Proper belief revision and equilibrium in dynamic games

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Abstract

We present a theory of rationality in dynamic games in which players, during the course of the game, may revise their beliefs about the opponents' utility functions. The theory is based upon the following three principles: (1) the players' *initial* beliefs about the opponents' utilities should agree on some profile u of utility functions, (2) every player should believe, at each of his information sets, that his opponents are carrying out optimal strategies and (3) a player at information set h should not change his belief about an opponent's ranking of strategies a and b if both a and b could have led to h. Scenarios with these properties are called *preference conjecture equilibria* for the profile u of utility functions. We show that every normal form proper equilibrium for u induces a preference conjecture equilibrium for u, thus implying existence of preference conjecture equilibrium.

1. Introduction

Proper equilibrium [9] and the related concept of proper rationalizability [15,1] are based on the following two assumptions: (1) a player should never exclude any opponent's strategy from consideration and (2) if player i believes that player j prefers strategy a over strategy b, then player i should deem strategy a "infinitely more likely" than strategy b. This may be formalized by considering sequences of full support beliefs, as is done in Myerson [9] and Schuhmacher [15], or equivalently by modeling the players' beliefs as lexicographic probability systems (LPSs) with

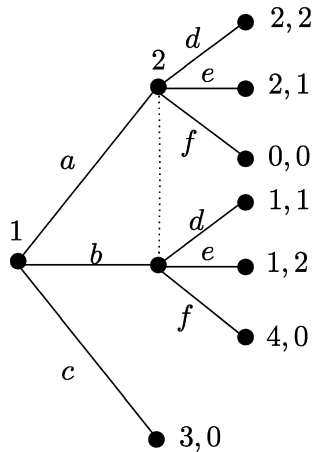*E-mail address:* a.perea@ke.unimaas.nl.

Fig. 1. An introductory example.

full support, following Blume et al. [6,7] and Asheim [1]. Both ways of modeling the players' beliefs have an important implication, namely that they induce well-defined conditional beliefs in dynamic games. Consider, for instance, the dynamic game in Fig. 1. Suppose that player 2's belief about player 1's strategy choice is given by an LPS in which he deems c infinitely more likely than a, and a infinitely more likely than b. At his information set, player 2 must conclude that player 1 has not chosen c. By updating his LPS, player 2 then forms a conditional belief at his information set in which he still believes that a is infinitely more likely than b.

In fact, this is the unique revised belief for player 2 that is selected by proper equilibrium. Namely, player 2 should realize that player 1 will deem player 2's strategy f infinitely less likely than the other two, since f is dominated by d and e. As such, player 2 should believe that player 1 prefers a over b, and hence should deem a infinitely more likely than b. Upon observing that player 1 has not chosen c, player 2's revised belief should therefore still deem a infinitely more likely than b.

One could complete the argument above by interpreting player 2's revised belief within the spirit of proper equilibrium. If player 2's revised belief deems a infinitely more likely than b, then this means that player 2, upon observing that player 1 has not chosen c, still believes that player 1 prefers a over b. With this additional interpretation, proper equilibrium thus states that player 2 should initially believe that player 1 prefers a over b, and that player 2 *should continue to believe so* when observing that player 1 has not chosen c. In a general game, proper equilibrium implies that a player who currently believes that an opponent prefers strategy a over strategy b should continue to believe so at information sets that do not exclude any of the strategies a and b. We refer to this condition on the players' belief revision policies as *proper belief revision*.

The latter condition certainly has some intuitive appeal when evaluated as an independent belief revision criterion. Consider a player i who, throughout the game, believes that player j chooses optimally at every information set. That is, player i never questions player j's rationality, no matter what happens in the game. In order to make this possible, it may be necessary for player i to revise his belief about player j's utility function during the game. Now, assume that player i believes at information set h_i^1 that player j at information set h_j strictly prefers strategy a over strategy b. Suppose the game moves from h_i^1 to a new player i information set h_i^2, and that h_i^2 can be reached

both by strategy a and strategy b. The new piece of information, namely that h_i^2 has been reached, does not contradict i's previous belief that j prefers a over b, since it does not rule out any of the strategies a and b. Proper belief revision states that in this case, player i should not revise his belief about j's relative ranking of a and b. To put it differently, if a player unconditionally believes in the opponents' rationality, then proper belief revision states that this player should never revise his belief about an opponent's ranking of two strategies a and b unless the observed play provides him with evidence against this belief.

To the best of my knowledge, proper equilibrium and proper rationalizability are the only existing rationality concepts in the literature that incorporate the idea of proper belief revision. However, it is only incorporated *implicitly* in these concepts, as both theories do not speak explicitly about belief revision. The objective of this paper is to develop a rationality concept for dynamic games that *explicitly* models the way in which players revise their beliefs during the game, and that *explicitly* imposes proper belief revision as a restriction on the players' belief revision policies. More precisely, players in our theory hold, initially and at each of their information sets, a point belief [1] about the opponents' utility functions and a probabilistic belief about the opponents' strategy choices. In particular, a player may revise his belief about an opponent's utility function as the game proceeds. We make the *equilibrium assumption* that all of these conditional beliefs are commonly believed among the players. That is, if player i holds certain conditional beliefs, then all other players believe throughout that he does so, and all players believe throughout that all players believe throughout that he does so, and so on. Choose a fixed profile u of utility functions at the terminal nodes. Our concept imposes the following conditions on the players' conditional beliefs:

1. (Initial belief in u): The players' *initial* beliefs about the opponents' utility functions should agree with u.
2. (Belief in sequential rationality): Every player should believe, at each of his information sets, that every opponent chooses optimally at every information set.
3. (Proper belief revision): A player at information set h should not revise his belief about an opponent's ranking of two strategies a and b if both a and b could have led to h.
4. (Bayesian updating): The players' conditional beliefs about the opponents' strategy choices should satisfy Bayes' rule.

