isomorphic state models the corresponding worlds induce the same information, posterior beliefs, and choices for all players. The subjective priors can be distinct yet the models qualify as isomorphic, because the players' belief hierarchies i.e. their full interactive thinking are fixed by the posterior beliefs. A difference in priors is not a relevant issue, as the posterior beliefs are the relevant doxastic mental configurations upon which the agents act. In that sense subjective prior beliefs could be viewed as artifacts of the state-based approach.

Take some state model $\mathcal{S}^\Gamma = \langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i\in I}\rangle$ as input and construct a state model $\hat{\mathcal{S}^\Gamma} = \langle \hat{\Omega}, (\hat{\mathcal{I}}_i, \hat{\sigma}_i, \hat{\pi}_i)_{i\in I}\rangle$ as output by first applying Definition 5 to $\mathcal{S}^\Gamma$ and then Definition 6 to the $\mathcal{S}^\Gamma$-generated type model. By counterexample it is now illustrated that such input and output state models are not necessarily isomorphic.

Example 2. Let Γ be a game with $I = \{i, j\}$, and $C_i = \{a\}$ as well as $C_j = \{b\}$ be choices of i and j, respectively. Consider the state model $\mathcal{S}^\Gamma$ of Γ with

- $\Omega = \{\omega_1, \omega_2\}$,
- $\mathcal{I}_i = \mathcal{I}_j = \{\Omega\}$,
- $\pi_i(\omega) = \pi_j(\omega) = \frac{1}{2}$ for all $\omega \in \Omega$,
- $\sigma_i(\omega) = a$ and $\sigma_j(\omega) = b$ for all $\omega \in \Omega$.

Then, $\mathcal{T}^\Gamma = \langle (T_i, b_i)_{i \in I} \rangle$ with

- $T_i = \{t_i^\Omega\}$ and $T_j = \{t_j^\Omega\}$,
- $b_i[t_i^\Omega](b, t_j^\Omega) = \sum_{\omega \in \Omega : \sigma_j(\omega) = b, \mathcal{I}_j(\omega) = \{\Omega\}} \pi_i(\omega \mid \{\Omega\}) = 1$,
- $b_j[t_j^\Omega](a, t_i^\Omega) = \sum_{\omega \in \Omega : \sigma_i(\omega) = a, \mathcal{I}_i(\omega) = \{\Omega\}} \pi_j(\omega \mid \{\Omega\}) = 1$,

constitutes the $\mathcal{S}^\Gamma$-generated type model of Γ. However, it directly follows that $\langle \hat{\Omega}, (\hat{\mathcal{I}}_i, \hat{\sigma}_i, \hat{\pi}_i)_{i \in I} \rangle$ with $\hat{\Omega} = \{\omega^{(c_i, t_i^\Omega, c_j, t_j^\Omega)}\}$ forms the unique $\mathcal{T}^\Gamma$-generated state model of Γ. Consequently, there exists no bijection $f : \Omega \to \hat{\Omega}$. The state models $\mathcal{S}^\Gamma$ and $\hat{\mathcal{S}}^\Gamma$ are consequently not isomorphic. ♣

In the preceding example the possible worlds ω_1 and ω_2 in the input state model $\mathcal{S}^\Gamma$ induce the same choices and beliefs for both players. In terms of interactive thinking one of them thus is superfluous. These kind of redundancies prevent the isomorphic relationship between input and output state models to hold in general.

We call a state model $\mathcal{S}^\Gamma$ of Γ *non-redundant*, if for all $\omega, \omega' \in \Omega$ such that $\omega \neq \omega'$ it is the case that $\mathcal{I}_i(\omega) \neq \mathcal{I}_i(\omega')$ or $\sigma_i(\omega) \neq \sigma_i(\omega')$ for some $i \in I$. Intuitively, any two distinct worlds in the structure carry some difference for at least one of the players. Observe that non-redundancy implies that $\cap_{i \in I} \mathcal{I}_i(\omega) = \{\omega\}$ for all $\omega \in \Omega$. Essentially, the latter says that if the players' information is pooled, then all uncertainty is resolved.

To get rid of any superfluous ingredients we also need a notion of reduction for state models in addition to non-redundancy.

