Int J Game Theory (2006) 34:457–473
DOI 10.1007/s00182-006-0035-4

ORIGINAL PAPER

Stochastic dominance equilibria in two-person noncooperative games

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Accepted: 31 August 2006 / Published online: 28 September 2006

Abstract Two-person noncooperative games with finitely many pure strategies are considered, in which the players have linear orderings over sure outcomes but incomplete preferences over probability distributions resulting from mixed strategies. These probability distributions are evaluated according to t-degree stochastic dominance. A t-best reply is a strategy that induces a t-degree stochastically undominated distribution, and a t-equilibrium is a pair of t-best replies. The paper provides a characterization and an existence proof of t-equilibria in terms of representing utility functions, and shows that for large t behavior converges to a form of max–min play. Specifically, increased aversion to bad outcomes makes each player put all weight on a strategy that maximizes the worst outcome for the opponent, within the supports of the strategies in the limiting sequence of t-equilibria.

Keywords Stochastic dominance · Two-person noncooperative games

1 Introduction

In this paper we consider noncooperative games between players with incomplete preferences over lotteries. More precisely, we assume that they have complete preferences over sure alternatives, and incomplete preferences over lotteries resulting from playing mixed strategies.

The paper has benefitted from the comments of four referees and an associate editor.

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An important motivation for considering incomplete preferences over lotteries is the intuition that completeness is a very strong assumption in many situations: it may be already quite demanding for players to be able to order all sure alternatives. It is not the objective of this paper to delve deep into a discussion about (in)completeness of preferences. A recent paper that both presents a brief history of this discussion and investigates the formal consequences of dropping the completeness assumption from the von Neumann-Morgenstern conditions, is Dubra et al. (2004). In fact, their main result applies to the preferences we consider in this paper.

To become more specific, we consider two-person games with finite (pure) strategy sets. The players have complete transitive antisymmetric preferences (linear orderings) over the pure outcomes of the game, and evaluate probability distributions over the outcomes induced by mixed strategies by a stochastic dominance criterion. This, of course, is a very specific way to deal with incomplete preferences, but there is an appropriate justification in terms of aversion to risk or, perhaps better, to bad outcomes, as we will see. With the exception of completeness of preferences, we do not deviate from the standard assumptions of noncooperative game theory, such as common knowledge between the players of the data of the game, including the (incomplete) preferences.

According to first-degree stochastic dominance, probability distributions that shift more probability to better outcomes are considered more attractive. Since, indeed, the stochastic dominance ordering is not complete, we call a mixed strategy of a player a best reply against the strategy of the opponent if the resulting probability distribution over the outcomes is not dominated by any other probability distribution that the player in question is able to realize, given the strategy of the opponent. An equilibrium is a pair of best replies. Fishburn (1978) established that the set of equilibria is equal to the union of all sets of Nash equilibria, taken over all possible utility representations of the preferences. This is an intuitive result in view of the familiar characterization of first-degree stochastic dominance which says that a distribution is undominated if and only if it maximizes expected utility for at least one utility representation of the preferences over certain outcomes.

We take this analysis a good deal further by assuming that the players' preferences over lotteries are more restricted or, equivalently, that the derived classes of utility functions which may represent their preferences are narrowed down. Specifically, we study so-called t-equilibria, where the natural number t is the degree of stochastic dominance used to evaluate probability distributions. As is well-known, a distribution is second-degree stochastically undominated if and only if it maximizes expected utility for at least one *concave* utility representation of the preferences over certain outcomes. Loosely speaking, higher degrees of stochastic dominance correspond to higher degrees of risk aversion or, more specifically, higher degrees of aversion to bad outcomes. As the degree of stochastic dominance becomes higher, preferences become more complete and, in the limit, would order lotteries by comparing worst outcomes.

After preliminaries about stochastic dominance, games and equilibria in Sects. 2 and 3, we consider an example in Sect. 4 which nicely illustrates these concepts and the main results of the paper. These results are, first, a characterization of t-best replies in terms of utility functions and existence of t-equilibria in Sect. 5, and, second, limit behavior as the degree of stochastic dominance t goes to infinity, in Sect. 6.

Existence of t-equilibria can be established directly by using a fixed point argument,[1] or indirectly by using representation by utility functions and existence of Nash equilibrium (Sect. 5).

It follows from the results of Sect. 5 that the sets of characterizing utility functions become smaller as t grows, as already indicated by the transition from $t = 1$ to $t = 2$. Consequently, the best reply correspondences and sets of t-equilibria decrease as well. In Sect. 6 we provide a complete characterization of the sets of pure strategies that can serve as supports for t-equilibria as t becomes large. In the limit, such equilibria converge to max–min play, in the sense that each player plays a pure strategy that, among the strategies in the supports, maximizes the worst outcome for the opponent. Observe that this is very different from what is usually meant by max–min play, namely that players maximize their own worst outcomes. Max–min play in the present setting is closer to equilibrium play: for large t, a player puts probability close to 1 on the pure strategy maximizing the worst outcome for the opponent among the strategies in the support of the opponent's mixed strategy, in order to keep all these strategies undominated. The intuition for this is that, as t becomes large, the opponent attaches increasing weights to worse outcomes, and to compensate for this a player should put low weights on those own strategies that possibly result in these worse outcomes for the opponent. In fact, this can be interpreted as altruism emerging in equilibrium as a consequence of high aversion to bad outcomes.

Section 7 concludes the paper with a brief discussion of possible extensions and of related literature. Appendix A collects the proofs of Sect. 2.

2 Stochastic dominance

Let $\ell \geq 1$ be an integer and let $O = \{1, \ldots, \ell\}$ be a set of ℓ *alternatives*. For $1 \leq k < l \leq \ell$ we assume that a decision maker strictly prefers alternative l to alternative k.