Profiles of conditional belief vectors that satisfy all of these requirements are called *preference conjecture equilibria* for u. Perea [13] provides a rationalizability concept that is based on similar conditions, but without the equilibrium assumption. As an illustration of our concept, consider again the game tree in Fig. 1. Suppose that the utilities at the terminal nodes represent both the players' actual utilities, and their *initial* beliefs about the opponent's utility function. That is, we take u to be the profile of utility functions as depicted at the terminal nodes. Note, however, that we allow player 2 to change his initial belief about player 1's utility function once his information set is reached. Let us derive the preference conjecture equilibria for u. Since player 1 initially believes in u and believes in sequential rationality, player 1 must believe that player 2 will not choose f. By the equilibrium assumption, player 2 initially believes that player 1 believes that player 2 will not choose f. Together with the requirement that player 2 initially believes in u and believes in sequential rationality, player 2 must initially believe that player 1 prefers c over a, and prefers a over b. By proper belief revision, player 2 must still believe at his information set that player 1 prefers a over b. By belief in sequential rationality, player 2 must also believe at

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[1] By point belief we mean that the player assigns probability 1 to *one* utility function for every opponent.

his information set that player 1 weakly prefers *a* over *c*, since player 2 must interpret *a* as an optimal choice for player 1. In order to achieve this, player 2 must revise his belief about player 1's utility function, for instance, by believing that player 1's utility after *c* was not 3 but 0. As such, player 2 must believe at his information set that player 1 has chosen *a*. By the equilibrium assumption, player 1 must believe that player 2, at his information set, believes that player 1 has chosen *a*. By belief in sequential rationality, player 1 must therefore believe that player 2 chooses *d*. Summarizing, there is basically a unique preference conjecture equilibrium for *u*, namely that player 1 believes that player 2 chooses *d*, player 2 initially believes that player 1 chooses *c* and player 2 believes at his information set that player 1 has chosen *a*. The only freedom left is that player 2 may revise his belief about player 1's utility function in many different ways as to ensure that player 2, at his information set, believes that player 1 weakly prefers *a* over *c*, and strictly prefers *a* over *b*. In each of these scenarios, the optimal choices for the players are *c* and *d*.

The organization and main results of this paper may be summarized as follows. In Section 2 we lay out our basic model, and develop the concept of preference conjecture equilibrium. Our main result, Theorem 3.4 in Section 3, roughly states that every normal form proper equilibrium induces a preference conjecture equilibrium. More precisely, consider an extensive form game tree with "observable deviators", a profile *u* of utility functions, a normal form proper equilibrium in mixed strategies for the game with utility functions *u*, together with a "proper" sequence of full support mixed strategy profiles converging to the proper equilibrium. Observable deviators states that a player, who currently expects his information set *h* not be reached, but who finds out later that *h* has actually been reached, knows exactly which player(s) is (are) to be held responsible for this surprise. By "proper" sequence, we mean that if in some mixed strategy profile in this sequence player *i* strictly prefers strategy *a* over strategy *b*, then the ratio between the probability of *b* and the probability of *a* should tend to zero within this sequence. Normal form proper equilibria are exactly those mixed strategy profiles that are the limit of some proper sequence of full support mixed strategy profiles. Theorem 3.4 states that we can always construct a preference conjecture equilibrium for *u* in which the conditional beliefs about the opponents' strategies are given by this proper sequence of full support mixed strategy profiles. This result thus guarantees the existence of preference conjecture equilibria for each profile of utility functions on which the players' initial beliefs must agree. Moreover, the result demonstrates that there is a strong formal connection between normal form proper equilibrium and preference conjecture equilibrium. Although this connection may appear natural after reading this Introduction, the formal proof is actually quite tedious. The condition of observable deviators is important for Theorem 3.4, as we show at the end of Section 3 that a preference conjecture equilibrium may fail to exist in a game that violates the observable deviators condition. In Section 4 we discuss the relationship between preference conjecture equilibrium and other rationality concepts.

2. Preference conjecture equilibrium

2.1. Preliminaries

We first present the notation to be used, as well as some preliminary definitions in extensive form games. The rules of the game are formalized by an *extensive form structure* $\mathcal{S}$, specifying a finite game tree, a finite set of players I, for every player i a finite collection H_i of information sets and at every information set $h \in H_i$ some finite set of actions $A(h)$. By A we denote the set of all actions, whereas Z denotes the set of terminal nodes. We assume that there are no chance moves and that the extensive form structure satisfies perfect recall.
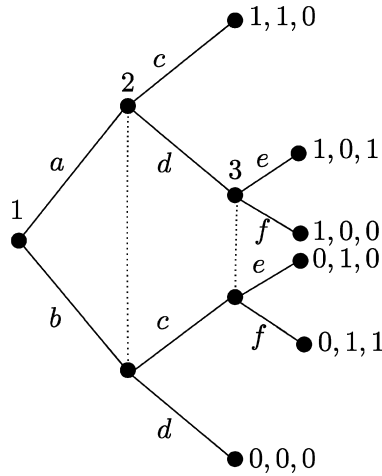
Fig. 2. A game that violates the observable deviators condition.

The objects of choice for the players are *strategies*. The notion of strategies we use coincides with the definition of a *plan of action*, as discussed in Rubinstein [14]. It differs slightly from the usual definition since we require players only to choose actions at those information sets which are *not avoided* by the strategy itself. Formally, let $H_i^* \subseteq H_i$ be a collection of player i information sets, not necessarily containing all player i information sets, and let $s_i : H_i^* \to A$ with $s_i(h) \in A(h)$ for all $h \in H_i^*$. We say that an information set $h \in H_i$ is *avoided* by s_i if for every node $x \in h$ the path from the root to x crosses some information set $h' \in H_i^*$ at which the prescribed action $s_i(h')$ deviates from this path. A *strategy* for player i is a function $s_i : H_i^* \to A$ with $s_i(h) \in A(h)$ for all $h \in H_i^*$ such that H_i^* is exactly the collection of player i information sets not avoided by s_i.[2] Let S_i be the set of player i strategies.

