Int J Game Theory (2014) 43:881–902
DOI 10.1007/s00182-013-0409-3

Utility proportional beliefs

Christian W. Bach · Andrés Perea

Accepted: 29 December 2013 / Published online: 11 January 2014

Abstract In game theory, basic solution concepts often conflict with experimental findings or intuitive reasoning. This fact is possibly due to the requirement that zero probability is assigned to irrational choices in these concepts. Here, we introduce the epistemic notion of common belief in utility proportional beliefs which also attributes positive probability to irrational choices, restricted however by the natural postulate that the probabilities should be proportional to the utilities the respective choices generate. Besides, we propose a procedural characterization of our epistemic concept. With regards to experimental findings common belief in utility proportional beliefs fares well in explaining observed behavior.

Keywords Algorithms · Epistemic game theory · Interactive epistemology · Solution concepts · Traveler's dilemma · Utility proportional beliefs

1 Introduction

Interactive epistemology, also called epistemic game theory when applied to games, provides a general framework in which epistemic notions such as knowledge and belief can be modeled for situations involving multiple agents. This rather recent discipline has been initiated by Harsanyi (1967–1968) as well as Aumann (1976) and first been adopted in the context of games by Aumann (1987), Brandenburger and Dekel (1987) as well as Tan and Werlang (1988). A comprehensive and in-depth introduction to

C. W. Bach · A. Perea (✉)
Department of Quantitative Economics, School of Business and Economics,
Maastricht University, 6200 MD Maastricht, The Netherlands
e-mail: a.perea@maastrichtuniversity.nl

C. W. Bach
e-mail: c.bach@maastrichtuniversity.nl

epistemic game theory is provided by Perea (2012). An epistemic approach to game theory analyzes the relation between knowledge, belief, and choice of rational game-playing agents. While classical game theory is based on the two basic primitives—game form and choice—epistemic game theory adds an epistemic framework as a third elementary component such that knowledge and beliefs can be explicitly modelled in games.

Intuitively, an epistemic model of a game can be interpreted as representing the players' reasoning. Indeed, before making a decision in a game, a player reasons about the game and his opponents, given his knowledge and beliefs. Precisely these epistemic mental states on which a player bases his decisions and which characterize his reasoning are described in an epistemically enriched game-theoretic framework.

A central idea in epistemic game theory is common belief in rationality, first explicitly formalized in an epistemic model for games by Tan and Werlang (1988). From an algorithmic perspective it corresponds to the concept of rationalizability, which is due to Bernheim (1984) and Pearce (1984). Intuitively, common belief in rationality assumes a player to believe his opponents to choose rationally, to believe his opponents to believe their opponents to choose rationally, etc. However, this basic concept gives counterintuitive as well as experimentally invalidated predictions in some games that have received a lot of attention. Possibly, the requirement that only rational choices are considered and zero probability is assigned to any irrational choice is too strong and does not reflect how real world agents reason.

Here, we introduce the epistemic concept of utility proportional beliefs, according to which a player assigns positive probability also to opponents' irrational choices, while at the same time for every opponent, differences in probability must be proportional to differences in utility. In particular, better opponents' choices receive higher probability than inferior choices. Intuitively, probabilities now confer the intrinsic meaning of how good the respective player deems his opponents' choices. The concept of common belief in utility proportional beliefs formalizes the idea that players do not only entertain utility proportional beliefs themselves, but also believe their opponents to do so, believe their opponents to believe their opponents to do so, etc. Philosophically, our concept can be seen as a way of formalizing cautious reasoning, since no choice is excluded from consideration. Rational choice under common belief in utility proportional beliefs fares well with regards to intuition and to explaining experimental findings in games of interest, where classical concepts such as rationalizability perform weakly.

As an illustration consider a simplified version of Basu's (1994) traveler's dilemma. Two persons have traveled with identical items on a plane, however when they arrive their items are damaged and they want to claim compensation by the airline. Both travelers are asked to simultaneously submit a discrete price between 1 and 10. The person with the lower price is then rewarded this value plus a bonus of 2, while the person with the higher price receives the lower price minus a penalty of 2. If the travelers submit the same price, then they both are compensated accordingly. Reasoning in line with common belief in rationality requires a traveler to rationally choose the lowest price 1. Intuitively, the highest price can never be optimal for neither traveler, and iteratively inferring that every respective lower price can then not be optimal either only leaves the very lowest price as rational choice. However, this conclusion conflicts with experimental findings as well as with intuition, possibly since people typically do not do all iterations or do not assign zero probability to opponents' irrational choices. Indeed, our concept of common belief in utility proportional beliefs leads to the choice of 6. Intuitively, if all prices receive a substantial positive probability, the very low prices perform quite badly and hence do so too with common belief in utility proportional beliefs.

In general, the classical concept of common belief in rationality is an idealized notion. There are two direct ways of modifying this concept to draw nearer to the reasoning of real-world agents. Firstly, the assumption of the players performing an infinite number of reasoning steps can be relaxed. Secondly, the hypothesis of belief in rationality may be weakened. Indeed, to fully understand the consequences of modifying one of the two basic building blocks of common belief in rationaliy, it is important to first study each of these separately. In fact, the two approaches can be viewed as orthogonal. Here, we follow the second route and replace belief in rationality by the more realistic concept of utility proportional beliefs. Formally, we still require players to do infinitely many reasoning steps. However, it turns out that typically a few reasoning steps already suffice to approximate the final beliefs selected by common belief in utility proportional beliefs.

The basic idea underlying utility proportional beliefs also appears in Rosenthal's (1989) t-solution, where players are required to assign probabilities to their own choices such that the probability differences are proportional to the utility differences using a proportionality factor t. In contrast to our model, Rosenthal uses the same proportionality factor t across all players; assumes that players consciously randomize, i.e. pick probability distributions over their choice sets; and builds in an equilibrium condition implying that players entertain correct beliefs about their opponents' randomized choices.

The intuition that better choices receive higher probabilities also occurs in McKelvey and Palfrey's (1995) quantal response equilibrium, where the utilities are subject to random errors. In contrast to our model, McKelvey and Palfrey do not assume probabilities to be proportional to utilities; require players to hold correct beliefs about the opponents' probabilities; and suppose agents to always choose optimally with respect to their beliefs, whereas their utilities are randomly perturbed.

The scheme of cautious reasoning—that is, no choice is completely discarded from consideration—is also present in Schuhmacher's (1999) and Asheim's (2001) concept of proper rationalizability, which assumes better choices to be infinitely more likely than worse choices. However, in our model every choice receives a substantial, non-infinitesimal positive probability, which is proportional to the utility the respective choice generates.

We proceed as follows. In Sect. 2, the concept of common belief in utility proportional beliefs is formalized in a type-based epistemic model for games. Also, a convenient way of stating utility proportional beliefs by means of an explicit formula is presented. Rational choice under common belief in utility proportional beliefs is defined as the decision-relevant notion for game-playing agents. Section 3 introduces the procedure of iterated elimination of utility-disproportional-beliefs, which recursively restricts the players' possible beliefs about the opponents' choices. Section 4 then establishes that iterated elimination of utility-disproportional-beliefs provides a procedural characterization of common belief in utility proportional beliefs, and can thus be used as a practical tool to compute the beliefs a player can hold when reasoning in line with common belief in utility proportional beliefs. In Sect. 5 it is shown that the procedure—surprisingly—yields unique beliefs for every player in two player games. By means of an example, non-uniqueness of beliefs is established for games with more than two players. Section 6 illustrates how well our concept fares with regards to intuition as well as experimental findings in some games that have received a lot of attention. Section 7 discusses utility proportional beliefs from a conceptual point of view and compares it to some related literature. Finally, Sect. 8 offers some concluding remarks and indicates possible directions for future research.