For a probability distribution $r = (r_1, \ldots, r_\ell)$ on O (so alternative l occurs with probability r_l) we define, recursively, for each $l \in \{1, \ldots, \ell\}$, $F_r^0(l) = r_l$ and

$$F_r^t(l) = \sum_{i=1}^{l} F_r^{t-1}(i) \quad (t \geq 1).$$

So F_r^1 is the cumulative distribution function of $F_r^0 = r$ and, similarly, F_r^t 'accumulates' the weights assigned by F_r^{t-1}. For probability distributions r and s on O, *r t-th degree stochastically dominates s* if

[1] This was done in an earlier version of the paper; the proof is available upon request.

$$F^t_r(l) \leq F^t_s(l) \quad \text{for every } l \in \{1, \dots, \ell\} .$$

Observe that at least one of these inequalities is strict if $r \neq s$. Hence, t-degree stochastic dominance is an asymmetric binary relation. It is easy to see that it is also reflexive and transitive but not complete. Clearly, it respects the presumed linear ordering on O.

For $t = 1$, this relation means that r puts more probability on better alternatives than s. It is well known that this is equivalent to the expected utility under r being at least as large as the expected utility under s for every utility representation of σ. For second degree stochastic dominance, an analogous equivalence holds if we restrict to concave utility functions, or, more generally, utility functions with non-increasing differences between adjacent alternatives. Note that tth degree stochastic dominance implies $(t+1)$th degree stochastic dominance. In a relative sense, a similar relation holds between $(t+1)$th and tth degree stochastic dominance as between second and first degree stochastic dominance. Thus, higher degree stochastic dominance can be associated with increased risk aversion of decision makers, who put increasing weight on bad outcomes when evaluating probability distributions.

Fishburn (1976, 1980) characterizes stochastic dominance in terms of utility functions and in terms of moments of distributions. Below, we provide a characterization of stochastic dominance in terms of utility functions for the context of this paper.

Denote $F^t_r = (F^t_r(1), \dots, F^t_r(\ell))$ and let $A = [a_{ij}]$ be the $\ell \times \ell$-matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases} \quad \text{for all } i, j \in \{1, \dots, \ell\}.$$

Write $A^t = A \cdot A \cdots A$ (t times, $t \in \boldsymbol{N}$). The following lemma gives a convenient representation of F^t_r.

Lemma 2.1 $F^t_r = rA^t$ *for every* $t \in \boldsymbol{N}$.

Denoting the element in row i and column j of A^t by a^t_{ij}, we derive the following expression for this number.

Lemma 2.2 *Let* $t \in \boldsymbol{N}$. *Then*

$$a^t_{ij} = \begin{cases} \frac{(j-i+t-1)!}{(j-i)!(t-1)!} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases} .$$

The following lemma applies to t approaching infinity.

Lemma 2.3 *Let* $i, i', j \in \{1, \dots, \ell\}$ *with* $i < i' \leq j$. *Then* $a^t_{ij} \geq a^t_{i'j}$ *for every* $t \in \boldsymbol{N}$ *and* $\lim\limits_{t \to \infty} a^t_{ij} / a^t_{i'j} = \infty$.

For every $t \in \boldsymbol{N}$, let[2]

$$U^t := \left\{u \in \mathbb{R}^\ell \mid u = -A^t c \text{ for some } c \in \mathbb{R}^\ell, c > 0\right\}$$

and

$$\bar{U}^t := \left\{u \in \mathbb{R}^\ell \mid u = -A^t c \text{ for some } c \in \mathbb{R}^\ell, c \geq 0\right\}.$$

An element u of U^t can be interpreted as a utility function representing the linear ordering on O by the assignment $i \mapsto u_i$, since $u_1 < u_2 < \cdots < u_\ell$. The set U^t is particularly relevant in Sect. 5 when we characterize and prove the existence of t-equilibria.

Note that U^1 contains essentially any utility representation of σ. This is consistent with remarks made earlier. The set U^t is decreasing in t.

The set $\bar{U}^t$ is a convex and closed set (the topological closure of U^t). Using Lemma 2.1, it is straightforward to derive the following proposition.

Proposition 2.4 *For all probability distributions r and s on O, r t-degree stochastically dominates s if and only if $\sum_{l=1}^{\ell} r_l u_l \geq \sum_{l=1}^{\ell} s_l u_l$ for all $u \in \bar{U}^t$.*

This result adapts Fishburn (1976) to our context, and is a special case of the main theorem in Dubra et al. (2004). Note that Proposition 2.4 would remain true if we replace $\bar{U}^t$ by U^t. Proofs of Lemmas 2.1–2.3 can be found in Appendix A.

3 Two-person games and t-equilibria

Consider two players. Player 1 has pure strategy set $M = \{1, \ldots, m\}$. A (mixed) strategy for player 1 is a probability distribution over M. Denote the set of strategies for player 1 by Δ^M. A pure strategy i is identified with the mixed strategy $e^i \in \Delta^M$, where $e^i_k = 1$ if $k = i$ and $e^i_k = 0$ otherwise. Similarly, player 2 has pure strategy set $N = \{1, \ldots, n\}$ and (mixed) strategy set Δ^N. A pure strategy j is identified with the mixed strategy $e^j \in \Delta^N$. If player 1 plays pure strategy i and player 2 pure strategy j, then the alternative o_{ij} results. If player 1 plays $p \in \Delta^M$ and player 2 plays $q \in \Delta^N$, then o_{ij} results with probability $p_i q_j$. Let $O := \{o_{ij} \mid i \in M, j \in N\}$ and assume that players 1 and 2 have preference relations represented, respectively, by bijections $\sigma, \tau : M \times N \to \{1, \ldots, mn\}$. Thus, player 1 strictly prefers o_{ij} to $o_{i'j'}$ when $\sigma(i,j) > \sigma(i',j')$ (and similarly for player 2 and τ).