For a given player i and information set h, not necessarily controlled by player i, let $S_i(h)$ be the set of those player i strategies s_i for which there is some profile $s_{-i} = (s_j)_{j \neq i}$ of opponents' strategies such that (s_i, s_{-i}) reaches h. Let $S(h)$ be the set of those strategy profiles $(s_i)_{i \in I}$ which reach h. Throughout the paper, we shall impose the requirement that $S(h) = \times_{i \in I} S_i(h)$ for all information sets h. In the literature, this condition is called the *observable deviators* condition (see, for instance, [3]). Intuitively, a game is with observable deviators if a player, who currently believes that his information set h will not be reached, but later finds out that h has been reached, knows exactly which player(s) is (are) to be held responsible for this surprise. Fig. 2 provides an example of a game violating the observable deviators condition. Let h be the information set controlled by player 3. By definition,

$$S(h) = \{(a, d, e), (a, d, f), (b, c, e), (b, c, f)\},$$
$$S_1(h) = \{(a, b)\}, \quad S_2(h) = \{(c, d)\} \quad \text{and} \quad S_3(h) = \{(e, f)\}$$

which implies that $S(h) \neq S_1(h) \times S_2(h) \times S_3(h)$, and hence the game is not with observable deviators. Intuitively, if player 3 initially believes that his opponents choose a and c, but later

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[2] A strategy s_i in our setting is thus obtained by taking a "classical" strategy $\tilde{s}_i$ (prescribing an action at *all* player i information sets) and restricting $\tilde{s}_i$ to those information sets that are not avoided by $\tilde{s}_i$.

finds out that his information set has been reached, he does not know which opponent is to be held responsible for this surprise. As such, player 3 does not know whether he should revise his belief about player 1's strategy choice, or about player 2's strategy choice, upon observing that his information set has been reached. As we will see at the end of Section 3, this problem appears to be crucial for our analysis. If a game only has two players, the observable deviators condition follows automatically from the perfect recall assumption. On the other hand, most games with more than two players that arise from applications in economics or other fields satisfy the observable deviators condition.

For a given information set $h_i \in H_i$, we write $S_{-i}(h_i) = \times_{j \neq i} S_j(h_i)$. By $\Delta(X)$ we denote the set of probability distributions on X.

2.2. Definition of preference conjecture equilibrium

Our basic assumption is that each player holds, at the beginning of the game as well as at each of his information sets, a preference relation over his own strategies that is of the expected utility type. Formally, let h_0 be the information set that marks the beginning of the game, and let $H_i^* = H_i \cup \{h_0\}$. Define $S_i(h_0) = S_i$. Every player i is endowed with a utility function u_i and holds at each information set $h_i \in H_i^*$ some conjecture $\mu_i(h_i) \in \Delta(S_{-i}(h_i))$ about the opponents' strategies, inducing for every strategy $s_i \in S_i(h_i)$ the expected utility

$$u_i(s_i, \mu_i(h_i)) = \sum_{s_{-i} \in S_{-i}(h_i)} \mu_i(h_i)(s_{-i}) u_i(z(s_i, s_{-i})).$$

Here, $z(s_i, s_{-i})$ is the terminal node reached by (s_i, s_{-i}). Hence, for every two strategies $s_i, s_i' \in S_i(h_i)$, player i weakly prefers s_i over s_i' at h_i if and only if $u_i(s_i, \mu_i(h_i)) \geqslant u_i(s_i', \mu_i(h_i))$. Throughout this paper, we assume that the conjecture $\mu_i(h_i) \in \Delta(S_{-i}(h_i))$ can be written as the product of its marginal probability distributions $\mu_{ij}(h_i) \in \Delta(S_j(h_i))$.

In our model, a player does not only have uncertainty about the opponents' strategy choices, but also about their utility functions. Moreover, a player may revise his conjecture about an opponent's utility function as the game proceeds. For a given player i, information set $h_i \in H_i^*$ and opponent j, let $u_{ij}(h_i) : Z \to \mathbb{R}$ represent player i's conjecture at h_i about player j's utility function. We thus assume that each player, at every instance of the game, assigns probability 1 to one particular utility function for every opponent. We do so in order to keep our model as simple as possible. A vector

$$c_i = (\mu_{ij}(h_i), u_{ij}(h_i))_{h_i \in H_i^*, j \neq i}$$

specifying at every player i information set a conjecture about the opponents' strategy choices and utility functions is called a *preference conjecture* for player i. A profile $c = (c_i)_{i \in I}$ is called a *preference conjecture profile*.

The *equilibrium assumption* we make is that there be common belief among the players about the preference conjecture profile c, that is, we assume that every player holds his preference conjecture in c, that every player believes throughout the game that every player holds his preference conjecture in c, and so on. Under this equilibrium assumption, we may now deduce from c player i's conjecture at h_i about player j's preference relation at h_j over his strategies in $S_j(h_j)$. By definition of c, player i believes at h_i that player j's utility function is given by $u_{ij}(h_i)$. Moreover, since player i believes that player j holds preference conjecture c_j, player i believes at h_i that player j's conjecture at h_j about the opponents' strategy choices is given by $\mu_j(h_j) = (\mu_{jk}(h_j))_{k \neq j}$.

Definition 2.1. We say that player i believes at h_i that player j at h_j weakly prefers strategy $s_j \in S_j(h_j)$ over strategy $s'_j \in S_j(h_j)$ if

$$u_{ij}(h_i)(s_j, \mu_j(h_j)) \geqslant u_{ij}(h_i)(s'_j, \mu_j(h_j)). \tag{2.1}$$

We say that player i believes at h_i that an opponent's strategy s_j is optimal at information set $h_j \in H_j(s_j)$ if player i believes at h_i that player j at h_j ranks s_j weakly over all other strategies in $S_j(h_j)$.

Here $u_{ij}(h_i)(s_j, \mu_j(h_j))$ denotes the expected utility for player j at h_j induced by the utility function $u_{ij}(h_i)$, strategy s_j and player j's conjecture $\mu_j(h_j)$ over the opponents' strategy choices. We now model the assumption that players, throughout the game, should believe that opponents choose optimally at each of their information sets.

Definition 2.2. A preference conjecture profile $c = (c_i)_{i \in I}$ is called *sequentially rational* if for every player i, information set $h_i \in H_i^*$, opponent j and strategy $s_j \in S_j(h_i)$ we have: $\mu_{ij}(h_i)(s_j) > 0$ only if player i believes at h_i that s_j is optimal at every information set $h_j \in H_j(s_j)$.

We next impose that players should update their beliefs about the opponents' strategy choices by Bayes' rule, whenever possible.

Definition 2.3. A preference conjecture profile $c = (c_i)_{i \in I}$ is said to satisfy *Bayesian updating* if for every player i, every two information sets $h_i^1, h_i^2 \in H_i^*$ where h_i^2 follows h_i^1 and every opponent j it holds that

$$\mu_{ij}(h_i^2)(s_j) = \frac{\mu_{ij}(h_i^1)(s_j)}{\mu_{ij}(h_i^1)(S_j(h_i^2))}$$

for every strategy $s_j \in S_j(h_i^2)$, whenever $\mu_{ij}(h_i^1)(S_j(h_i^2)) > 0$.