2 Common belief in utility proportional beliefs

In order to model reasoning in line with utility proportional beliefs, infinite belief hierarchies need to be considered. Here, we restrict attention to static games and follow the type-based approach to epistemic game theory, which represents belief hierarchies as types. More precisely, a set of types is assigned to every player, where each player's type induces a belief on the opponents' choices and types. Then, the whole infinite belief hierarchy can be derived from a given type. Note that the notion of type was originally introduced by Harsanyi (1967–1968) to model incomplete information, but can actually be more generally used for any interactive uncertainty. Indeed, the context we consider is the uncertainty about choice in finite normal form games.

Notationally, a finite normal form game is represented by the tuple

$$\Gamma = (I, (C_i)_{i \in I}, (U_i)_{i \in I}),$$

where I denotes a finite set of players, C_i denotes player i's finite choice set, and $U_i : \times_{j \in I} C_j \to \mathbb{R}$ denotes player i's utility function.

The notion of an epistemic model constitutes the framework in which various epistemic mental states of players can be described.

Definition 1 An *epistemic model* of a game Γ is a tuple $\mathcal{M}^{\Gamma} = ((T_i)_{i \in I}, (b_i)_{i \in I})$, where

- T_i is a set of types for player $i \in I$,
- $b_i : T_i \to \Delta(C_{-i} \times T_{-i})$ assigns to every type $t_i \in T_i$ a probability measure with finite support on the set of opponents' choice-type combinations.

Here, $C_{-i} := \times_{j \in I \setminus \{i\}} C_j$ and $T_{-i} := \times_{j \in I \setminus \{i\}} T_j$ denote the set of opponents' choice and type combinations, respectively. Besides, by imposing the finite support condition, we focus on types that only assign positive probability to *finitely* many types for each of their respective opponents. Hence, no topological assumptions concerning the type spaces are needed. Note that although according to Definition 1 the probability measure $b_i(t_i)$ represents type t_i's belief function on the set of opponents' choice-type pairs, for sake of notational convenience we often also use $b_i(t_i)$ to denote any projected belief function for type t_i.[1] It should always be clear from the context which belief function $b_i(t_i)$ refers to.

Besides, in this paper we follow the one-player perspective approach to epistemic game theory advocated by Perea (2007a, b, 2012). Accordingly, all epistemic concepts including iterated ones are understood and defined as mental states inside the mind of a single person. Indeed, a one-player approach seems natural, since reasoning is formally represented by epistemic concepts and any reasoning process prior to choice takes place entirely *within* the reasoner's mind.

Some further notions and notation are now introduced. For that purpose consider a game Γ, an epistemic model $\mathcal{M}^{\Gamma}$ of it, and fix two players $i, j \in I$ such that $i \neq j$. A type $t_i \in T_i$ of i is said to *deem possible* some type $t_j \in T_j$ of his opponent j, if $b_i(t_i)$ assigns positive probability to an opponents' choice-type combination that includes t_j. By $T_j(t_i)$ we then denote the set of types of player j deemed possible by t_i. Furthermore, given a type $t_i \in T_i$ of player i, and given an opponent's type $t_j \in T_j(t_i)$,

$$(b_i(t_i))(c_j \mid t_j) := \frac{(b_i(t_i))(c_j, t_j)}{(b_i(t_i))(t_j)}$$

is type t_i's *conditional belief* that player j chooses c_j given his belief that j is of type t_j. Note that the conditional belief $(b_i(t_i))(c_j \mid t_j)$ is only defined for types of j deemed possible by t_i.

Moreover, a choice combination for player i's opponents is denoted by $c_{-i} \in \times_{j \in I \setminus \{i\}} C_j$. For each of his choices $c_i \in C_i$ type t_i's expected utility given his belief on his opponents' choice combinations is given by

$$u_i(c_i, t_i) = \Sigma_{c_{-i}} (b_i(t_i))(c_{-i}) U_i(c_i, c_{-i}).$$

Besides, let $C := \times_{i \in I} C_i$ be the set of all choice combinations in the game. Then, $\overline{u}_i := max_{c \in C} u_i(c)$ and $\underline{u}_i := min_{c \in C} u_i(c)$ denote the best and worst possible utilities player i can obtain in the game, respectively. Farther, type t_i's average expected utility is denoted by

$$u_i^{average}(t_i) := \frac{1}{|C_i|} \Sigma_{c_i \in C_i} u_i(c_i, t_i).$$

The idea that a player entertains beliefs on his opponents' choices proportional to the respective utilities these choices yield for the opponents can be formalized within the framework of an epistemic model for normal form games.

Definition 2 Let $i \in I$ be some player, and $\lambda_i = (\lambda_{ij})_{j \in I \setminus \{i\}} \in \mathbb{R}^{|I \setminus \{i\}|}$ such that $\lambda_{ij} \geq 0$ for all $j \in I \setminus \{i\}$. A type $t_i \in T_i$ of player i expresses λ_i*-utility-proportional-beliefs*, if

[1] A type's belief function projected on some opponent's type space or projected on the set of opponents' choice combinations are examples for projected belief functions.

$$(b_i(t_i))(c_j \mid t_j) - (b_i(t_i))(c'_j \mid t_j) = \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}(u_j(c_j, t_j) - u_j(c'_j, t_j)) \qquad (UPB)$$

for all $t_j \in T_j(t_i)$, for all $c_j, c'_j, \in C_j$, for all $j \in I \backslash \{i\}$.

Accordingly, the difference between player i's conditional belief probabilities attached to two choices of any of his opponents is proportional to the difference between the respective utilities the opponent derives from them, with proportionality factor $\frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}$ for every opponent $j \in I \backslash \{j\}$.

Note that λ_i-utility-proportional-beliefs may not exist if any of the λ_{ij}'s is too large, as this may force some of the conditional probabilities $(b_i(t_i))(c_j \mid t_j)$ to become negative and hence would no longer qualify as probabilities. However, in Sect. 3 an upper bound on λ_{ij} will be defined which guarantees that λ_i-utility-proportional-beliefs always exist in the sense of being well-defined probability measures.

Intuitively, λ_{ij} measures the sensitivity of the beliefs to differences in utility. If it is too large, some of the beliefs will become negative or greater than 1 in order to satisfy equation (UPB), thus violating the property of being probabilities. Hence, there exists an upper bound for every λ_{ij}, which furnishes maximally dispersed belief probabilities while at the same time still complying with equation (UPB) for every opponent's type. Farther, the minimal value λ_{ij} can assume is zero, which then implies the conditional beliefs about the respective opponent's choice to be uniformly distributed. In other words, if $\lambda_{ij} = 0$, then utility differences are not at all reflected in the beliefs, as all choices are being assigned the same probability. Besides, note that in the context of modeling reasoning in line with utility proportional beliefs, choosing λ_{ij} as large as possible seems plausible, as the idea of utility proportional beliefs then unfolds its maximal possible effect.

Moreover, λ_i-utility-proportional-beliefs are invariant with respect to affine transformations of any player's utility function. Indeed, suppose $a \in \mathbb{R}$, $b > 0$ and $j \in I \backslash \{i\}$ such that $\hat{u}_j(c_j, t_j) = a + bu_j(c_j, t_j)$ for all $c_j \in C_j$ and for all $t_j \in T_j$. Assume that t_i expresses λ_i-utility-proportional-beliefs with respect to $\hat{u}_j$. Then, observe that

$$\begin{aligned}
&(b_i(t_i))(c_j \mid t_j) - (b_i(t_i))(c'_j \mid t_j) \\
&= \frac{\lambda_{ij}}{\overline{\hat{u}}_j - \underline{\hat{u}}_j}(\hat{u}_j(c_j, t_j) - \hat{u}_j(c'_j, t_j)) \\
&= \frac{\lambda_{ij}}{(a + b\overline{u}_j) - (a + b\underline{u}_j)}\Big(\Big(a + bu_j(c_j, t_j)\Big) - \Big(a + bu_j(c'_j, t_j)\Big)\Big) \\
&= \frac{\lambda_{ij}}{b(\overline{u}_j - \underline{u}_j)} b\Big(u_j(c_j, t_j) - u_j(c'_j, t_j)\Big) \\
&= \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}\Big(u_j(c_j, t_j) - u_j(c'_j, t_j)\Big)
\end{aligned}$$

for all $c_j, c'_j \in C_j$ and for all $t_j \in T_j$. The invariance of utility proportional beliefs with regards to affine transformations of the players' utilities strengthens the concept, since it does not depend on the particular cardinal payoff structure but only on the underlying ordinal preferences.