For $p \in \Delta^M$ and $q \in \Delta^N$ we denote by $pq\sigma$ the vector of probabilities with lth coordinate $pq\sigma_l = p_i q_j$ such that $\sigma(i,j) = l$, for all $l \in \{1, \ldots, mn\}$. We assume that the players evaluate strategies according to a stochastic dominance criterion. More precisely, let $t \in \boldsymbol{N}$ and fix a strategy $q \in \Delta^N$ for player 2. Then

[2] For vectors x and y, $x > y$ $[x \geq y]$ means $x_i > y_i$ $[x_i \geq y_i]$ for every coordinate i. Similarly for $x < y, x \leq y$.

a strategy $p \in \Delta^M$ of player 1 results in the weight vector $F^t_{pq\sigma}$, which depends on σ and assigns weight $F^t_{pq\sigma}(\sigma(i,j))$ to alternative o_{ij}.

We call p a *t-best reply against* q if there is no $p' \in \Delta^M$ such that $p'q \neq pq$ and $p'q$ tth degree stochastically dominates pq. The definition of a t-best reply q against p is analogous. A pair $(p,q) \in \Delta^M \times \Delta^N$ is a *t-equilibrium* if p is a t-best reply against q and vice versa. By E^t we denote the set of t-equilibria.[3]

4 An example

The example presented here is illustrative of the main results of this paper, namely (1) existence and characterization of t-equilibria; and (2) asymptotic behavior for t approaching infinity.

Let $m = n = 2$ and consider the game

$$\begin{bmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{bmatrix}$$

where the rows are the pure strategies of player 1, the columns those of player 2, and the preferences are given by $\sigma(1,2) = 1$, $\sigma(2,2) = 2$, $\sigma(2,1) = 3$, and $\sigma(1,1) = 4$ for player 1 and $\tau(1,2) = 1$, $\tau(1,1) = 2$, $\tau(2,1) = 3$, and $\tau(2,2) = 4$ for player 2.

We concentrate on player 1. The matrix A^t $(t \geq 1)$ can be computed using Lemma 2.2. This results in

$$A^t = \begin{bmatrix} 1 & t & \frac{1}{2}t(t+1) & \frac{1}{6}t(t+1)(t+2) \\ 0 & 1 & t & \frac{1}{2}t(t+1) \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider strategies $p = (p_1, p_2)$ and $q = (q_1, q_2)$ for players 1 and 2, respectively. In order to examine t-best replies of player 1 we compute (cf. Lemma 2.1)

$$F^t_{pq\sigma} = (p_1q_2, p_2q_2, p_2q_1, p_1q_1)A^t.$$

Dropping the part that does not depend on p and which therefore is not needed to compute t-best replies of player 1, this results in the vector

$$p_1\left(q_2, (t-1)q_2, \frac{1}{2}(t^2-t)q_2 - q_1, \frac{1}{6}(t^3-t)q_2 - q_1(t-1)\right). \qquad (1)$$

For $t = 1$, (1) reduces to $p_1(q_2, 0, -q_1, 0)$. Since player 1 wants to 'minimize' this vector, the t-best responses are $p_1 = 0$ if $q_1 = 0$, $p_1 = 1$ if $q_1 = 1$, and any

[3] We assume here that both players use the same order t of stochastic dominance, but our results easily generalize to the case where these orders are different (see Sect. 7).

$0 \leq p_1 \leq 1$ if $0 < q_1 < 1$. With a similar argument for player 2 (not reproduced here) we find the set E^1: it contains the two pure Nash equilibria of the game, resulting in o_{11} and in o_{22}, and *all* strategy combinations where no player plays a pure strategy. This is no surprise: in general, E^1 consists of all strategy combinations that are a Nash equilibrium for at least one choice of utility functions representing σ and τ. This is a consequence of the familiar characterization of first degree stochastic dominance using utility representations, mentioned in Sect. 2. See also Fishburn (1978), where the result is derived formally.

Next, consider $t = 2$, so best replies are second degree stochastically undominated. By substituting $t = 2$ in (1) it follows that for $q_1 = 1$ the 2-best reply is $p_1 = 1$, for $1 > q_1 > \frac{1}{2}$ any $0 \leq p_1 \leq 1$ is a 2-best reply, and for $q_1 \leq \frac{1}{2}$ the 2-best reply is $p_1 = 0$. Again after a similar argument for player 2 it follows that E^2 consists of the two pure Nash equilibria of the game plus the set

$$\left\{(p, q) \mid \frac{1}{2} < q_1 < 1,\ 0 < p_1 < \frac{1}{2}\right\}.$$

In these mixed strategy equilibria player 1 puts a larger weight on row 2. Row 2 is player 1's max–min pure strategy: he prefers the worst alternative in row 2, o_{22}, to the worst alternative in row 1, o_{12}. Thus, one might be tempted to conclude that a higher t leads to max–min play. This, however, is deceptive. As will turn out later, what is important is that row 2 is the max–min row from the point of view of player 2: player 2 prefers the worst alternative (for him) of row 2, o_{21}, to the worst alternative of row 1, o_{12}. (A similar consideration holds for the strategy of player 2.)

Observe also that the 2-best reply correspondences of the players are not upper semi-continuous (their graphs are not closed).

For $t > 2$, let $\hat{q}_1 = (t^3 - t)/(t^3 + 5t - 6)$. For $0 \leq q_1 \leq \hat{q}_1$ the t-best reply is $p_1 = 0$, for $\hat{q}_1 < q_1 < 1$ any p_1 is a t-best reply, and for $q_1 = 1$ the t-best reply is $p_1 = 1$. The t-equilibria are again the two pure Nash equilibria in the game together with the collection

$$\left\{(p, q) \mid \frac{t^3 - t}{t^3 + 5t - 6} < q_1 < 1,\ 0 < p_1 < 1 - \frac{t^3 - t}{t^3 + 5t - 6}\right\},$$

For $t \to \infty$ these mixed strategy t-equilibria converge to the pure strategy combination of row 2 and column 1.

5 Existence and characterization of *t*-equilibria

The existence of t-equilibria can be proved directly by applying a fixed point argument to the best-reply correspondences. This proof is not completely straightforward since the best reply correspondences do not have to be upper semi-continuous, see the example in the previous section, so that the argument has to be applied to a suitable sub-correspondence. Details can be found in an earlier version of the paper (see Perea et al. 2005).