Here, $\mu_{ij}(h_i^1)(S_j(h_i^2))$ is the sum of the probabilities that $\mu_{ij}(h_i^1)$ assigns to strategies in $S_j(h_i^2)$. We now formalize proper belief revision, stating that a player at information set h should not change his belief about an opponent's relative ranking of two strategies a and b if both a and b could have led to h.

Definition 2.4. A preference conjecture profile $c = (c_i)_{i \in I}$ is said to satisfy *proper belief revision* if for every two information sets $h_i^1, h_i^2 \in H_i^*$ where h_i^2 follows h_i^1, every opponent's information set $h_j \in H_j^*$ and every two opponent's strategies $s_j, s'_j \in S_j(h_j) \cap S_j(h_i^2)$ the following holds: player i believes at h_i^2 that player j at h_j weakly prefers s_j over s'_j if and only if player i believes so at h_i^1.

Note that player i, upon observing that the game has moved from h_i^1 to h_i^2, cannot distinguish between the strategies s_j and s'_j in $S_j(h_j) \cap S_j(h_i^2)$, since both lead to the information set h_i^2. Proper belief revision states that in this case, player i should maintain his previous belief about player j's relative ranking of these two strategies. Combining sequential rationality, Bayesian updating and proper belief revision leads to the concept of preference conjecture equilibrium.

Definition 2.5. A preference conjecture profile $c = (c_i)_{i \in I}$ is called a *preference conjecture equilibrium* if it is sequentially rational and satisfies Bayesian updating and proper belief revision.

The last requirement we impose is that the players' *initial* beliefs about the opponents' utility functions coincide with a given profile of utility functions. We do so in order to compare the concept of preference conjecture equilibrium to existing concepts in the literature that assume a fixed profile of utility functions.

Definition 2.6. Let $\mathcal{S}$ be an extensive form structure and $u = (u_i)_{i \in I}$ a profile of utility functions. A preference conjecture profile $c = (c_i)_{i \in I}$ is called a *preference conjecture equilibrium for* $(\mathcal{S}, u)$ if (1) c is a preference conjecture equilibrium and (2) $u_{ij}(h_0) = u_j$ for all players i and j.

Note that the pair $(\mathcal{S}, u)$ corresponds to what is normally called an *extensive form game*.

3. Relation with proper equilibrium

3.1. Some preparatory lemmas

In this section we prove that for any extensive form game $(\mathcal{S}, u)$, every normal form proper equilibrium [9] in mixed strategies for $(\mathcal{S}, u)$ induces a preference conjecture equilibrium for $(\mathcal{S}, u)$. For the proof of our theorem, we need some preparatory lemmas. The proofs of the second and the third lemma can be found in the Appendix. The first result is a characterization of normal form proper equilibrium which can be found in Perea [11, Lemma 3.6.2]. Consider a sequence $(\lambda^n)_{n \in \mathbb{N}}$ of strictly positive mixed strategy profiles $(\lambda_i^n)_{i \in I}$, that is, λ_i^n assigns strictly positive probability to every strategy s_i. Say that the sequence $(\lambda^n)_{n \in \mathbb{N}}$ is *proper* if for every $n \in \mathbb{N}$, every player i and all strategies $s_i, t_i \in S_i$ with $u_i(s_i, \lambda_{-i}^n) < u_i(t_i, \lambda_{-i}^n)$ it holds that $\lim_{n \to \infty} \lambda_i^n(s_i)/\lambda_i^n(t_i) = 0$. Here, $u_i(s_i, \lambda_{-i}^n)$ denotes player i's expected utility induced by (s_i, λ_{-i}^n).

Lemma 3.1. *A mixed strategy profile $\lambda = (\lambda_i)_{i \in I}$ is a normal form proper equilibrium if and only if it is the limit of some proper sequence $(\lambda^n)_{n \in \mathbb{N}}$ of strictly positive mixed strategy profiles.*

Consider a normal form proper equilibrium λ for $(\mathcal{S}, u)$ with supporting proper sequence $(\lambda^n)_{n \in \mathbb{N}}$. Assume that for every player i, every information set $h_i \in H_i^*$, every opponent j and strategy $s_j \in S_j(h_i)$ the limit

$$\lambda_j(h_i)(s_j) := \lim_{n \to \infty} \frac{\lambda_j^n(s_j)}{\lambda_j^n(S_j(h_i))} \tag{3.1}$$

exists. Then, this number may be interpreted as the induced subjective probability that player i assigns to strategy s_j, conditional on the event that his information set h_i has been reached. It therefore represents a natural candidate for the subjective probability $\mu_{ij}(h_i)(s_j)$ that player i at information set h_i assigns to the strategy s_j in a preference conjecture profile induced by λ. The following result will play a key role in the proof of Theorem 3.4.

Lemma 3.2. *Let λ be a normal form proper equilibrium for $(\mathcal{S}, u)$ with supporting sequence $(\lambda^n)_{n \in \mathbb{N}}$. Suppose that the limits $\lambda_j(h_i)(s_j)$, as defined in (3.1), exist for all $j \in I$, $h_i \in H_i^*$ and $s_j \in S_j(h_i)$. Then, $\lambda_j(h_i)(s_j) > 0$ implies that $u_j(s_j, \lambda_{-j}(h_j)) \geqslant u_j(s_j', \lambda_{-j}(h_j))$ for all $h_j \in H_j(s_j)$ and all $s_j' \in S_j(h_i) \cap S_j(h_j)$.*

Here, $\lambda_{-j}(h_j)$ denotes the profile $(\lambda_k(h_j))_{k \neq j}$. The following lemma highlights a technical property that extensive form structures with observable deviators have. In order to state that lemma, we need some additional notation. Let $h_i \in H_i^*$ and $h_j \in H_j$. If h_j precedes h_i, let $A(h_j, h_i)$ be the set of actions at h_j which lead to the information set h_i, that is, $a \in A(h_j, h_i)$ if and only if there is some path from the root to h_i at which a is chosen at h_j. If h_j does not precede h_i, then define $A(h_j, h_i) = A(h_j)$. Let $Z_j(h_i)$ be the set of terminal nodes that can be reached by strategies in $S_j(h_i)$.