Definition 10. *Let Γ be a game, and $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ as well as $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i \in I} \rangle$ be state models of Γ.*

(a) *The state model $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i \in I} \rangle$ is a* reduction *of the state model $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$, if there exists a reduction function $r : \Omega \to \tilde{\Omega}$ such that r is surjective and for all $i \in i$*

$$\tilde{\mathcal{I}}_i\big(r(\omega)\big) = \{r(\omega') : \omega' \in \mathcal{I}_i(\omega)\} \text{ for all } \omega \in \Omega, \tag{9}$$

$$\tilde{\sigma}_i\big(r(\omega)\big) = \sigma_i(\omega) \text{ for all } \omega \in \Omega \text{ such that } \pi_j\big(\omega \mid \mathcal{I}_j(\omega)\big) > 0 \text{ for some } j \in I \setminus \{i\}, \tag{10}$$

$$\tilde{\pi}_i\Big(\tilde{\omega} \mid \tilde{\mathcal{I}}_i\big(r(\omega)\big)\Big) = \pi_i\big(r^{-1}(\tilde{\omega}) \mid \mathcal{I}_i(\omega)\big) \text{ for all } \omega \in \Omega \text{ and for all } \tilde{\omega} \in \tilde{\Omega}. \tag{11}$$

(b) *The state model $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i \in I} \rangle$ is a* strict reduction *of the state model $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$, if $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i \in I} \rangle$ is a reduction of $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ and $\mid \tilde{\Omega} \mid < \mid \Omega \mid$.*
(c) *The state model $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i \in I} \rangle$ is a* maximal reduction *of the state model $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$, if $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i \in I} \rangle$ is a reduction of $\langle \Omega, (\mathcal{I}_i, \sigma_i, \pi_i)_{i \in I} \rangle$ and there exists no strict reduction of $\langle \tilde{\Omega}, (\tilde{\mathcal{I}}_i, \tilde{\sigma}_i, \tilde{\pi}_i)_{i \in I} \rangle$.*

Some results about reductions of state models are developed before an isomorphic relationship between state models and their two-fold transformed counterparts emerges.

Lemma 3. *Let Γ be a game, and $\mathcal{S}^\Gamma$ a state model of Γ. If there exists no strict reduction of $\mathcal{S}^\Gamma$, then $\mathcal{S}^\Gamma$ is non-redundant.*

Proof. We proceed by contraposition. Suppose that $\mathcal{S}^\Gamma$ is redundant. Then there exist distinct worlds $\omega', \omega'' \in \Omega$ such that $\mathcal{I}_i(\omega') = \mathcal{I}_i(\omega'')$ as well as $\sigma_i(\omega') = \sigma_i(\omega'')$ for every player $i \in I$. Construct a state model $\tilde{\mathcal{S}^\Gamma}$ of Γ as follows:

- $\tilde{\Omega} := \Omega \setminus \{\omega', \omega''\} \cup \{\omega^*\}$

and for every player $j \in I$,

- $\tilde{\mathcal{I}_j}(\omega^*) := \mathcal{I}_j(\omega') \setminus \{\omega', \omega''\} \cup \{\omega^*\}$,
- $\tilde{\mathcal{I}_j}(\omega) := \begin{cases} \mathcal{I}_j(\omega), & \text{if } \omega', \omega'' \notin \mathcal{I}_j(\omega), \\ \tilde{\mathcal{I}_j}(\omega^*), & \text{otherwise,} \end{cases}$ for all $\omega \in \tilde{\Omega} \setminus \{\omega^*\}$,
- $\tilde{\sigma}_j(\omega^*) = \sigma_j(\omega')$,
- $\tilde{\sigma}_j(\omega) = \sigma_j(\omega)$ for all $\omega \in \tilde{\Omega} \setminus \{\omega^*\}$,
- $\tilde{\pi}_j(\omega^*) = \pi_j(\omega') + \pi_j(\omega'')$,
- and $\tilde{\pi}_j(\omega) = \pi_j(\omega)$ for all $\omega \in \tilde{\Omega} \setminus \{\omega^*\}$.