Alternatively, t-equilibria can be characterized as Nash equilibria for suitably chosen utility functions. Existence then follows from the standard existence result for Nash equilibrium. This is the approach taken here. For $t = 1$, this has already been done in Fishburn (1978).

In the next lemma we consider the game as defined in Sect. 3. The bijection σ represents the preference relation of player 1. The set U^t was defined in Sect. 2.

Lemma 5.1 *Let $p \in \Delta^M$, $q \in \Delta^N$, and $t \geq 1$. Then p is a t-best reply against q if and only if there is a $u^t \in U^t$ such that*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j u^t_{\sigma(i,j)} \geq \sum_{i=1}^{m}\sum_{j=1}^{n} p'_i q_j u^t_{\sigma(i,j)}$$

for all $p' \in \Delta^M$.

Proof p is a t-best reply against q if and only if there is no $p' \in \Delta^M$ such that $F^t_{p'q\sigma} \leq F^t_{pq\sigma}$ with at least one coordinate strictly smaller. This is the case if and only if

$$\left\{x \in \mathbb{R}^{mn} \mid x \leq F^t_{pq\sigma}\right\} \cap \left\{x \in \mathbb{R}^{mn} \mid x = F^t_{p'q\sigma} \text{ for some } p' \in \Delta^M\right\} = \left\{F^t_{pq\sigma}\right\}.$$

By a standard separation argument it follows that the two sets on the left-hand side of this identity can be separated by a hyperplane through $F^t_{pq\sigma}$ with a nonnegative normal $c \in \mathbb{R}^{mn}$ such that $F^t_{pq\sigma} \cdot c \leq F^t_{p'q\sigma} \cdot c$ for all $p' \in \Delta^M$. Since the second set is a polytope, this normal can be chosen positive (see Shapley 1959, for a detailed argument). By Lemma 2.1 this is equivalent to $(pq\sigma)A^t c \leq (p'q\sigma)A^t c$ for all $p' \in \Delta^M$. The proof is complete by taking $u^t := -A^t c$. □

Lemma 5.1 can be formulated for player 2 in an analogous way. Then t-equilibria can be characterized as follows.

Corollary 5.2 *Let σ and τ represent the preferences of players 1 and 2, respectively. Let $t \geq 1$, $p^* \in \Delta^M$, and $q^* \in \Delta^N$. Then (p^*, q^*) is a t-equilibrium if and only if there are $u, v \in U^t$ such that (p^*, q^*) is a Nash equilibrium for the payoff functions $O \to \mathbb{R}$ defined by $o_{ij} \mapsto u_{\sigma(i,j)}$ and $o_{ij} \mapsto v_{\tau(i,j)}$ for players 1 and 2, respectively.*

Since Nash equilibria always exist, Corollary 5.2 implies existence of t-equilibria.

Corollary 5.3 *$E^t \neq \emptyset$ for every $t \geq 1$.*

6 Limit behavior of *t*-equilibria

The example in Sect. 4 suggests some kind of max–min behavior of the players in a t-equilibrium for t going to infinity. In this section we consider this in detail. The setting is the general game model as defined in Sect. 3. Unless stated otherwise, the number t is arbitrary but fixed. The preferences of the players are represented by the bijections σ for player 1 and τ for player 2.

For $p \in \Delta^M$, the *support* of p is the set

$$\mathrm{supp}(p) = \{i \in M \mid p_i > 0\}.$$

For $q \in \Delta^N$, $\mathrm{supp}(q)$ is defined in the same way. We start with an auxiliary result.

Lemma 6.1 *Let $p^t \in \Delta^M$ and let $q^t \in \Delta^N$. Let $p \in \Delta^M$ and $q \in \Delta^N$ with* $\mathrm{supp}(p) \subseteq \mathrm{supp}(p^t)$ *and* $\mathrm{supp}(q) \subseteq \mathrm{supp}(q^t)$. *Then*

(i) *if p^t is a t-best reply against q^t, then p is a t-best reply against q^t;*
(ii) *if q^t is a t-best reply against p^t, then q is a t-best reply against p^t.*

Proof Apply Lemma 5.1. □

Our main results are established in a series of three propositions. Theorem 6.6 summarizes these propositions.

Proposition 6.2 *Let $I \subseteq M$, $J \subseteq N$, and let $(p^t, q^t)_{t\in \boldsymbol{N}}$ be a sequence of pairs of mixed strategies such that $I = \mathrm{supp}(p^t)$ and $J = \mathrm{supp}(q^t)$ for all $t \in \boldsymbol{N}$.*

(i) *Let p^t be a t-best reply against q^t for every $t \in \boldsymbol{N}$. Then, for every $i \in M$, there is a $j \in J$ such that $\sigma(i,j) < \sigma(i',j)$ for all $i' \in I\backslash\{i\}$.*
(ii) *Let q^t be a t-best reply against p^t for every $t \in \boldsymbol{N}$. Then, for every $j \in N$, there is an $i \in I$ such that $\tau(i,j) < \tau(i,j')$ for all $j' \in J\backslash\{j\}$.*

Proof We only prove (i), the proof of (ii) is analogous. Suppose (i) were not true. Then there is an $\hat{\imath} \in M$ such that for every $j \in J$

$$I_j := \{i \in I \mid \sigma(i,j) < \sigma(\hat{\imath},j)\} \neq \emptyset.$$

For every $j \in J$, choose an $i_j \in I_j$. Let $\hat{I} := \{i_j \mid j \in J\}$, $s := |\hat{I}|$, [4] and define $\hat{p} \in \Delta^M$ by $\hat{p}_i = 1/s$ if $i \in \hat{I}$ and $\hat{p}_i = 0$ otherwise. We will show that, for t sufficiently large,

$$F^t_{e^{\hat{\imath}} q^t \sigma} \leq F^t_{\hat{p} q^t \sigma}. \tag{2}$$

Since, clearly, the two probability distributions $e^{\hat{\imath}} q^t \sigma$ and $\hat{p} q^t \sigma$ in (2) are different, this means that at least one of the inequalities must be strict if (2) holds. Since $\mathrm{supp}(\hat{p}) \subseteq I = \mathrm{supp}(p^t)$ for all t, (2) contradicts Lemma 6.1. This proves (i).