Lemma 3.3. *Let $\mathcal{S}$ be an extensive form structure with observable deviators. Then, the following is true for every $h_i \in H_i^*$ and every player j:*

(a) $s_j \in S_j(h_i)$ *if and only if* $s_j(h_j) \in A(h_j, h_i)$ *for every* $h_j \in H_j(s_j)$;
(b) $z \in Z_j(h_i)$ *if and only if for every player j information set h_j on the path to z, the unique action at h_j leading to z belongs to $A(h_j, h_i)$.*

3.2. Proper equilibrium induces preference conjecture equilibrium

Theorem 3.4. *Let $\mathcal{S}$ be an extensive form structure with observable deviators, $u = (u_i)_{i \in I}$ a profile of utility functions and $\lambda = (\lambda_i)_{i \in I}$ a normal form proper equilibrium in the game $(\mathcal{S}, u)$ with supporting proper sequence $(\lambda^n)_{n \in \mathbb{N}}$. Then, there is a preference conjecture equilibrium c for $(\mathcal{S}, u)$ such that every player i, at each of his information sets $h_i \in H_i^*$, believes that every opponent j chooses each of his strategies $s_j \in S_j(h_i)$ with probability*

$$\mu_{ij}(h_i)(s_j) = \lim_{n \to \infty} \frac{\lambda_j^n(s_j)}{\lambda_j^n(S_j(h_i))}.$$

In this theorem, we thus implicitly assume that $\lim_{n \to \infty} \lambda_j^n(s_j) / \lambda_j^n(S_j(h_i))$ always exists. This can be done without loss of generality, since every proper sequence $(\lambda^n)_{n \in \mathbb{N}}$ contains a proper subsequence for which these limits always exist.

Proof. Let $(\mathcal{S}, u)$ be given and let $\lambda = (\lambda_i)_{\in I}$ be a normal form proper equilibrium for $(\mathcal{S}, u)$ with supporting proper sequence $(\lambda^n)_{n \in \mathbb{N}}$ of strictly positive mixed strategy profiles converging to λ. For every player i, information set $h_i \in H_i^*$ and opponent $j \neq i$, define player i's conjecture $\mu_{ij}(h_i)$ at h_i about player j's strategy choice by

$$\mu_{ij}(h_i)(s_j) = \lim_{n \to \infty} \frac{\lambda_j^n(s_j)}{\lambda_j^n(S_j(h_i))} \tag{3.2}$$

for all strategies $s_j \in S_j(h_i)$. We shall now define the players' beliefs about the opponents' utility functions, and prove that these beliefs, together with the conjectures about the opponents' strategy choices defined in (3.2), constitute a preference conjecture equilibrium for $(\mathcal{S}, u)$. For a given player i, information set $h_i \in H_i^*$ and opponent j, let player i's belief at h_i about player j's utility function be given by $u_{ij}(h_i) : Z \to \mathbb{R}$ where

$$u_{ij}(h_i)(z) = \begin{cases} u_j(z) & \text{if } z \in Z_j(h_i), \\ u_j(z) - K_j(h_i) & \text{if } z \notin Z_j(h_i). \end{cases} \tag{3.3}$$

Here, the constant $K_j(h_i) > 0$ is chosen such that $u_j(z_1) > u_j(z_2) - K_j(h_i)$ for all $z_1 \in Z_j(h_i)$ and all $z_2 \notin Z_j(h_i)$.

Now, let $c = (c_i)_{i \in I}$ be the preference conjecture profile given by (3.2) and (3.3). We prove that c is a preference conjecture equilibrium for $(\mathcal{S}, u)$. Hence, we must show that c satisfies sequential rationality, Bayesian updating and proper belief revision, and that the initial beliefs about the opponents' utility functions coincide with u.

Sequential rationality: Let $h_i \in H_i^*$ and let s_j be a strategy in $S_j(h_i)$ such that $\mu_{ij}(h_i)(s_j) > 0$. Choose an information set h_j in $H_j(s_j)$. We must show that $u_{ij}(h_i)(s_j, \mu_j(h_j)) \geqslant u_{ij}(h_i) (s_j', \mu_j(h_j))$ for all $s_j' \in S_j(h_j)$. We prove this result in two steps.

Claim 1. $u_{ij}(h_i)(s_j, \mu_j(h_j)) \geqslant u_{ij}(h_i)(s_j', \mu_j(h_j))$ *for all* $s_j' \in S_j(h_i) \cap S_j(h_j)$.

This claim follows easily from (3.1)–(3.3) and Lemma 3.2. Note that, for every $s_j' \in S_j(h_i) \cap S_j(h_j)$, the profile $(s_j', \mu_j(h_j))$ only leads to terminal nodes in $Z_j(h_i)$, and therefore, by (3.3), $u_{ij}(h_i)(s_j', \mu_j(h_j)) = u_j(s_j', \mu_j(h_j))$. The formal proof can be found in the working paper version [12].

Claim 2. *For every* $s_j'' \in S_j(h_j) \backslash S_j(h_i)$ *there is some* $s_j' \in S_j(h_i) \cap S_j(h_j)$ *with* $u_{ij}(h_i)(s_j', \mu_j(h_j)) \geqslant u_{ij}(h_i)(s_j'', \mu_j(h_j))$.

Proof of claim 2. For a given $s_j'' \in S_j(h_j) \backslash S_j(h_i)$, let

$$\hat{H}_j = \{ h_j' \in H_j(s_j'') \mid s_j''(h_j') \notin A(h_j', h_i) \}.$$

By definition of $A(h_j', h_i)$ it holds that, if $a \in A(h_j') \backslash A(h_j', h_i)$, then a avoids h_i. Hence, every h_j'' following both h_j' and a cannot precede h_i which implies that $A(h_j'', h_i) = A(h_j'')$. We may thus conclude that, if $h_j', h_j'' \in \hat{H}_j$, then h_j'' cannot precede or follow h_j'. Let s_j' be such that $s_j'(h_j') = s_j''(h_j')$ for all $h_j' \in H_j(s_j'') \backslash \hat{H}_j$ and $s_j'(h_j') \in A(h_j', h_i)$ for all $h_j' \in H_j(s_j'') \cap \hat{H}_j$. Then, $s_j'(h_j') \in A(h_j', h_i)$ for all $h_j' \in H_j(s_j')$ and hence, by Lemma 3.3, $s_j' \in S_j(h_i) \cap S_j(h_j)$. By definition,