Define a function $r : \Omega \to \tilde{\Omega}$ by $r(\omega') = r(\omega'') = \omega^*$ and $r(\omega) = \omega$ for all $\omega \in \Omega \setminus \{\omega', \omega''\}$. Observe that r is surjective and also satisfies conditions (9), (10), and (11). As $\mid \tilde{\Omega} \mid = \mid \Omega \mid -1 < \mid \Omega \mid$, the state model $\tilde{\mathcal{S}^\Gamma}$ constitutes a strict reduction of $\mathcal{S}^\Gamma$. ■

Accordingly, maximal reduction in the sense of the impossibility of strict reduction implies non-redundancy.

By considering maximally reduced models, the existence of superfluous worlds such as in Example 2 is blocked and an isomorphic relationship between input and output state models ensues.

Theorem 6. *Let Γ be a game, $\mathcal{S}^\Gamma$ a state model of Γ, and $\hat{\mathcal{S}^\Gamma}$ a state model of Γ generated by the $\mathcal{S}^\Gamma$-generated type model. Then, every maximal reduction of $\mathcal{S}^\Gamma$ is isomorphic to every maximal reduction of $\hat{\mathcal{S}^\Gamma}$.*

Proof. Consider a maximal reduction $\mathcal{S}^\Gamma_\downarrow$ of $\mathcal{S}^\Gamma$ and a maximal reduction $\hat{\mathcal{S}}^\Gamma_\downarrow$ of $\hat{\mathcal{S}^\Gamma}$. The set $\hat{\Omega}$ from $\hat{\mathcal{S}}^\Gamma_\downarrow$ is a subset of $\{\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}} : c_i \in C_i \text{ for all } i \in I, \omega \in \Omega\}$, which is from $\hat{\mathcal{S}}^\Gamma$, and where Ω and $\mathcal{I}_i$ for all $i \in I$ belong to $\mathcal{S}^\Gamma_\downarrow$. It is first shown that for every world $\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}} \in \hat{\Omega}$, it is the case that $c_i = \sigma_i(\omega)$, where σ_i belongs to $\mathcal{S}^\Gamma_\downarrow$, for all $i \in I$. Towards a contradiction suppose that there exists a world $\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}} \in \hat{\Omega}$ such that $c_j \neq \sigma_j(\omega)$ for some player $j \in I$. By definition of the two transformation procedures,

$$\hat{\pi}_k\big(\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}} \mid \hat{\mathcal{I}}_k(\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}})\big)$$

$$= b_k[t_k^{\mathcal{I}_k(\omega)}](c_{-k}, t_{-k}^{\mathcal{I}_{-k}(\omega)})$$

$$= \sum_{\omega' \in \mathcal{I}_k(\omega) : \sigma_{-k}(\omega') = c_{-k}, \mathcal{I}_{-k}(\omega') = \mathcal{I}_{-k}(\omega)} \pi_k\big(\{\omega'\} \mid \mathcal{I}_k(\omega)\big)$$

for all $k \in I \setminus \{j\}$. Since $c_j \neq \sigma_j(\omega)$ the $\mathcal{I}_j$-measurability of σ_j implies that $\sigma_j(\omega'') \neq c_j$ for all $\omega'' \in \mathcal{I}_j(\omega)$. Consequently, there exists no world $\omega' \in \mathcal{I}_k(\omega)$ such that $\sigma_j(\omega') = c_j$ and $\mathcal{I}_j(\omega') = \mathcal{I}_j(\omega)$. It follows that $\pi_k\big(\omega' \mid \mathcal{I}_k(\omega)\big) = 0$ for all $\omega' \in \mathcal{I}_k(\omega)$ such that $\sigma_{-k}(\omega') = c_{-k}$ and $\mathcal{I}_{-k}(\omega') = \mathcal{I}_{-k}(\omega)$. Thus, $\hat{\pi}_k\big(\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}} \mid \hat{\mathcal{I}}_k(\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}})\big) = 0$ for all $k \in I \setminus \{j\}$. Next define a state model $\tilde{S}^{\Gamma}$ based on $\tilde{\Omega} := \{\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i \in I}} : \omega \in \Omega\}$ as set of all possible worlds and a surjection $r : \hat{\Omega} \to \tilde{\Omega}$ with $r(\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}}) = \hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i \in I}}$ for all $\hat{\omega}^{(c_i, t_i^{\mathcal{I}_i(\omega)})_{i \in I}} \in \hat{\Omega}$ such that for all $i \in I$:

- $\tilde{\mathcal{I}}_i\big(r(\hat{\omega})\big) := \{r(\hat{\omega}') : \hat{\omega}' \in \hat{\mathcal{I}}_i(\hat{\omega})\}$ for all $r(\hat{\omega}) \in \tilde{\Omega}$,
- $\tilde{\sigma}_i(\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i \in I}}) := \sigma_i(\omega)$ for all $\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i \in I}} \in \tilde{\Omega}$,
- $\tilde{\pi}_i\Big(\tilde{\omega} \mid \tilde{\mathcal{I}}_i\big(r(\hat{\omega})\big)\Big) := \hat{\pi}_i\big(r^{-1}(\tilde{\omega}) \mid \hat{\mathcal{I}}_i(\hat{\omega})\big)$ for all $\tilde{\omega} \in \tilde{\Omega}$ and for all $\hat{\omega} \in \hat{\Omega}$.

Note that whenever $\hat{\pi}_j\big(\hat{\omega} \mid \hat{\mathcal{I}}_j(\hat{\omega})\big) > 0$ for some $j \in I \setminus \{i\}$, it is the case that $\hat{\omega} \in \tilde{\Omega}$ hence $\tilde{\sigma}_i\big(r(\hat{\omega})\big) = \hat{\sigma}_i(\hat{\omega}) = \sigma_i(\omega)$, and thus equation (10) is satisfied. As $\mid \tilde{\Omega} \mid < \mid \hat{\Omega} \mid$, the state model $\tilde{S}^{\Gamma}$ forms a strict reduction of $\hat{\mathcal{S}_{\downarrow}^{\Gamma}}$, a contradiction.

Construct a function $f : \Omega \to \hat{\Omega}$ such that $f(\omega) := \hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i \in I}}$ for all $\omega \in \Omega$. The function f is surjective, as for every world $\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i \in I}} \in \hat{\Omega}$ the pre-image $f^{-1}\left(\hat{\omega}^{\left(\sigma_i(\omega), t_i^{\mathcal{I}_i(\omega)}\right)_{i \in I}}\right) \supseteq \{\omega\}$ contains $\{\omega\}$ and is thus non-empty by the successive application of the two transformation procedures, i.e. by first applying Definition 5 to $\mathcal{S}_{\downarrow}^{\Gamma}$ and then Definition 6 to the $\mathcal{S}_{\downarrow}^{\Gamma}$-generated type model to induce $\hat{\mathcal{S}_{\downarrow}^{\Gamma}}$. Suppose that $f(\omega') = f(\omega'')$, i.e. $\hat{\omega}^{\left(\sigma_i(\omega'), t_i^{\mathcal{I}_i(\omega')}\right)_{i \in I}} = \hat{\omega}^{\left(\sigma_i(\omega''), t_i^{\mathcal{I}_i(\omega'')}\right)_{i \in I}}$, for some worlds $\omega', \omega'' \in \Omega$. Then, $\sigma_i(\omega') = \sigma_i(\omega'')$ as well as $\mathcal{I}_i(\omega') = \mathcal{I}_i(\omega'')$ for all $i \in I$. As $\mathcal{S}_{\downarrow}^{\Gamma}$ is non-redundant by Lemma 3, it follows that $\omega' = \omega''$. Hence, f is injective too and thus bijective.

We now show that the bijection f satisfies equations (6), (7), and (8) of Definition 10. First, observe that

$$\hat{\mathcal{I}}_i\big(f(\omega)\big) = \hat{\mathcal{I}}_i\big(\hat{\omega}^{\left(\sigma_j(\omega), t_j^{\mathcal{I}_j(\omega)}\right)_{j \in I}}\big)$$

$$= \{\hat{\omega}^{\left(\sigma_j(\omega'), t_j^{\mathcal{I}_j(\omega')}\right)_{j \in I}} \in \hat{\Omega} : \sigma_i(\omega') = \sigma_i(\omega), \mathcal{I}_i(\omega') = \mathcal{I}_i(\omega)\}$$

$$= \{f(\omega') : \sigma_i(\omega') = \sigma_i(\omega), \mathcal{I}_i(\omega') = \mathcal{I}_i(\omega)\} = \{f(\omega') : \omega' \in \mathcal{I}_i(\omega)\}$$