Utility proportional beliefs as formalized in Definition 2 can be expressed by means of an explicit formula for a given opponent's choice conditional of him being of a given type. This convenient alternative way of stating utility proportional beliefs relates the conditional belief in a specific opponent's choice to the utility this choice generates for the respective opponent.

Lemma 1 *Let $i \in I$ be some player, and $\lambda_i = (\lambda_{ij})_{j \in I \setminus \{i\}} \in \mathbb{R}^{|I \setminus \{i\}|}$. A type $t_i \in T_i$ of player i expresses λ_i-utility-proportional-beliefs if and only if*

$$(b_i(t_i))(c_j \mid t_j) = \frac{1}{\mid C_j \mid} + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}(u_j(c_j, t_j) - u_j^{average}(t_j)),$$

for all $t_j \in T_j(t_i)$, for all $c_j \in C_j$, for all $j \in I \setminus \{i\}$.

Proof Let $j \in I \setminus \{i\}$ be some opponent of player i, $t_j \in T_j(t_i)$ be some type of j deemed possible by i and $c_j^* \in C_j$ be some choice of j. Note that

$$\begin{aligned} 1 &= \Sigma_{c_j \in C_j}(b_i(t_i))(c_j \mid t_j) \\ &= \Sigma_{c_j \in C_j}((b_i(t_i))(c_j^* \mid t_j) + (b_i(t_i))(c_j \mid t_j) - (b_i(t_i))(c_j^* \mid t_j)) \\ &= \Sigma_{c_j \in C_j}((b_i(t_i))(c_j^* \mid t_j) + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}(u_j(c_j, t_j) - u_j(c_j^*, t_j))) \\ &= \big(|C_j|(b_i(t_i)\big)(c_j^* \mid t_j) + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}\Sigma_{c_j \in C_j}(u_j(c_j, t_j) - u_j(c_j^*, t_j))) \\ &= |C_j|(b_i(t_i))(c_j^* \mid t_j) + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}\big(|C_j| u_j^{average}(t_j) - |C_j| u_j(c_j^*, t_j)\big), \end{aligned}$$

which is equivalent to

$$(b_i(t_i))(c_j^* \mid t_j) = \frac{1}{|C_j|} + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}\big(u_j(c_j^*, t_j) - u_j^{average}(t_j)\big).$$

□

Intuitively, the formula provided by Lemma 1 assigns to every opponents' type the uniform distribution on the respective opponents' choice set plus or minus an adjustment for each choice depending on its goodness relative to the average utility.

Reasoning in line with utility proportional beliefs requires a player not only to entertain utility proportional beliefs himself, but also to believe his opponents to do so, to believe his opponents to believe their opponents to do so, etc. Within an epistemic framework this reasoning assumption can be formally expressed by common belief in utility proportional beliefs.

Definition 3 Let $i \in I$ be some player, $t_i \in T_i$ be some type of player i, and $\lambda = (\lambda_i)_{i \in I} \in \times_{i \in I} \mathbb{R}^{|I \setminus \{i\}|}$.

- Type t_i expresses *1-fold belief in λ-utility-proportional-beliefs*, if t_i expresses λ_i-utility-proportional-beliefs.
- Type t_i expresses *k-fold belief in λ-utility-proportional-beliefs*, if $(b_i(t_i))$ only deems possible types $t_j \in T_j$ for all $j \in I \setminus \{i\}$ such that t_j expresses $k-1$-fold belief in λ-utility-proportional-beliefs, for all $k > 1$.
- Type t_i expresses *common belief in λ-utility-proportional-beliefs*, if t_i expresses k-fold belief in λ-utility-proportional-beliefs for all $k \geq 1$.

Intuitively, a player i expressing common belief in λ-utility-proportional-beliefs holds λ_{ij}-utility-proportional-beliefs on opponent j's choices, he believes opponent j to entertain $\lambda_{jj'}$-utility-proportional-beliefs on opponent j''s choices, etc. In other words, a belief hierarchy of player i satisfying λ-utility-proportional-beliefs exhibits no level at which it is not iteratively believed that every opponent j holds λ_j-utility-proportional-beliefs.

The choices a player can reasonably make under common belief in utility proportional beliefs are those that are rational under his respectively restricted beliefs on the opponents' choices.

Definition 4 Let $i \in I$ be some player, and $\lambda = (\lambda_i)_{i \in I} \in \times_{i \in I} \mathbb{R}^{|I \setminus \{i\}|}$. A choice $c_i \in C_i$ of player i is *rational under common belief in λ-utility-proportional-beliefs*, if there exists an epistemic model $\mathcal{M}^\Gamma$ and some type $t_i \in T_i$ of player i such that c_i is optimal given $(b_i(t_i))$ and t_i expresses common belief in λ-utility-proportional-beliefs.

3 Recursive procedure

A recursive procedure is introduced that iteratively deletes beliefs and that—as will be shown later in Sect. 4—yields precisely those beliefs that are possible under common belief in λ-utility proportional beliefs.

Before we formally define our recursive procedure some more notation needs to be fixed. Let $P_i^0 := \Delta(C_{-i})$ denote the set of i's beliefs about his opponents' choice combinations. Besides, given $p_i \in P_i^0$ we define $u_i^{average}(p_i) := \frac{1}{|C_i|} \Sigma_{c_i \in C_i} u_i(c_i, p_i)$. Moreover, for every player i, each of his opponents $j \in I \setminus \{i\}$, every $p_j \in P_j^0$, and every $c_j \in C_j$, we define the number

$$(p_{ij}^*(p_j))(c_j) := \frac{1}{|C_j|} + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} \big(u_j(c_j, p_j) - u_j^{average}(p_j)\big).$$

Let λ_{ij}^{max} be the largest λ_{ij} such that $(p_{ij}^*(p_j))(c_j) \geq 0$ for all $p_j \in P_j^0$ and for all $c_j \in C_j$. Note that by construction of $(p_{ij}^*(p_j))(c_j)$, it holds that

$$\Sigma_{c_j \in C_j} (p_{ij}^*(p_j))(c_j) = 1.$$

Hence, if $\lambda_{ij} \leq \lambda_{ij}^{max}$, then the vector $p_{ij}^*(p_j) := ((p_{ij}^*(p_j))(c_j))_{c_j \in C_j}$ constitutes a well-defined probability measure on C_j. In that case, $p_{ij}^* : P_j^0 \to \Delta(C_j)$ becomes a function mapping beliefs p_j of player j on his opponents' choice combinations to beliefs $p^*_{ij}(p_j)$ on j's choice.

The recursive procedure, which we call *iterated elimination of utility-disproportional-beliefs*, can now be formally stated.

Definition 5 For all $i \in I$ and for all $k \geq 0$ the set P_i^k of i's beliefs about his opponents' choice combinations is inductively defined as follows:

$$P_i^0 := \Delta(C_{-i}),$$
$$P_i^k := \{p_i \in \Delta(C_{-i}) : marg_{C_j} p_i \in p^*_{ij}(P_j^{k-1}) \text{ for all } j \in I \setminus \{i\}\}.$$

The set of beliefs $P_i^\infty = \bigcap_{k \geq 0} P_i^k$ contains the beliefs that survive *iterated elimination of utility-disproportional beliefs*.

Here, $p^*_{ij}(P_j^{k-1})$ denotes the set $\{p^*_{ij}(p_j) : p_j \in P_j^{k-1}\}$. Intuitively, $p^*_{ij}(p_j)$ is the utility-proportional-belief on j's choice generated by p_j. The recursive procedure then iteratively deletes beliefs p_i for which the marginal on j's choices cannot be obtained by the function p^*_{ij} for some opponent j. In other words, beliefs are repeatedly eliminated which are not utility proportional with respect to beliefs from the preceding set of beliefs in the recursive procedure.