$$u_{ij}(h_i)(s_j'', \mu_j(h_j)) = \sum_{h_j' \in \hat{H}_j} \mathbb{P}_{(s_j'', \mu_j(h_j))}(h_j') u_{ij}(h_i)(s_j'', \mu_j(h_j'))$$
$$+ \sum_{z \notin Z(\hat{H}_j)} \mathbb{P}_{(s_j'', \mu_j(h_j))}(z) u_{ij}(h_i)(z), \tag{3.4}$$

where $Z(\hat{H}_j)$ is the set of terminal nodes preceded by some information set in $\hat{H}_j$, and $\mathbb{P}_{(s_j'', \mu_j(h_j))}(h_j')$ is the probability that $(s_j'', \mu_j(h_j))$ reaches h_j'. Similarly, $\mathbb{P}_{(s_j'', \mu_j(h_j))}(z)$ is the probability that $(s_j'', \mu_j(h_j))$ reaches the terminal node z. By construction of s_j', we have that $\mathbb{P}_{(s_j', \mu_j(h_j))}(h_j') = \mathbb{P}_{(s_j'', \mu_j(h_j))}(h_j')$ for all $h_j' \in \hat{H}_j$ and $\mathbb{P}_{(s_j', \mu_j(h_j))}(z) = \mathbb{P}_{(s_j'', \mu_j(h_j))}(z)$ for all $z \notin Z(\hat{H}_j)$. Let $h_j' \in \hat{H}_j$ be given. Then, $s_j''(h_j') \notin A(h_j', h_i)$. Hence, $(s_j'', \mu_j(h_j'))$ only leads to terminal nodes following information set h_j' and some action $a \notin A(h_j', h_i)$. But then, from Lemma 3.3(b), we may conclude that $(s_j'', \mu_j(h_j'))$ only leads to terminal nodes that are not in $Z_j(h_i)$. On the other hand, $(s_j', \mu_j(h_j'))$ reaches only terminal nodes in $Z_j(h_i)$ since $s_j' \in S_j(h_i)$. By (3.3) it then follows that

$$u_{ij}(h_i)(s_j'', \mu_j(h_j')) < u_{ij}(h_i)(s_j', \mu_j(h_j'))$$

for all $h'_j \in \hat{H}_j$. Combining all these insights with (3.4) leads to

$$u_{ij}(h_i)(s''_j, \mu_j(h_j)) \leqslant \sum_{h'_j \in \hat{H}_j} \mathbb{P}_{(s'_j, \mu_j(h_j))}(h'_j) u_{ij}(h_i)(s'_j, \mu_j(h'_j))$$

$$+ \sum_{z \notin Z(\hat{H}_j)} \mathbb{P}_{(s'_j, \mu_j(h_j))}(z) u_{ij}(h_i)(z)$$

$$= u_{ij}(h_i)(s'_j, \mu_j(h_j)),$$

with $s'_j \in S_j(h_i) \cap S_j(h_j)$. This completes the proof of claim 2. $\quad\square$

From Claims 1 and 2 it immediately follows that $u_{ij}(h_i)(s_j, \mu_j(h_j)) \geqslant u_{ij}(h_i)(s'_j, \mu_j(h_j))$ for all $s'_j \in S_j(h_j)$, and hence sequential rationality holds.

Bayesian updating: From (3.2) it follows immediately that the preference conjecture profile c satisfies Bayesian updating.

Proper belief revision: We next prove that the preference conjecture profile c given by (3.2) and (3.3) satisfies proper belief revision. Consider two player i information sets h_i^1, h_i^2 in H_i^* such that h_i^2 follows h_i^1, and consider a player j information set h_j. Moreover, let s_j, s'_j be two player j strategies in $S_j(h_j) \cap S_j(h_i^2)$. We prove that player i believes at h_i^2 that player j at h_j weakly prefers s_j over s'_j if and only if player i believes so at h_i^1. Since $s_j, s'_j \in S_j(h_i^2) \subseteq S_j(h_i^1)$, it follows that $(s_j, \mu_j(h_j))$ and $(s'_j, \mu_j(h_j))$ only lead to terminal nodes in $Z_j(h_i^2) \subseteq Z_j(h_i^1)$. By (3.3), it follows that

$$u_{ij}(h_i^2)(s_j, \mu_j(h_j)) = u_{ij}(h_i^1)(s_j, \mu_j(h_j)) = u_j(s_j, \mu_j(h_j))$$

and

$$u_{ij}(h_i^2)(s'_j, \mu_j(h_j)) = u_{ij}(h_i^1)(s'_j, \mu_j(h_j)) = u_j(s'_j, \mu_j(h_j)).$$

But then, player i believes at h_i^2 that player j at h_j weakly prefers s_j over s'_j if and only if player i believes so at h_i^1.

Initial beliefs about utilities coincide with u: In order to see this, note that, by definition, $Z_j(h_0) = Z$ and hence $u_{ij}(h_0)(z) = u_j(z)$ for all terminal nodes z in Z.

As such, we may conclude that the preference conjecture profile c given by (3.2) and (3.3) is a preference conjecture equilibrium for $(\mathcal{S}, u)$. This completes the proof of this theorem. $\quad\square$

At this stage, the reader may wonder whether the observable deviators condition is needed in the theorem. We shall illustrate, by means of a counterexample, that preference conjecture equilibria may fail to exist in games that do not satisfy the observable deviators condition. Consider again the game in Fig. 2, which violates the observable deviators condition. Let u_1, u_2 and u_3 be the utility functions depicted at the terminal nodes. We show that there is no preference conjecture equilibrium for $(\mathcal{S}, u)$. Suppose, on the contrary, that c would be a preference conjecture equilibrium for $(\mathcal{S}, u)$. Since player 3, at the beginning of the game, believes that the opponents' utility functions are given by u_1 and u_2, player 3 initially believes that player 1 strictly prefers a over b and that player 2 strictly prefers c over d. Let h be player 3's information set. Proper belief revision states that player 3, at information set h, should maintain his initial belief about player 1's preference relation over strategies in $S_1(h)$ and about player 2's preference relation over strategies in $S_2(h)$. Since $S_1(h) = \{a, b\}$ and $S_2(h) = \{c, d\}$, player 3 should then still believe at h that player 1 strictly prefers a over b, and that player 2 strictly prefers c over d. Sequential

rationality then implies that player 3 must believe at h that player 1 has chosen a and player 2 has chosen c, which is incompatible with the event that h has been reached. We may thus conclude that there is no preference conjecture equilibrium for $(\mathcal{S}, u)$.