We are left to prove (2), hence we are left to prove

$$F^t_{e^{\hat{\imath}} q^t \sigma}(k) \leq F^t_{\hat{p} q^t \sigma}(k) \quad \text{for all } k = 1, \ldots, mn. \tag{3}$$

[4] $|\cdot|$ denotes the cardinality of a set.

Fix $k \in \{1, \ldots, mn\}$ and define $J_k := \{j \in J \mid \sigma(\hat{\imath}, j) \leq k\}$. Then

$$F^t_{e^{\hat{\imath}} q^t \sigma}(k) = \sum_{l=1}^{mn} (e^{\hat{\imath}} q^t \sigma)_l a^t_{lk} = \sum_{l=1}^{k} (e^{\hat{\imath}} q^t \sigma)_l a^t_{lk} = \sum_{j \in J_k} q^t_j a^t_{\sigma(\hat{\imath}, j), k}. \tag{4}$$

Here, the first equality follows from Lemma 2.1, the second equality from Lemma 2.2, and the last equality by Lemma 2.2 and the definition of J_k.

Now

$$\begin{aligned} F^t_{\hat{p} q^t \sigma}(k) &= \sum_{l=1}^{k} (\hat{p} q^t \sigma)_l a^t_{lk} \geq \sum_{j \in J_k} q^t_j \hat{p}_{i_j} a^t_{\sigma(i_j, j), k} \\ &= \sum_{j \in J_k} \frac{1}{s} q^t_j a^t_{\sigma(i_j, j), k} \geq \sum_{j \in J_k} \frac{1}{s} q^t_j a^t_{\sigma(\hat{\imath}, j) - 1, k}. \end{aligned} \tag{5}$$

The first equality follows again by Lemmas 2.1 and 2.2. The first inequality follows since some terms are left out. The second equality follows by definition of $\hat{p}$ since $i_j \in \hat{I}$ for every $j \in J$. The last inequality follows by the first statement in Lemma 2.3 since $\sigma(i_j, j) \leq \sigma(\hat{\imath}, j) - 1$ for every $j \in J$.

If $J_k = \emptyset$ then (3) follows immediately from (4) and (5). Otherwise, by the second statement in Lemma 2.3 there is a t sufficiently large such that for every $k = 1, 2, \ldots, mn$ and $j \in J_k$ we have

$$\frac{1}{s} q^t_j a^t_{\sigma(\hat{\imath}, j) - 1, k} \geq q^t_j a^t_{\sigma(\hat{\imath}, j), k}. \tag{6}$$

Then (3) follows from (4), (5), and (6). □

Proposition 6.2(i) states that for every row (pure strategy) i in I there must be a column (pure strategy) j in J such that the resulting outcome o_{ij} is the worst outcome for player 1 in that column restricted to the rows in I. In turn, this implies $|I| \leq |J|$. Similarly, Proposition 6.2(ii) implies $|J| \leq |I|$. So we have the following result.

Corollary 6.3 *Let $I \subseteq M$, $J \subseteq N$, and let $(p^t, q^t)_{t \in \boldsymbol{N}}$ be a sequence of t-equilibria such that $I = \operatorname{supp}(p^t)$ and $J = \operatorname{supp}(q^t)$ for all $t \in \boldsymbol{N}$. Then $|I| = |J|$.*

If $|I| = |J| = 1$ in Corollary 6.3, then the sequence of t-equilibria reduces to the constant Nash equilibrium in which player 1 picks the best element from the column played by player 2 and player 2 picks the best element from the row played by player 1.

The next result implies that in Corollary 6.3 the t-equilibria must converge to pure strategy combinations.

Proposition 6.4 *Let $I \subseteq M$, $J \subseteq N$, and let $(p^t, q^t)_{t \in \boldsymbol{N}}$ be a sequence of t-equilibria such that $I = \operatorname{supp}(p^t)$ and $J = \operatorname{supp}(q^t)$ for all $t \in \boldsymbol{N}$. Let $\hat{\imath} \in I$ and*

$\hat{\jmath} \in J$ be such that $\min\{\tau(i,j) \mid j \in J\} < \min\{\tau(\hat{\imath},j) \mid j \in J\}$ *for all* $i \in I\backslash\{\hat{\imath}\}$ *and* $\min\{\sigma(i,j) \mid i \in I\} < \min\{\sigma(i,\hat{\jmath}) \mid i \in I\}$ *for all* $j \in J\backslash\{\hat{\jmath}\}$. *Then* (*i*) $\lim_{t\to\infty} p^t_{\hat{\imath}} = 1$ *and* (*ii*) $\lim_{t\to\infty} q^t_{\hat{\jmath}} = 1$.

Proof We only prove (ii), the proof of (i) is analogous. By Proposition 6.2 and Corollary 6.3 we may renumber the strategies of the players such that

(a) $I = J = \{1,\ldots,s\}$ for some $s \geq 1$;
(b) $\sigma(j,j) < \sigma(i,j)$ for every $j \in J$ and $i \in I\backslash\{j\}$;
(c) $\sigma(1,1) < \sigma(2,2) < \cdots < \sigma(s,s)$.