Typically, our recursive procedure does not terminate within a finite number of steps. Indeed, it will be shown in Sect. 5 for the case of two players that the recursive procedure converges to a singleton belief set for both players, whereas the sets P_i^k normally contain infinitely many beliefs for all rounds $k \geq 1$.

Some important properties of our recursive procedure are now presented, which will be used later.

Lemma 2 *For all $i \in I$ and for all $k \geq 0$, the set P_i^k is non-empty, convex and compact.*

Proof We proceed by induction on k. Note that for $k = 0$ the statement holds as $P_i^0 = \Delta(C_{-i})$. Let $k \geq 1$ and assume that P_i^{k-1} is non-empty, convex and compact for all players $i \in I$. Consider some player $i \in I$ and recall that

$$P_i^k := \{p_i \in \Delta(C_{-i}) : marg_{C_j} p_i \in p^*_{ij}(P_j^{k-1}) \text{ for all } j \in I \setminus \{i\}\}.$$

First of all, we show that P_i^k is non-empty. As P_j^{k-1} is non-empty by the inductive assumption, $p^*_{ij}(P_j^{k-1})$ is non-empty too for every opponent j of i. Thus, for every opponent j we can take some probability measure $p_{ij} \in p^*_{ij}(P_j^{k-1})$ and define $p_i \in \Delta(C_{-i})$ as the product measure of these p_{ij}'s. Then, $p_i \in P_i^k$ and therefore P_i^k is non-empty.

Secondly, it is argued that P_i^k is convex. Let $p_i, p_i' \in P_i^k$ and $\alpha \in [0, 1]$. We now prove that $p_i'' := \alpha p_i + (1 - \alpha) p_i' \in P_i^k$. Observe that for every opponent j of i it holds that $marg_{C_j} p_i'' = \alpha marg_{C_j} p_i + (1 - \alpha) marg_{C_j} p_i'$. By definition of P_i^k, the two marginal probability distributions $marg_{C_j} p_i$ and $marg_{C_j} p_i'$ are both elements of

$p^*_{ij}(P^{k-1}_j)$. Note that the set $p^*_{ij}(P^{k-1}_j)$ is a convex set by virtue of being the affine image of the convex set P^{k-1}_j. Hence, $marg_{C_j} p''_i \in p^*_{ij}(P^{k-1}_j)$ for all opponents j of i and therefore $p''_i \in P^k_i$.

Thirdly, we show that P^k_i is compact, i.e. bounded and closed. Indeed, $P^k_i \subseteq \Delta(C_{-i})$ is bounded. Now, take some converging sequence $(p^n_i)_{n\in\mathbb{N}} \in P^k_i$ with limit point $p_i \in P^o_i$. For every $n \in \mathbb{N}$ and for every opponent j of i, it holds that $marg_{C_j} p^n_i \in p^*_{ij}(P^{k-1}_j)$. Note that P^{k-1}_j is closed by the inductive assumption and that p^*_{ij} is a continuous function. Therefore, $p^*_{ij}(P^{k-1}_j)$ is a closed set which guarantees that $marg_{C_j} p_i \in p^*_{ij}(P^{k-1}_j)$ for every opponent j of i. Consequently, $p_i \in P^k_i$ which ensures that P^k_i is closed. □

The preceding result implies that the recursive procedure yields a non-empty output.

Corollary 1 *The set P^∞_i is non-empty for all players $i \in I$.*

Proof Observe that $P^k_i \subseteq P^{k-1}_i$ for all $k \geq 1$ and for all $i \in I$. Moreover, by Lemma 2 the sets P^k_i are non-empty and compact for $k \geq 1$ and for all $i \in I$. Hence, $P^\infty_i = \bigcap_{k\geq 0} P^k_i$ is non-empty for all players $i \in I$. □

Finally, it is established that the sets P^∞_i for all players $i \in I$ are the largest fixed points of our recursive procedure.

Lemma 3 *For all players i it holds that*

$$P^\infty_i = \{p_i \in \Delta(C_{-i}) : marg_{C_j} p_i \in p^*_{ij}(P^\infty_j) \textit{ for all } j \in I\backslash\{i\}\}.$$

Proof First, we show that $P^\infty_i \subseteq \{p_i \in \Delta(C_{-i}) : marg_{C_j} p_i \in p^*_{ij}(P^\infty_j)$ for all $j \in I\backslash\{i\}\}$. Let $p_i \in P^\infty_i$. Since $p_i \in P^{k+1}_i$ for all $k \geq 0$ it follows that $marg_{C_j} p_i \in p^*_{ij}(P^k_j)$ for all $k \geq 0$ and for all opponents j of i. Fix an opponent j of i and define $p_{ij} := marg_{C_j} p_i$. It suffices to show that $p_{ij} \in p^*_{ij}(P^\infty_j)$. As $p_{ij} \in p^*_{ij}(P^k_j)$ for all $k \geq 0$, there exists for every $k \geq 0$ some $p^k_j \in P^k_j$ such that $p_{ij} = p^*_{ij}(p^k_j)$. Since P^0_j is compact, the sequence $(p^k_j)_{k\geq 0}$ has a convergent subsequence. Without loss of generality, assume that $(p^k_j)_{k\geq 0}$ is itself convergent to some $p_j \in P^0_j$. As the function p^*_{ij} is continuous it follows that $p_{ij} = p^*_{ij}(p_j)$. Fix $m \geq 0$ and consider the subsequence $(p^k_j)_{k\geq m} \in P^m_j$. By closedness of P^m_j it holds that the limit point p_j must be in P^m_j. Since $p_j \in P^m_j$ for all $m \geq 0$, $p_j \in P^\infty_j$ follows. As $p_{ij} = p^*_{ij}(p_j)$ it obtains that $p_{ij} \in p^*_{ij}(P^\infty_j)$.

Secondly, we establish that $\{p_i \in \Delta(C_{-i}) : marg_{C_j} p_i \in p^*_{ij}(P^\infty_j)$ for all $j \in I\backslash\{i\}\} \subseteq P^\infty_i$. Take $p_i \in \{p_i \in \Delta(C_{-i}) : marg_{C_j} p_i \in p^*_{ij}(P^\infty_j)$ for all $j \in I\backslash\{i\}\}$. Then, $marg_{C_j} p_i \in p^*_{ij}(P^k_j)$ for all $k \geq 0$ and for all opponents j of i. It follows that $p_i \in P^{k+1}_i$ for all $k \geq 0$, and thus $p_i \in P^\infty_i$. □

4 Procedural characterization of common belief in utility proportional beliefs

It is now established that the recursive procedure yields precisely those beliefs that a player can entertain under common belief in λ-utility proportional beliefs.

Theorem 1 *Let $\lambda = (\lambda_i)_{i \in I} \in \times_{i \in I} \mathbb{R}^{|I \setminus \{i\}|}$ such that $\lambda_{ij} \leq \lambda_{ij}^{max}$ for all two distinct players i and j. A belief $p_i \in \Delta(C_{-i})$ can be held by a type $t_i \in T_i$ that expresses common belief in λ-utility-proportional beliefs in some epistemic model $\mathcal{M}^{\Gamma}$ of Γ if and only if p_i survives iterated elimination of utility-disproportional beliefs.*

Proof For the *only if* direction of the theorem, we prove by induction on k that a belief that can be held by a type that expresses up to k-fold belief in λ-utility-proportional beliefs survives k rounds of iterated elimination of utility-disproportional beliefs. It then follows that a type expressing common belief in λ-utility-proportional-beliefs holds a belief which survives iterated elimination of utility-disproportional beliefs.