4. Relation with other rationality concepts

Nash equilibrium: Consider a preference conjecture equilibrium c for $(\mathcal{S}, u)$ with the additional property that every two different players j and k have the same initial belief about player i's strategy $(i \neq j, k)$. Then, it can easily be shown that the (common) initial beliefs in c about the opponents' strategies constitute a Nash equilibrium in mixed strategies for $(\mathcal{S}, u)$. For a proof of this result, as well as a detailed comparison with Aumann and Brandenburger's [2] epistemic foundation for Nash equilibrium, the reader is referred to the working paper version [12].

Sequential equilibrium: Say that a preference conjecture profile has *common beliefs about future actions* if for every player i, every $h_i \in H_i^*$, all players $j, k \neq i$ and all $h_j \in H_j^*$, $h_k \in H_k^*$ preceding h_i with $\mu_{ji}(h_j)(S_i(h_i)) > 0$ and $\mu_{ki}(h_k)(S_i(h_i)) > 0$, we have that j's belief at h_j about player i's action choice at h_i is the same as k's belief at h_k about player i's action choice at h_i. It can be shown that every preference conjecture equilibrium c for $(\mathcal{S}, u)$ with common beliefs about future actions induces a weak sequential equilibrium. A precise statement and proof of this result can be found in the working paper version [12]. Here, by a weak sequential equilibrium we mean an assessment that satisfies sequential rationality (as defined by Kreps and Wilson [8]) and Bayesian updating . The difference with Kreps and Wilson's original definition of sequential equilibrium is that consistency is replaced by the weaker condition of Bayesian updating.

Conversely, not every weak sequential equilibrium for $(\mathcal{S}, u)$ corresponds to a preference conjecture equilibrium for $(\mathcal{S}, u)$. Consider, for instance, the game in Fig. 1. The behavioral strategy profile (c, e) with belief vector $(0, 1)$ at player 2's information set is a weak sequential equilibrium, but does not correspond to a preference conjecture equilibrium for $(\mathcal{S}, u)$. Since (c, e) is also a Nash equilibrium for $(\mathcal{S}, u)$, this implies that also not every Nash equilibrium corresponds to a preference conjecture equilibrium.

Extensive form rationalizability: Until now, we have been reviewing rationality concepts that are either refinements or weakenings of preference conjecture equilibrium. Our main result, Theorem 3.4, states that normal form proper equilibrium may be seen as a refinement of preference conjecture equilibrium, whereas the paragraphs above show that Nash equilibrium and weak sequential equilibrium may be viewed as weakenings of preference conjecture equilibrium. There are other rationality concepts, like extensive form rationalizability [10,4] that belong to neither category. Let us consider the game in Fig. 1 again. The reasoning of extensive form rationalizability in this game is as follows: Since f is dominated by d and e, player 2 should initially believe that player 1 believes that player 2 will not choose f. As such, player 2 should initially believe that player 1 chooses c. However, if player 2 finds out that player 1 has not chosen c, player 2 looks for a theory about player 1 that (1) explains the event that player 1 has not chosen c, (2) maintains player 2's original belief about player 1's utility function and (3) comes as close as possible to common belief in sequential rationality. See Battigalli and Siniscalchi [5] for a precise statement of (3). Basically, there are two possible theories that satisfy (1) and (2). Either, player 1 did not choose rationally, or player 1 rationally chose b but believed, with sufficiently high probability, that player 2 would irrationally choose f. Among these two theories, the second is closer to common belief in sequential rationality. Hence, player 2 must believe, upon observing that player 1 has not chosen c, that he chose b. As such, extensive form rationalizability leads player 2 to choose e and player 1 to choose c. In particular, extensive form rationalizability uniquely selects

strategy e for player 2, whereas we have seen that preference conjecture equilibrium uniquely selects strategy d for player 2. The same holds if we would use iterated maximal elimination of weakly dominated strategies instead of extensive form rationalizability.

Acknowledgments

Appendix

Proof of Lemma 3.2. Suppose that $u_j(s_j, \lambda_{-j}(h_j)) < u_j(s'_j, \lambda_{-j}(h_j))$ for some $h_j \in H_j(s_j)$ and $s'_j \in S_j(h_i) \cap S_j(h_j)$. We show that $\lambda_j(h_i)(s_j) = 0$.

Define

$$\lambda_k^n(h_j)(s_k) := \frac{\lambda_k^n(s_k)}{\lambda_k^n(S_k(h_j))}$$

for all $s_k \in S_k(h_j)$, and let $\lambda_{-j}^n(h_j) := (\lambda_k^n(h_j))_{k \neq j}$. Since $u_j(s_j, \lambda_{-j}(h_j)) < u_j(s'_j, \lambda_{-j}(h_j))$, we have that $u_j(s_j, \lambda_{-j}^n(h_j)) < u_j(s'_j, \lambda_{-j}^n(h_j))$ for some n. Fix such n.

Let $Z(h_j)$ be the set of terminal nodes that follow the information set h_j, and let $H_j(h_j)$ be the collection of player j information sets that precede $Z(h_j)$. Let s''_j be the strategy which coincides with s'_j at every $h_j \in H_j(h_j)$, and coincides with s_j at all other information sets. Then, by construction, $s''_j \in S_j(h_i) \cap S_j(h_j)$. For a given terminal node z, let $\mathbb{P}_{(s_j, \lambda_{-j}^n)}(z)$ be the probability that z is reached by (s_j, λ_{-j}^n). It holds that

$$
\begin{aligned}
u_j(s_j, \lambda_{-j}^n) &= \sum_{z \in Z} \mathbb{P}_{(s_j, \lambda_{-j}^n)}(z)\, u_j(z) \\
&= \mathbb{P}_{(s_j, \lambda_{-j}^n)}(h_j)\, u_j(s_j, \lambda_{-j}^n(h_j)) + \sum_{z \notin Z(h_j)} \mathbb{P}_{(s_j, \lambda_{-j}^n)}(z)\, u_j(z) \\
&< \mathbb{P}_{(s_j, \lambda_{-j}^n)}(h_j) u_j(s'_j, \lambda_{-j}^n(h_j)) + \sum_{z \notin Z(h_j)} \mathbb{P}_{(s_j, \lambda_{-j}^n)}(z) u_j(z) \\
&= \mathbb{P}_{(s'_j, \lambda_{-j}^n)}(h_j)\, u_j(s'_j, \lambda_{-j}^n(h_j)) + \sum_{z \notin Z(h_j)} \mathbb{P}_{(s_j, \lambda_{-j}^n)}(z)\, u_j(z) \\
&= u_j(s''_j, \lambda_{-j}^n).
\end{aligned}
$$