Note that $\hat{\jmath} = s$. Let $s' \in \{1,\ldots,s-1\}$ arbitrary. To prove (ii), it is sufficient to prove that $\lim_{t\to\infty} q^t_{s'} = 0$. Suppose that this is not the case. Then we may assume that there is an $\alpha > 0$ such that $q^t_{s'} \geq \alpha$ for all t (otherwise there is a subsequence with this property and we can apply the following argument to this subsequence). Define the strategy $\hat{p} \in \Delta^M$ by $\hat{p}_i = 1/s'$ for $i = 1,\ldots,s'$ and $\hat{p}_i = 0$ otherwise. We will show that $\hat{p}q^t\sigma$ is t-dominated by $e^{s'+1}q^t\sigma$ for sufficiently large t, which contradicts Lemma 6.1 and therefore completes the proof. So we are left to show that for t sufficiently large

$$F^t_{e^{s'+1}q^t\sigma}(k) \leq F^t_{\hat{p}q^t\sigma}(k) \quad \text{for every } k = 1,\ldots,mn. \tag{7}$$

(Since the probability distributions $e^{s'+1}q^t\sigma$ and $\hat{p}q^t\sigma$ are clearly different, at least one of the inequalities in (7) must be strict.)

Let $k \in \{1,\ldots,mn\}$. By (b) and Lemma 2.3 we can choose t_1 such that for all $t \geq t_1$ we have

$$a^t_{\sigma(j,j),k} \geq s\, a^t_{\sigma(s'+1,j),k} \quad \text{for all } j = 1,\ldots,s'-1. \tag{8}$$

Also by (b) and Lemma 2.3 we can choose t_2 such that for all $t \geq t_2$ we have

$$\alpha a^t_{\sigma(s',s'),k} \geq 2s\, q^t_{s'} a^t_{\sigma(s'+1,s'),k}. \tag{9}$$

By (b), (c), and Lemma 2.3 we can choose t_3 such that for all $t \geq t_3$ we have

$$\alpha a^t_{\sigma(s',s'),k} \geq 2s \sum_{j\in J:\, j\geq s'+1} q^t_j a^t_{\sigma(s'+1,j),k}. \tag{10}$$

Then, for $t \geq \max\{t_1, t_2, t_3\}$, we have

$$\begin{aligned} F^t_{e^{s'+1}q^t\sigma}(k) &= \sum_{j\in J:\ \sigma(s'+1,j)\leq k} q^t_j a^t_{\sigma(s'+1,j),k} \\ &= \sum_{j\in J:\ j<s',\ \sigma(s'+1,j)\leq k} q^t_j a^t_{\sigma(s'+1,j),k} \\ &\quad + \sum_{j\in J:\ j\geq s',\ \sigma(s'+1,j)\leq k} q^t_j a^t_{\sigma(s'+1,j),k} \\ &\leq \sum_{j\in J:\ j<s'} \frac{1}{s} q^t_j a^t_{\sigma(j,j),k} + \frac{1}{2s}\alpha a^t_{\sigma(s',s'),k} + \frac{1}{2s}\alpha a^t_{\sigma(s',s'),k} \\ &\leq F^t_{\hat{p}q^t\sigma}(k), \end{aligned}$$

where the first inequality follows from (8), (9), and (10). This implies (7) and completes the proof of the lemma. □

Proposition 6.4 has the following converse.

Proposition 6.5 *Let $I \subseteq M$ and $J \subseteq N$ satisfy*

(i) *for every $i \in M$, there is a $j \in J$ such that $\sigma(i,j) < \sigma(i',j)$ for all $i' \in I\backslash\{i\}$;*
(ii) *for every $j \in N$, there is an $i \in I$ such that $\tau(i,j) < \tau(i,j')$ for all $j' \in J\backslash\{j\}$.*

Let $t \in \boldsymbol{N}$. Then there are $p^t \in \Delta^M$ and $q^t \in \Delta^N$ with $\mathrm{supp}(p^t) = I$, $\mathrm{supp}(q^t) = J$, *and $(p^t, q^t) \in E^t$.*

Proof Note that, as before, (i) and (ii) imply $|I| = |J|$. If $|I| = |J| = 1$, then there is a pure Nash equilibrium $(p^t, q^t) \in E^t$ with supports I and J. Assume now that $|I| = |J| \geq 2$. As in the proof of Proposition 6.4 we may renumber the pure strategies of the players such that

(a) $I = J = \{1, \ldots, s\}$ for some $s \geq 2$;
(b) $\sigma(j,j) < \sigma(i,j)$ for every $j \in J$ and $i \in I\backslash\{j\}$;
(c) $\sigma(1,1) < \sigma(2,2) < \cdots < \sigma(s,s)$.

Define $q^t \in \Delta^N$ with $\mathrm{supp}(q^t) = \{1, \ldots, s\}$ such that

$$q^t_j / q^t_{j-1} = m\,n\,a^t_{1,mn} \quad \text{for every } j = 2, \ldots, s. \tag{11}$$

We will show that every $p \in \Delta^M$ with $\mathrm{supp}(p) \subseteq I$ is a t-best reply against q^t. Since we can define p^t analogously and show that every $q \in \Delta^N$ with $\mathrm{supp}(q) \subseteq J$ is a t-best reply against p^t, the proof is complete.

So let $p \in \Delta^M$ with $\mathrm{supp}(p) \subseteq I$. Assume, contrary to what we wish to prove, that there is a $p' \in \Delta^M$ such that pq^t is t-dominated by $p'q^t$. We first argue that without loss of generality $\mathrm{supp}(p) \cap \mathrm{supp}(p') = \emptyset$. For, suppose that i is an element in this intersection, and let $\alpha := \min\{p_i, p'_i\}$. Define $\bar{p} := 1/(1-\alpha)\,(p - \alpha e^i)$ and $\bar{p}' := 1/(1-\alpha)\,(p' - \alpha e^i)$. Then $\bar{p}, \bar{p}' \in \Delta^M$, and $\bar{p}q^t$ is still t-dominated by $\bar{p}'q^t$ (since $\bar{p}q^t$ and $\bar{p}'q^t$ arise from pq^t and $p'q^t$, respectively, by first subtracting the same amount from the same coordinates and next rescaling, so that the inequalities of the stochastic dominance relation do not change), whereas $i \notin \operatorname{supp}(\bar{p}) \cap \operatorname{supp}(\bar{p}')$ and $\operatorname{supp}(\bar{p}) \subseteq I$.