First of all, let $k = 1$ and consider $t_i \in T_i$ that expresses 1-fold belief in λ-utility-proportional beliefs. Then,

$$(b_i(t_i))(c_j, t_j) = \left(\frac{1}{|C_j|} + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} (u_j(c_j, t_j) - u_j^{average}(t_j)) \right) (b_i(t_i))(t_j)$$

for all $c_j \in C_j$, $t_j \in T_j$ and for all $j \in I \setminus \{i\}$. It follows that

$$(b_i(t_i))(c_j) = \Sigma_{t_j \in T_j(t_i)} (b_i(t_i))(t_j) \left(\frac{1}{|C_j|} + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} (u_j(c_j, t_j) - u_j^{average}(t_j)) \right)$$

for all $c_j \in C_j$. Written as a vector,

$$\begin{aligned} marg_{C_j}(b_i(t_i)) = \; & \Sigma_{t_j \in T_j(t_i)} (b_i(t_i))(t_j) \\ & \times \left(\frac{1}{|C_j|}(1, \ldots, 1) + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} (u_j(c_j, t_j) - u_j^{average}(t_j))_{c_j \in C_j} \right) \end{aligned}$$

obtains. Note that, by definition,

$$\frac{1}{|C_j|}(1, \ldots, 1) + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} (u_j(c_j, t_j) - u_j^{average}(t_j))_{c_j \in C_j} \in p_{ij}^*(P_j^0)$$

holds. Thus, $marg_{C_j}(b_i(t_i))$ is a convex combination of elements in the set $p_{ij}^*(P_j^0)$. Since P_j^0 is convex and p_{ij}^* is a linear function, it follows that the image set $p_{ij}^*(P_j^0)$ is convex too. Therefore, $marg_{C_j}(b_i(t_i)) \in p_{ij}^*(P_j^0)$. As this holds for all opponents $j \in I \setminus \{i\}$, if follows that $marg_{C_{-i}}(b_i(t_i)) \in P_i^1$.

Now let $k \geq 1$ and consider $t_i \in T_i$ that expresses up to k-fold belief in λ-utility-proportional beliefs. Then,

$$(b_i(t_i))(c_j, t_j) = \left(\frac{1}{|C_j|} + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} (u_j(c_j, t_j) - u_j^{average}(t_j)) \right) (b_i(t_i))(t_j)$$

for all $c_j \in C_j$ and for all $t_j \in T_j$. Therefore,

$$(b_i(t_i))(c_j) = \Sigma_{t_j \in T_j(t_i) \subseteq B_j^{k-1}} (b_i(t_i))(t_j) \left(\frac{1}{|C_j|} + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} (u_j(c_j, t_j) - u_j^{average}(t_j)) \right)$$

for all $c_j \in C_j$, where B_j^{k-1} denotes the set of j's types that express up to $k-1$-fold belief in λ-utility-proportional-beliefs. Written as a vector,

$$\begin{aligned} marg_{C_j}(b_i(t_i)) = \Sigma_{t_j \in T_j(t_i) \subseteq B_j^{k-1}} (b_i(t_i))(t_j) \\ \times \left(\frac{1}{|C_j|}(1, \ldots, 1) + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} (u_j(c_j, t_j) - u_j^{average}(t_j))_{c_j \in C_j} \right) \end{aligned}$$

obtains. Since every $t_j \in T_j(t_i)$ is in B_j^{k-1} and hence by the induction hypothesis $marg_{C_{-j}}(b_j(t_j)) \in P_j^{k-1}$, it follows that

$$\begin{aligned} &\frac{1}{|C_j|}(1, \ldots, 1) + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} (u_j(c_j, t_j) - u_j^{average}(t_j))_{c_j \in C_j} \\ &= p_{ij}^*(marg_{C_{-j}} b_j(t_j)) \in p_{ij}^*(P_j^{k-1}). \end{aligned}$$

Thus, $marg_{C_j}(b_i(t_i))$ is a convex combination of elements in the set $p_{ij}^*(P_j^{k-1})$. Since, by Lemma 2 P_j^{k-1} is convex and p_{ij}^* is a linear function, it follows that the image set $p_{ij}^*(P_j^{k-1})$ is convex too. Therefore, $marg_{C_j}(b_i(t_i)) \in p_{ij}^*(P_j^{k-1})$. As this holds for all opponents $j \in I \backslash \{i\}$, $marg_{C_{-i}}(b_i(t_i)) \in P_i^k$ results. By induction on k it follows that for every type t_i that expresses common belief in λ-utility proportional beliefs, $marg_{C_{-i}}(b_i(t_i)) \in P_i^{\infty}$, and consequently the *only if* direction of the theorem obtains.

For the *if* direction of the theorem, we construct for every player $i \in I$ and for every belief $p_i \in P_i^{\infty}$ some type $t_i^{p_i}$ as follows. Let $p_i \in P_i^{\infty}$ be some belief that survives the recursive procedure, which exists by Corollary 1. By Lemma 3 it then holds for every opponent $j \in I \backslash \{i\}$ that $marg_{C_j} p_i = p_{ij}^*(p_j)$ for some $p_j \in P_j^{\infty}$. We define $t_i^{p_i}$ to be the type that (i) for every opponent $j \in I \backslash \{i\}$ only deems possible the type $t_j^{p_j}$ where p_j is the belief just fixed above, and (ii) that entertains belief p_i about his opponents' choice combinations.

We now show that every such type $t_i^{p_i}$ expresses λ_i-utility-proportional-beliefs. Observe that for every opponent $j \in I \backslash \{i\}$ type $t_i^{p_i}$ only considers possible a unique type, namely $t_j^{p_j}$ which holds belief p_j about his opponents' choice combinations. As the belief of $t_i^{p_i}$ about j's choice equals $marg_{C_j} p_i = p_{ij}^*(p_j)$, it follows by definition of the function p_{ij}^* that $t_i^{p_i}$ entertains λ_i-utility-proportional-beliefs.

Consider the epistemic model consisting of all such types $t_i^{p_i}$ for every player $i \in I$. Since this epistemic model only contains types $t_i^{p_i}$ that express λ_i-utility-proportional-beliefs for every player $i \in I$, any type also expresses common belief in λ-utility-proportional-beliefs. Therefore, for every player $i \in I$ and for every belief $p_i \in P_i^\infty$ there exists a type $t_i^{p_i}$ such that $t_i^{p_i}$ holds belief p_i about his opponents' choice combinations and expresses common belief in λ-utility-proportional-beliefs, which concludes the proof. □

According to the preceding theorem the recursive procedure thus provides a convenient way to compute the beliefs a player can hold when reasoning in line with common belief in utility proportional beliefs.

Farther, note that the proof of the *if* direction of Theorem 1 establishes that common belief in utility proportional beliefs is always possible in every game.

Corollary 2 *Let $\lambda = (\lambda_i)_{i \in I} \in \times_{i \in I} \mathbb{R}^{|I \setminus \{i\}|}$. There exists an epistemic model $\mathcal{M}^\Gamma$ of Γ, and a type $t_i \in T_i$ for every every player $i \in I$ such that t_i expresses common belief in λ-utility-proportional-beliefs.*

It is thus guaranteed that common belief in utility proportional beliefs is a logically sound concept, which can be adopted to describe players' reasoning in any game.

5 Uniqueness of beliefs

We first show that in the case of two players, our concept of common belief in λ-utility-proportional-beliefs yields unique beliefs for both players. However, by means of a counterexample it is then established that uniqueness of beliefs does no longer hold with more than two players.

5.1 Games with two players

It is now shown that the recursive procedure returns unique beliefs in two player games, whenever $\lambda_{ij} < \lambda_{ij}^{max}$ for both players i and j. Recall that λ_{ij}^{max} is the largest possible proportionality factor such that $p_{ij}^*(p_j)$ is a probability distribution for all $p_j \in P_j^0$, and hence $\lambda_{ij} < \lambda_{ij}^{max}$ does not constitute a strong assumption. Note that whenever $\lambda_{ij} < \lambda_{ij}^{max}$, the utility proportional beliefs always assign positive probability to every opponent choice. By Theorem 1 it then follows that the concept of common belief in λ-utility-proportional-beliefs yields unique beliefs for both players, if $\lambda_{ij} < \lambda_{ij}^{max}$ for i and j.