In order to see why the second equality holds, note that by the observable deviators assumption, $(s_j, (s_k)_{k \neq j})$ reaches h_j if and only if $s_k \in S_k(h_j)$ for all $k \neq j$. (Recall that s_j is fixed). As such, $\mathbb{P}_{(s_j, \lambda_{-j}^n)}(h_j) = \prod_{k \neq j} \lambda_k^n(S_k(h_j))$. Since $\lambda_k^n(h_j)(s_k) = \lambda_k^n(s_k)/\lambda_k^n(S_k(h_j))$ for all $s_k \in S_k(h_j)$, the second equality follows. The inequality follows from $u_j(s_j, \lambda_{-j}^n(h_j)) < u_j(s'_j, \lambda_{-j}^n(h_j))$ and the observation that $\mathbb{P}_{(s_j, \lambda_{-j}^n)}(h_j) > 0$, since $s_j \in S_j(h_j)$ and λ_{-j}^n is strictly positive. For the third equality, note that $\mathbb{P}_{(s_j, \lambda_{-j}^n)}(h_j) = \mathbb{P}_{(s'_j, \lambda_{-j}^n)}(h_j)$ since $s_j, s'_j \in S_j(h_j)$ and hence, by perfect recall, s_j and s'_j coincide on the player j information sets preceding h_j. The fourth equality simply follows from the definition of s''_j.

We thus have shown that $u_j(s_j, \lambda^n_{-j}) < u_j(s''_j, \lambda^n_{-j})$ for some $s''_j \in S_j(h_i)$. Since $(\lambda^n)_{n\in\mathbb{N}}$ is a proper sequence, it follows that $\lim_{n\to\infty} \lambda^n_j(s_j)/\lambda^n_j(s''_j) = 0$. Since $s_j, s''_j \in S_j(h_i)$, it follows that

$$\lambda_j(h_i)(s_j) = \lim_{n\to\infty} \frac{\lambda^n_j(s_j)}{\lambda^n_j(S_j(h_i))} = 0,$$

which completes the proof. $\square$

Proof of Lemma 3.3. (a) Let $s_j \in S_j(h_i)$. Suppose that there is some $h_j \in H_j(s_j)$ with $s_j(h_j) \notin A(h_j, h_i)$. Then, necessarily, h_j precedes h_i. Hence, by the definition of $A(h_j, h_i)$, the action $s_j(h_j)$ avoids h_i. On the other hand, since h_j precedes h_i, there is some node $x \in h_j$ which leads to h_i. By perfect recall, there is some strategy profile s_{-j} such that (s_j, s_{-j}) reaches x. Hence, there is some strategy profile $(\tilde{s}_j, \tilde{s}_{-j})$ such that $(\tilde{s}_j, \tilde{s}_{-j})$ reaches x and h_i. Since $(\tilde{s}_j, \tilde{s}_{-j}) \in S(h_i)$ and, by the observable deviators condition, $S(h_i) = \times_{k\in I} S_k(h_i)$, it follows that $\tilde{s}_{-j} \in \times_{k\neq j} S_k(h_i)$. Since $(\tilde{s}_j, \tilde{s}_{-j})$ reaches $x \in h_j$, we know, by perfect recall, that $\tilde{s}_j$ coincides with s_j on the player j information sets preceding h_j. Hence, $(s_j, \tilde{s}_{-j})$ reaches h_j. Since $s_j(h_j)$ avoids h_i, we have that $(s_j, \tilde{s}_{-j})$ does not reach h_i, and hence $(s_j, \tilde{s}_{-j}) \notin S(h_i)$. Since, by the observable deviators condition, $S(h_i) = \times_{k\in I} S_k(h_i)$ and $\tilde{s}_{-j} \in \times_{k\neq j} S_k(h_i)$, it thus follows that $s_j \notin S_j(h_i)$, which is a contradiction. We may thus conclude that $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$.

Now, let s_j be such that $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$. We prove that $s_j \in S_j(h_i)$. We distinguish two cases. Suppose first that there is no player j information set preceding h_i. Then, obviously, $s_j \in S_j(h_i)$. Suppose now that there is some player j information set preceding h_i. Let $h_j \in H_j(s_j)$ be a player j information set preceding h_i such that there is no other player j information set in $H_j(s_j)$ between h_j and h_i. By assumption, $s_j(h_j) \in A(h_j, h_i)$, hence there exists a node $x \in h_j$ such that h_i can be reached through x via action $s_j(h_j)$. By perfect recall, there is some strategy profile $\tilde{s}_{-j}$ for the opponents such that $(s_j, \tilde{s}_{-j})$ reaches x. Since there is no $h'_j \in H_j(s_j)$ between h_j and h_i, and since h_i can be reached through x via $s_j(h_j)$, we can choose $\tilde{s}_{-j}$ such that $(s_j, \tilde{s}_{-j})$ reaches h_i. But then, by definition, $s_j \in S_j(h_i)$. This completes the proof of part (a).

(b) Suppose that $z \in Z_j(h_i)$ and h_j is an information set on the path to z. Then, obviously, the unique action at h_j leading to z belongs to $A(h_j, h_i)$. Suppose, on the other hand, that the terminal node z is such that for every player j information set h_j on the path to z, the unique action at h_j leading to z belongs to $A(h_j, h_i)$. Let s_j be a strategy such that at every information set $h_j \in H_j(s_j)$ on the path to z, the strategy s_j chooses the unique action at h_j leading to z, and at every other information set $h_j \in H_j(s_j)$ the strategy s_j chooses some action in $A(h_j, h_i)$. Then, $s_j(h_j) \in A(h_j, h_i)$ for all $h_j \in H_j(s_j)$, and hence, by part (a), $s_j \in S_j(h_i)$. Since z can be reached by strategy s_j, it follows that $z \in Z_j(h_i)$. This completes the proof. $\square$

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