for all $\omega \in \Omega$ and for all $i \in I$. Therefore, f satisfies equation (6). Second, $\hat{\sigma}_i\big(f(\omega)\big) = \hat{\sigma}_i(\hat{\omega}^{\left(\sigma_j(\omega), t_j^{\mathcal{I}_j(\omega)}\right)_{j \in I}}) = \sigma_i(\omega)$ for all $\omega \in \Omega$ and for all $i \in I$. Hence, f satisfies equation (7). Third,

$$\hat{\pi}_i\Big(\{f(\omega)\} \mid \hat{\mathcal{I}}_i\big(f(\omega)\big)\Big) = \hat{\pi}_i\Big(\hat{\omega}^{\left(\sigma_j(\omega), t_j^{\mathcal{I}_j(\omega)}\right)_{j \in I}} \mid \hat{\mathcal{I}}_i\big(f(\omega)\big)\Big)$$

$$\hat{\pi}_i\Big(\hat{\omega}^{\left(\sigma_j(\omega), t_j^{\mathcal{I}_j(\omega)}\right)_{j \in I}} \mid \hat{\mathcal{I}}_i\big(\hat{\omega}^{\left(\sigma_j(\omega), t_j^{\mathcal{I}_j(\omega)}\right)_{j \in I}}\big)\Big)$$

$$= b_i[t_i^{\mathcal{I}_i(\omega)}]\big(\sigma_{-i}(\omega), t_{-i}^{\mathcal{I}_i(\omega)}\big)$$

$$= \sum_{\omega' \in \mathcal{I}_i(\omega) : \sigma_{-i}(\omega') = \sigma_{-i}(\omega), \mathcal{I}_{-i}(\omega') = \mathcal{I}_{-i}(\omega)} \pi_i\big(\{\omega'\} \mid \mathcal{I}_i(\omega)\big)$$

$$= \sum_{\omega' \in \Omega : \mathcal{I}_i(\omega') = \mathcal{I}_i(\omega), \sigma_i(\omega') = \sigma_i(\omega), \sigma_{-i}(\omega') = \sigma_{-i}(\omega), \mathcal{I}_{-i}(\omega') = \mathcal{I}_{-i}(\omega)} \pi_i\big(\{\omega'\} \mid \mathcal{I}_i(\omega)\big)$$

$$= \pi_i\big(f^{-1}(\hat{\omega}^{\left(\sigma_j(\omega), t_j^{\mathcal{I}_j(\omega)}\right)_{j \in I}}) \mid \mathcal{I}_i(\omega)\big) = \pi_i(\{\omega\}) \mid \mathcal{I}_i(\omega)\big)$$

for all $\omega \in \Omega$, and for all $i \in I$. Thus, f satisfies equation (8).

Consequently, $\mathcal{S}^{\Gamma}_{\downarrow}$ and $\hat{\mathcal{S}}^{\Gamma}_{\downarrow}$ are isomorphic. ■

Hence, a state model is structurally equivalent to its two-fold transformed counterpart modulo superfluous ingredients.

6 Conclusion

Belief hierarchies as well as the common prior assumption are structurally preserved across the two epistemic frameworks by the two proposed transformation procedures. With regards to modelling interactive thinking in games the state-based and type-based approaches can thus be viewed as equivalent. None of the two models thus contains any *relevant* structure that the respective other lacks. The two transformation procedures can be viewed as practical tools to switch back and forth between state-based and type-based interactive epistemology.

A somewhat more subtle difference between the two epistemic approaches surfaces, as the transformation procedures fail to constitute inverses. The underlying reason is attributable to the richer structure of state models compared to type models. While the latter only specify interactive thinking the former also fixes the players' choices. Once superfluous ingredients are wiped out, by restricting attention to maximally reduced models, the two transformation procedures turn out to be inverse to each other .

While this disparity between the state and type models does not make a difference with respect to interactive thinking at all, the particular usage could determine which epistemic approach is more appropriate. If the focus is put on reasoning in games before decisions are made or the perspective of a particular player is considered, then type models might be more suitable. In contrast for analyses that are conducted from the perspective of a modeller the state-based framework may be preferable. After all there remains a degree of subjectivity whether the specifaction of beliefs only *or* beliefs and behaviour is desired in an epistemic framework for games. Besides, the notions of maximal reduction for state and type models can also serve to simplify a given epistemic structure while retaining the same interactive thinking in applications.

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