In order to prove uniqueness of beliefs the following lemma is adduced. Recall that P_i^k denotes the set of beliefs generated for player i in round k of the recursive procedure. Moreover, for any two sets $A, B \subseteq P_i^0$ and for all $\alpha \in [0, 1]$, we define the set $\alpha A + (1 - \alpha)B := \{\alpha a + (1 - \alpha)b : a \in A \text{ and } b \in B\}$. Further, $p_{ij}^*(\cdot, \lambda_{ij})$ denotes the function p_{ij}^* induced by the proportionality factor λ_{ij}.

Lemma 4 *Let Γ be a two player game with $I = \{1, 2\}$, $\lambda_{12} < \lambda_{12}^{max}$, and $\lambda_{21} < \lambda_{21}^{max}$. Moreover, define $\alpha_{12} := \frac{\lambda_{12}}{\lambda_{12}^{max}} < 1$, $\alpha_{21} := \frac{\lambda_{21}}{\lambda_{21}^{max}} < 1$, and $\alpha := max\{\alpha_{12}, \alpha_{21}\} < 1$. Then, for every player $i \in I$ and every round $k \geq 0$ there exists $p_i \in P_i^0$ such that $P_i^k \subseteq \alpha^k P_i^0 + (1-\alpha^k)\{p_i\}$.*

Proof We prove the statement by induction on k. First of all, let $k = 0$ and $i \in I$ be some player. Then, $P_i^0 \subseteq \alpha^0 P_i^0 + (1-\alpha^0)\{p_i\}$ holds for all $p_i \in P_i^0$, as $\alpha^0 = 1$.

Now, suppose $k > 0$ and assume that for both players $i \in I$ there exists $p_i \in P_i^0$ such that $P_i^{k-1} \subseteq \alpha^{k-1} P_i^0 + (1-\alpha^{k-1})\{p_i\}$. Consider some player $i \in I$. Since there are only two players, $P_i^k = p_{ij}^*(P_j^{k-1}, \lambda_{ij})$ holds by construction of the recursive procedure. As $P_j^{k-1} \subseteq \alpha^{k-1} P_j^0 + (1-\alpha^{k-1})\{p_j\}$ by the inductive assumption, it follows that

$$\begin{aligned} P_i^k &\subseteq p_{ij}^*(\alpha^{k-1} P_j^0 + (1-\alpha^{k-1})\{p_j\}, \lambda_{ij}) \\ &= \alpha^{k-1} p_{ij}^*(P_j^0, \lambda_{ij}) + (1-\alpha^{k-1})\{p_{ij}^*(p_j, \lambda_{ij})\}, \end{aligned}$$

where the latter equality holds by linearity of p_{ij}^* on P_j^0. Therefore,

$$P_i^k \subseteq \alpha^{k-1} p_{ij}^*(P_j^0, \lambda_{ij}) + (1-\alpha^{k-1})\{p_{ij}^*(p_j, \lambda_{ij})\} \tag{1}$$

obtains. Observe that, by definition of α_{ij}, it holds that $\lambda_{ij} = \alpha_{ij}\lambda_{ij}^{max} + (1-\alpha_{ij})0$. As p_{ij}^* also is linear in λ_{ij}, it follows that

$$p_{ij}^*(P_j^0, \lambda_{ij}) = \alpha_{ij} p_{ij}^*(P_j^0, \lambda_{ij}^{max}) + (1-\alpha_{ij}) p_{ij}^*(P_j^0, 0).$$

Note that, by definition of p_{ij}^*,

$$p_{ij}^*(p_j', 0) = p_i^{uniform} \text{ for all } p_j' \in P_j^0,$$

where $p_i^{uniform}$ is the uniform distribution on C_j. Consequently, $p_{ij}^*(P_j^0, 0) = \{p_i^{uniform}\}$. Since also $p_{ij}^*(P_j^0, \lambda_{ij}^{max}) \subseteq P_i^0$, it holds that

$$p_{ij}^*(P_j^0, \lambda_{ij}) \subseteq \alpha_{ij} P_i^0 + (1-\alpha_{ij})\{p_i^{uniform}\}.$$

As $\alpha_{ij} \leq \alpha$,

$$\alpha_{ij} P_i^0 + (1-\alpha_{ij})\{p_i^{uniform}\} \subseteq \alpha P_i^0 + (1-\alpha)\{p_i^{uniform}\},$$

and hence

$$p_{ij}^*(P_j^0, \lambda_{ij}) \subseteq \alpha P_i^0 + (1-\alpha)\{p_i^{uniform}\} \tag{2}$$

obtains. Combining (1) and (2) yields

$$\begin{aligned}P_i^k &\subseteq \alpha^{k-1}(\alpha P_i^0 + (1-\alpha)\{p_i^{uniform}\}) + (1-\alpha^{k-1})\{p_{ij}^*(p_j, \lambda_{ij})\}\\ &= \alpha^k P_i^0 + \alpha^{k-1}(1-\alpha)\{p_i^{uniform}\} + (1-\alpha^{k-1})\{p_{ij}^*(p_j, \lambda_{ij})\}.\end{aligned}$$

Recall that $\alpha < 1$. Then, by defining

$$\hat{p}_i := \frac{\alpha^{k-1}(1-\alpha)p_i^{uniform} + (1-\alpha^{k-1})p_{ij}^*(p_j, \lambda_{ij})}{1-\alpha^k},$$

it follows that

$$P_i^k \subseteq \alpha^k P_i^0 + (1-\alpha^k)\{\hat{p}_i\},$$

which concludes the proof. □

The preceding lemma can be used to show that the beliefs surviving the recursive procedure are unique in the case of two players.

Theorem 2 *Let Γ be a two player game with $I = \{1, 2\}$, $\lambda_{12} < \lambda_{12}^{max}$, and $\lambda_{21} < \lambda_{21}^{max}$. Then, $| P_1^\infty |= 1$ and $| P_2^\infty |= 1$.*

Proof It follows directly from Lemma 4 that the set P_i^k converges to a singleton as $k \to \infty$ for every player $i \in I$. □

Note that Lemma 4 shows that in the case of two players the recursive procedure consists of recursively applying a kind of contraction mapping on the set of the players' beliefs. Hence, uniqueness of beliefs obtains, which constitutes a highly convenient property of our recursive procedure for two player games.

Moreover, Lemma 4 can be used to determine how many steps are needed to approach the unique beliefs up to n decimal points, for any $n \geq 1$.

5.2 Games with more than two players

The following example establishes that uniqueness of beliefs does no longer necessarily hold for games with more than two players.

Example 1 Consider the three player game Γ depicted in Fig. 1.

Fig. 1 A 3-player game

Claire: e

Alice \ Bob	c	d
a	1, 1, 1	0, 0, 0
b	0, 0, 0	1, 1, 1

Claire: f

Alice \ Bob	c	d
a	0, 0, 0	1, 1, 1
b	1, 1, 1	0, 0, 0

First of all, note that $\lambda_{ij}^{max} = 1$ for all $i, j \in I$ such that $i \neq j$. Fix some $\lambda < 1$ and let $\lambda_{ij} = \lambda$ for all $i, j \in I$ such that $i \neq j$. Suppose the epistemic model $\mathcal{M}^{\Gamma}$ of Γ given by the sets of types $T_{Alice} = \{t_1\}$, $T_{Bob} = \{t_2, t_2'\}$, $T_{Claire} = \{t_3, t_3^e, t_3^f\}$, and the following induced belief functions

- $b_{Alice}(t_1) = \frac{1}{4}((c, t_2), (e, t_3)) + \frac{1}{4}((c, t_2), (f, t_3)) + \frac{1}{4}((d, t_2), (e, t_3)) + \frac{1}{4}((d, t_2), (f, t_3))$,
- $b_{Bob}(t_2) = \frac{1}{4}((a, t_1), (e, t_3)) + \frac{1}{4}((a, t_1), (f, t_3)) + \frac{1}{4}((b, t_1), (e, t_3)) + \frac{1}{4}((b, t_1), (f, t_3))$,
- $b_{Bob}(t_2') = (\frac{1}{4} + \frac{1}{4}\lambda)((a, t_1), (e, t_3^e)) + (\frac{1}{4} - \frac{1}{4}\lambda)((a, t_1), (f, t_3^e)) + (\frac{1}{4} + \frac{1}{4}\lambda)((b, t_1), (e, t_3^e)) + (\frac{1}{4} - \frac{1}{4}\lambda)((b, t_1), (f, t_3^e))$,
- $b_{Claire}(t_3) = \frac{1}{4}((a, t_1), (c, t_2)) + \frac{1}{4}((a, t_1), (d, t_2)) + \frac{1}{4}((b, t_1), (c, t_2)) + \frac{1}{4}((b, t_1), (d, t_2))$,
- $b_{Claire}(t_3^e) = \frac{1}{2}((a, t_1), (c, t_2)) + \frac{1}{2}((b, t_1), (d, t_2))$,
- $b_{Claire}(t_3^f) = \frac{1}{2}((a, t_1), (d, t_2)) + \frac{1}{2}((b, t_1), (c, t_2))$.