So assume that $\operatorname{supp}(p) \cap \operatorname{supp}(p') = \emptyset$, and take $\hat{\imath} \in \operatorname{supp}(p')$ such that $p'_{\hat{\imath}} \geq 1/m$. If $\hat{\imath} \in I$ then let $\hat{\jmath} := \hat{\imath}$. Then, by condition (b), $\sigma(\hat{\imath}, \hat{\jmath}) < \sigma(i, \hat{\jmath})$ for all $i \in I \backslash \{\hat{\imath}\}$, hence for all $i \in \operatorname{supp}(p)$ in particular. If $\hat{\imath} \in M \backslash I$ then take $\hat{\jmath} \in J$ such that $\sigma(\hat{\imath}, \hat{\jmath}) < \sigma(i, \hat{\jmath})$ for all $i \in I$, hence for all $i \in \operatorname{supp}(p)$: this is possible by condition (i) in Proposition 6.2. Together with conditions (b) and (c) this implies

$$\sigma(\hat{\imath}, \hat{\jmath}) < \sigma(i, j) \text{ for all } i \in \operatorname{supp}(p) \text{ and } j \in \{\hat{\jmath}, \ldots, s\}. \tag{12}$$

Let $k := \sigma(\hat{\imath}, \hat{\jmath})$. Then

$$\begin{aligned} F^t_{pq^t\sigma}(k) &= \sum_{l=1}^{k} (pq^t\sigma)_l a^t_{lk} = \sum_{j=1}^{\hat{\jmath}-1} q^t_j \left(\sum_{i \in I:\, p_i > 0,\, \sigma(i,j) < k} p_i\, a^t_{\sigma(i,j),k} \right) \\ &\leq \sum_{j=1}^{\hat{\jmath}-1} q^t_j a^t_{1,mn} < n q^t_{\hat{\jmath}-1}\, a^t_{1,mn} = \frac{q^t_{\hat{\jmath}}}{m} \\ &\leq \frac{1}{m} q^t_{\hat{\jmath}} a^t_{\sigma(\hat{\imath},\hat{\jmath}),k} \leq F^t_{p'q^t\sigma}(k) \end{aligned} \tag{13}$$

where the first equality follows from Lemma 2.1; the second equality by (12); the first inequality by Lemma 2.2; the second (strict) inequality and the third equality by (11); the third inequality since $a^t_{\sigma(\hat{\imath},\hat{\jmath}),k} \geq 1$; and the final inequality by Lemma 2.1 and the choice of $p'_{\hat{\imath}} \geq 1/m$.

Since (13) contradicts the assumption that pq^t is t-dominated by $p'q^t$, the proof of the lemma is complete. □

Propositions 6.2, 6.4, and 6.5, and Corollary 6.3 can be summarized as follows.

Theorem 6.6 *If $I \subseteq M$, $J \subseteq N$, and if $(p^t, q^t)_{t \in \mathbb{N}}$ is a sequence of t-equilibria such that $I = \operatorname{supp}(p^t)$ and $J = \operatorname{supp}(q^t)$ for all $t \in \mathbb{N}$, then (i) and (ii) in Proposition 6.5 hold, $|I| = |J|$, and the sequence of t-equilibria converges to the pure strategy combination $(\hat{\imath}, \hat{\jmath})$, where $\hat{\imath}$ and $\hat{\jmath}$ are as in Proposition 6.4. Conversely, if $\emptyset \neq I \subseteq M$ and $\emptyset \neq J \subseteq N$ satisfy (i) and (ii) in Proposition 6.5, then a sequence of t-equilibria with supports I and J for players 1 and 2, respectively, exists.*

As announced earlier, the results in this section imply that, as t becomes large, the equilibrium behavior of the players converges to max–min play in a specific sense. Take any sequence of t-equilibria with (without loss of generality) constant supports I and J of the players' strategies. Then, in the limit, player 1 puts all weight on that strategy (row) in I in which the worst outcome for player 2 with respect to the strategies (columns) in J is maximal among all rows in I; and player 2 puts all weight on that column in J in which the worst outcome for player 1 with respect to the rows in I is maximal among all columns in J. Observe that this can be interpreted as altruistic behavior in equilibrium. The next examples provide further illustrations.

Example 6.7 Consider the following 3×3 game, in which the numbers express the ordinal preferences of the players:

$$\begin{bmatrix} 1,1 & 6,5 & 9,4 \\ 5,6 & 2,2 & 8,7 \\ 4,9 & 7,8 & 3,3 \end{bmatrix}.$$

In this game, $I := M = \{1,2,3\}$ and $J := N = \{1,2,3\}$ clearly satisfy (i) and (ii) in Proposition 6.5. Obviously, $\hat{\imath} = \hat{\jmath} = 3$. For every $t \in \boldsymbol{N}$ the t-equilibrium used in the proof of Proposition 6.5 is defined by

$$p^t = q^t = \left(\frac{1}{1+A+A^2}, \frac{A}{1+A+A^2}, \frac{A^2}{1+A+A^2}\right)$$

where $A = 9(7+t)!/8!(t-1)!$. In this equilibrium the weights on the first two rows (and columns) converge to 0 and p_1^t/p_2^t (and q_1^t/q_2^t) converges to zero as well. That is, the weight on the first row (column) goes to 0 much faster than the weight on the second row (column). The latter phenomenon is not a necessary one: in the present example, for instance, it is also possible to have a sequence of t-equilibria with equal weights on the first two rows (columns). It can be verified that taking $p_1^t = p_2^t = q_1^t = q_2^t =: \alpha^t$ such that $(1-2\alpha^t)/\alpha^t > a_{1,3}^t + a_{1,8}^t$ for every $t \in \boldsymbol{N}$ is again a t-equilibrium. Finally, $I = M$ and $J = N$ are the only subsets of pure strategies satisfying (i) and (ii) in Proposition 6.5, hence the only supports of t-equilibria. Hence, in the limit each player plays his third strategy, resulting in the 'payoffs' $(3,3)$.