Observe that types t_1, t_2, and t_3 are each indifferent between their respective two choices. As type t_1 only deems possible opponents' types t_2 and t_3, as well as assigns equal probability to both choices for each opponent, it follows that t_1 entertains λ-utility-proportional beliefs. Similarly, it can be verified that types t_2, t_3, t_3^e, and t_3^f also hold λ-utility-proportional-beliefs.

We now show that type t_2' expresses λ-utility-proportional beliefs, too. First of all, note that t_2' only deems possible *Alice*'s type t_1 which is indifferent between choices a and b, while assigning equal probability to her choices a and b. Secondly, t_2' only deems possible *Claire*'s type t_3^e, for which the choices e and f generate expected utilities of 1 and 0, respectively. More precisely, $u_{Claire}(e, t_3^e) = 1$ and $u_{Claire}(f, t_3^e) = 0$. At the same time, $(b_{Bob}(t_2'))(e \mid t_3^e) = \frac{1}{2} + \frac{1}{2}\lambda$ as well as $(b_{Bob}(t_2'))(f \mid t_3^e) = \frac{1}{2} - \frac{1}{2}\lambda$ hold, which imply that

$$(b_{Bob}(t_2'))(e \mid t_3^e) - (b_{Bob}(t_2'))(f \mid t_3^e) = \lambda.$$

Since $\overline{u}_{Claire} - \underline{u}_{Claire} = 1$ and $\lambda_{Bob,Claire} = \lambda$, it follows that

$$\begin{aligned}&(b_{Bob}(t_2'))(e \mid t_3^e) - (b_{Bob}(t_2'))(f \mid t_3^e)\\ &\quad = \frac{\lambda_{Bob,Claire}}{\overline{u}_{Claire} - \underline{u}_{Claire}}(u_{Claire}(e, t_3^e) - u_{Claire}(f, t_3^e)).\end{aligned}$$

Hence, t_2' holds λ-utility-proportional-beliefs by definition.

As every type in the epistemic model $\mathcal{M}^{\Gamma}$ entertains λ-utility-proportional-beliefs, every type also expresses common belief in λ-utility-proportional-beliefs. However, observe that *Bob*'s types t_2 and t_2' hold distinct beliefs about *Claire*'s choice, whenever $\lambda > 0$. ♣

In fact, note that in the case of two players it holds that $P_i^{k+2} = (p_{ij}^* \circ p_{ji}^*)(P_i^k)$, where $(p_{ij}^* \circ p_{ji}^*)$ is a contraction mapping by Lemma 4, which implies that P_i^k converges to a singleton set. However, such a construction is not possible for more than two players, as in that case P_i^{k+2} does not only depend on one set P_j^{k+1}, but on several such sets, one for every opponent j.

6 Illustration

The recursive procedure is easy and conveniently implementable. Indeed, without any difficulties we wrote a small program for two player games, which computes the unique belief vector there. We now illustrate in some well-known two player games how well common belief in utility proportional beliefs fares with respect to intuition as well as to experimental findings—in contrast to classical concepts which run into problems when applied to these games. In each example we use λ_{ij} slightly smaller than λ_{ij}^{max} such that the differences in utilities have the largest possible effect on the players' beliefs, while still guaranteeing these beliefs to be unique. In fact, from these unique beliefs it is possible to directly read off the rational choices under common belief in λ-utility-proportional-beliefs, since these choices must receive the highest probability under those beliefs.

Example 2 Consider again the traveler's dilemma which has already been introduced in Sect. 1. Consider three variants of the game, according to which the players can choose between 10, 30, and 100 prices. Reasoning in line with common belief in rationality requires the travelers to opt for the minimum price of 1 in each of the three variations. However, it neither seems plausible to exclude any irrational choice completely from consideration nor do experiments confirm such results. For instance, in an experiment with members of the game theory society by Becker et al. (2005), where prices between 2 and 100 could be chosen, most persons opted for a high price of at least 90. In fact, contrary to common belief in rationality, our concept yields the much more natural choices of 6, 26, and 96 for the games with 10, 30, and 100 prices, respectively. Besides, note that common belief in utility proportional beliefs is actually sensitive to the cardinality of the choice sets. Indeed, it seems intuitive that when there are few prices to choose from rather lower prices will be opted for, and when there are many prices available then the ones picked will be higher. ♣

Example 3 Figure 2 depicts an asymmetric matching pennies game that is taken from Goeree and Holt (2001).

In the unique Nash equilibrium of the game, *Row Player* chooses $\left(\frac{1}{2}, \frac{1}{2}\right)$ and *Column Player* chooses $\left(\frac{1}{8}, \frac{7}{8}\right)$. Intuitively, it seems reasonable for *Row Player* to opt for *top* due to the very high possible payoff of 320, while *Column Player* might tend to pick *right* anticipating *Row Player*'s temptation for *top*. Indeed, in experiments by Goeree and Holt (2001) approximately 95 % of the row players choose *top*, while approximately

Fig. 2 Asymmetric matching pennies

		Column Player: *left*	Column Player: *right*
Row Player	*top*	320, 40	40, 80
	bottom	40, 80	80, 40

Fig. 3 A coordination game with a secure outside option

		Column Player		
		left	*middle*	*right*
Row Player	*top*	90, 90	0, 0	0, 40
	bottom	0, 0	180, 180	0, 40

		Column Player			
		left	*middle*	*non-Nash*	*right*
Row Player	*top*	20, 5	0, 4	1, 3	$2, -10^4$
	bottom	$0, -10^4$	$1, -10^3$	3, 3	5, 10

Fig. 4 Kreps game

85 % of the column players opt for *right*. Here, close to the experimental findings our concept of common belief in utility proportional beliefs yields choices *top* and *right* for *Row Player* and *Column Player*, respectively. ♣.

Example 4 Suppose the normal form in Fig. 3 which models a coordination game with a secure outside option that is taken from Goeree and Holt (2001).

The game contains multiple Nash equilibria, among which there is the focal high-payoff one (*bottom*, *middle*), while *Column Player* has access to a secure outside option guaranteeing him a payoff of 40. In experiments by Goeree and Holt (2001), approximately 95 % of the row players choose *bottom*, while approximately 95 % of the column players pick *middle*. Close to the results from the laboratory, common belief in utility proportional beliefs yields *bottom* and *middle*. ♣

Example 5 The Kreps (1995) is represented in Fig. 4.

The game exhibits three Nash equilibria, two pure and one mixed, and in none of them *Column Player* chooses *Non-Nash* with positive proabability. However, *Non-Nash* appears to be reasonable, as all other options only yield a slightly higher payoff, but might lead to considerable losses. When anticipating this reasoning *Row player* would optimally choose *bottom*. Indeed, in informal experiments by Kreps the row players pick *bottom*, while the column players choose *Non-Nash* in the majority of cases. Also in this game common belief in utility proportional beliefs performs intuitively by generating *top* for *Row Player* and *non-Nash* for *Column Player*, respectively. Indeed, *top* seems reasonable for the *Row Player* as long as he assigns a substantial probability to the *Column Player* choosing *left*, which is what our concept does. ♣

7 Discussion

Utility proportional beliefs The concept of utility proportional beliefs seems quite a natural and basic way of reasoning in the context of games. Indeed, it does appear plausible to assign non-zero probability to opponents' irrational choices due to causes such as the complexity of the interactive situation, uncertainty about the opponents' utilities and choice rules, possibility of mistakes, caution etc. However, at the same time it is intuitive that the opponents' relative utilities are reflected in a player's beliefs about their choice and to thus assign probabilities proportional to the respective utilities.