Example 6.8 Consider the following 3×3 game:

$$\begin{bmatrix} 4,4 & 8,5 & 3,6 \\ 5,8 & 7,7 & 2,9 \\ 6,3 & 9,2 & 1,1 \end{bmatrix}.$$

The following combinations satisfy (i) and (ii) in Proposition 6.5:

(a) $I = \{3\}, J = \{1\}$, resulting in $(3,1)$ in the limit;
(b) $I = \{1\}, J = \{3\}$, resulting in $(1,3)$ in the limit;
(c) $I = \{1,3\}, J = \{2,3\}$, resulting in $(1,2)$ in the limit;
(d) $I = \{2,3\}, J = \{1,3\}$, resulting in $(2,1)$ in the limit;
(e) $I = M, J = N$, resulting in $(2,2)$ in the limit.

This means that the 'payoff pairs' that can arise as limits of t-equilibria are $(6,3)$, $(3,6)$, $(8,5)$, $(5,8)$, and $(7,7)$.

7 Concluding remarks

7.1 Extensions

There are some obvious possible extensions of the results in this paper.

First, the definitions and results in Sects. 3 and 5 can be generalized quite easily to games with more than two players. The same, however, is far from obvious for the asymptotic results of Sect. 6. For this reason, we chose to present the entire paper only for two-person games.

Second, the implicit assumption in the concept of t-equilibrium that the value of t is the same for both players can be omitted. All results would also hold for (t_1, t_2)-equilibria, where $(p, q) \in \Delta^M \times \Delta^N$ is called a (t_1, t_2)-equilibrium if p is a t_1-best reply against q and q is a t_2-best reply against p.

Third, the degree of stochastic dominance t can be varied continuously instead of in discrete steps. Again, all results would continue to hold for this extension.

Fourth, the assumption of the players having a linear ordering (no indifferences) on the certain alternatives is not an essential one, but it makes the asymptotic results of Sect. 6 much cleaner.

Fifth, modelling incomplete preferences by stochastic dominance is an in our view justifiable but also quite specific choice. Dubra et al. (2004) characterize incomplete preferences satisfying the von Neumann–Morgenstern axioms by identifying characterizing classes of utility functions that play the same role as the classes $\bar{U}^t$ in our paper, cf. the end of Sect. 2. This points at a general approach that can be used to extend some of the results of the paper.

Sixth (and related to the second and third points above), we conjecture that the precise way in which increasing aversion to bad outcomes is modelled in the present paper – by t-degree stochastic dominance for increasing t – is not essential for the limit result. This is left for future research.

7.2 Related literature

There is quite some literature on noncooperative games with only ordinal preferences: many economic games (for instance, Cournot or Bertrand oligopoly games) belong to this category, but also games used for implementing social choice correspondences, to name just a few examples. However, apart from Fishburn (1978) the only references to ordinal games with mixed strategies that we know of are Börgers (1993) and Rothe (1995). Börgers (1993) proposes a definition of rationalizability in which only ordinal preferences over outcomes are assumed to be common knowledge. Rothe (1995) considers equilibrium selection in 2×2-games under the first-degree stochastic dominance criterion.

A Remaining proofs

Proof of Lemma 2.2. The proof is by induction on t . For $t = 1$ the formula holds by definition of A. Let the formula be true for all $k < t$, where $t \geq 2$. Then for all $i, j \in \{1, \ldots, \ell\}$,

$$a_{ij}^t = \sum_{l=1}^{\ell} a_{il}^{t-1} a_{lj} = \sum_{l=i}^{\ell} a_{il}^{t-1} a_{lj}, \tag{14}$$

where the second equality holds by induction. If $i > j$ then every $a_{lj} = 0$ in the RHS of (14) since $l > j$. If $i \leq j$ then (14) implies by induction

$$a_{ij}^t = \sum_{l=i}^{j} a_{il}^{t-1} = \sum_{l=i}^{j} \frac{(t-2+l-i)!}{(l-i)!(t-2)!}. \tag{15}$$

We are done if we can prove

$$\sum_{l=i}^{j} \frac{(t-2+l-i)!}{(l-i)!(t-2)!} = \frac{(t-1+j-i)!}{(j-i)!(t-1)!}.$$

We show this again by induction. For $j = i$ it is immediate. Let the equality hold for $i, \ldots, j-1$, then

$$\begin{aligned}
\sum_{l=i}^{j} \frac{(t-2+l-i)!}{(l-i)!(t-2)!} &= \frac{(t-1+j-1-i)!}{(j-1-i)!(t-1)!} + \frac{(t-2+j-i)!}{(j-i)!(t-2)!} \\
&= \frac{(t-1+j-i-1)!(j-i) + (t-2+j-i)!(t-1)}{(j-i)!(t-1)!} \\
&= \frac{(t-1+j-i)!}{(j-i)!(t-1)!}.
\end{aligned}$$

This completes the proof. □

Proof of Lemma 2.1. The proof is by induction on t . For $t = 1$ the identity in the lemma holds by definition. Assume it holds for every $k < t$ ($t \geq 2$). Let $l \in \{1, \ldots, \ell\}$. Then

$$F_r^t = F_r^{t-1} A = rA^{t-1} A = rA^t,$$

where the second equality follows by induction. □

Proof of Lemma 2.3. By Lemma 2.2,

$$\begin{aligned}\frac{a^t_{ij}}{a^t_{i'j}} &= \frac{(t-1+j-i)!}{(j-i)!(t-1)!} \cdot \frac{(j-i')!(t-1)!}{(t-1+j-i')!} \\ &= \frac{(j-i+1)\cdot(j-i+2)\cdots(j-i+t-1)}{(j-i'+1)\cdot(j-i'+2)\cdots(j-i'+t-1)} \\ &= \frac{(j-i'+t)\cdot(j-i'+t+1)\cdots(j-i+t-1)}{(j-i'+1)\cdot(j-i'+2)\cdots(j-i)},\end{aligned}$$

hence $a^t_{ij} \geq a^t_{i'j}$ and $\lim_{t\to\infty} a^t_{ij}/a^t_{i'j} = \infty$. □

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