Moreover, utility proportional beliefs furnishes probabilities with intrinsic meaning in the sense of measuring how good a player deems some choice for the respective opponent, and thus also provides an account of *how* agents form their beliefs. In contrast, basic classical concepts like common belief in rationality treat every choice that receives positive probability as equally plausible.

Besides, utility proportional beliefs does not only appear reasonable from intuitive as well as theoretical perspectives, but also fares well with regards to experimental findings, as indicated in Sect. 6. In this context, also note that experimental findings can often not be explained by the basic concept of common belief in rationality, which implies that any irrational choice always receives zero probability.

Bounded reasoning Although the proposed notion of common belief in utility proportional beliefs invokes an infinite number of reasoning steps on behalf of the players, note that typically a few reasoning steps already suffice for the players to appproximate the beliefs selected by our concept. In this sense, common belief in utility proportional beliefs can actually be considered to be in line with the viewpoint that players only reason a finite number of steps.

t-Solutions Rosenthal's (1989) class of t-solutions for two player games formalizes the idea that players do not exclusively play best responses. Intuitively, given a fixed parameter $t \in \mathbb{R}$, a pair of randomized choices constitutes a t-solution, if each of them satisfies the property that if positive probability is assigned to some pure choice, then the difference in probability with any other pure choice of the same player equals t times the difference in the respective utilities given the opponent's randomized choice.[2] In other words, players assign probabilities to their choices such that the probability differences are proportional to the utility difference multiplied by the proportionality factor t.

In contrast to our concept of utility proportional beliefs, Rosenthal's t-solutions employs a proportionality factor which is the same across all players. It seems more desirable to permit different agents to entertain distinct proportionality factors, in order to represent heterogenous states of mind, and to thus provide a more realistic account of reasoning. Also, t-solutions are not invariant to affine translations of the utilities, which is a serious drawback not arising in our model. Moreover, the players' probability distributions which are restricted by a utility proportionality condition are distinct objects in Rosenthal's and our models. While in the former randomized choices, i.e. conscious randomizations of the players are considered, beliefs on the opponents' choices are used in the latter. Since assuming probabilities to be objects of choice constitutes a problematic assumption for at least most game-theoretic contexts, probabilities interpreted as players' beliefs seems more plausible and realistic. Besides, by keeping the opponents' choices fixed, an equilibrium condition is built into Rosenthal's t-solution concept. However, from an epistemic point of view fix-

[2] Given a game Γ and a player $i \in I$, a *randomized choice* for i is a probability distribution $\sigma_i \in \Delta(C_i)$ on i's choice space.

ing the opponents' choices seems highly unreasonable, as it means that the reasoner already knows what his opponents will do in the game. Note that in our model we admit players to be erroneous about their opponents' choices as well as beliefs, which again is closer to real life, where people are frequently not correct about their fellow men's choices in interactive situations.

Quantal response equilibrium McKelvey and Palfrey (1995) introduce the concept of quantal response equilibrium as a statistical version of equilibrium, where each player chooses deterministically, however his utility for each of his choices is subject to random error. Given a rational decision rule players are assumed to apply, the random error induces a probability distribution over the players' observed choices. In their model these probabilities satisfy the intuitive property that better choices are more likely to be chosen than worse choices.

In contrast to our concept of utility proportional beliefs, McKelvey and Palfrey do not require the probability of a given choice to be *proportional* to the expected utility it generates. Yet, in terms of reasoning it appears natural that a player assigns utility proportional probabilities to his opponents' choices—as in our model—when deliberating about what his opponents might choose. Moreover, the probabilities in quantal response equilibrium are not invariant to affine translations of the utilities. This serious drawback is avoided in our model. Besides, from an epistemic point of view McKelvey and Palfrey's equilibrium condition implicitly assumes that players know their opponents' random error induced probabilities. This seems rather implausible, as players can never have direct access to opponents' minds. Farther, in McKelvey and Palfrey's model agents are assumed to always choose best responses with respect to their beliefs but not with respect to their utilities , which are randomly perturbed. However, in our model the utilities are kept fixed, but we allow players to assign positive probability to opponents' suboptimal choices.

Proper rationalizability The concept of proper rationalizability, introduced by Schumacher (1999) as well as Asheim (2001), and algorithmically characterized by Perea (2011), formalizes cautious reasoning in games. Intuitively, a choice is properly rationalizable for a player, if he is cautious, i.e. does not exclude any opponent's choice from consideration; respects his opponents' preferences, i.e. if he believes an opponent to prefer some choice c to c', then he deems c infinitely more likely than c'; as well as expresses common belief in the event that his opponents are cautious and respect their opponents' preferences. A standard tool to model infinitely-more-likely relations are lexicographic beliefs.[3] Loosely speaking, a reasoner is then said to be cautious, if for every opponent each of his choices occur in the support of the probability distribution of some lexicographic level of the reasoner's lexicographic belief. Hence, all opponents' choices receive positive probability somewhere in a cautious lexicographic belief.

In the sense of modeling cautious reasoning that considers all choices including irrational ones, proper rationalizability and utility proportional beliefs are similar

[3] Given some set W a *lexicographic belief* is a finite sequence $\rho = (\rho^1, \rho^2, \ldots, \rho^K)$ of probability distributions such that $\rho^k \in \Delta(W)$ for all $k \in \{1, 2, \ldots, K\}$.

concepts. However, on the one hand, utility proportional beliefs can be viewed as a milder version than proper rationalizability, since the former assigns substantial, non-infinitesimal positive probability to any choice including non-optimal ones, while the latter assigns only infinitesimal probabilities to non-optimal choices. On the other hand, the two concepts can be viewed as opposite ways of cautious reasoning, since utility proportional beliefs reflects the utility differences of choices, while proper rationalizability treats all non-optimal choices as infinitely less likely than optimal choices. Moreoever, on the purely formal level both ways of cautious reasoning are distinct, as utility proportional beliefs employs standard beliefs, whereas proper rationalizability models lexicographically minded agents.

8 Conclusion

Utility proportional beliefs provides a basic and natural way of reasoning in games. The underlying intuitions that irrational choices should not be completely neglected, and beliefs ought to reflect how good a player deems his opponents' choices, seem plausible. The surprising property that the iterated elimination of utility-disproportional-beliefs procedure yields unique beliefs in two player games strengthens the suitability of common belief in utility proportional beliefs to be used for descriptions there. Moreover, in various games of interest our concept matches well intuition and experimental findings.

The idea of utility proportional beliefs opens up a new direction of research. Naturally, the concept can be extended to dynamic games. Besides, the effects of allowing uncertainty about the opponents' λ's can be studied. Moreover, applications of our epistemic concept to well-known games or economic problems such as auctions may be highly interesting.

Acknowledgments We are grateful to conference participants at the Eleventh Conference of the Society for the Advancement of Economic Theory (SAET2011) as well as to seminar participants at Maastricht University, City University of New York (CUNY), and University of Helsinki for useful and constructive comments. Besides, valuable remarks by two anonymous referees are highly appreciated.

References

Asheim GB (2001) Proper rationalizability in lexicographic beliefs. Int J Game Theory 30:453–478
Aumann RJ (1976) Agreeing to disagree. Ann Stat 4:1236–1239
Aumann RJ (1987) Correlated equilibrium as an expression of Bayesian rationality. Econometrica 55:1–18
Basu K (1994) The traveler's dilemma: paradoxes of rationality in game theory. Am Econ Rev 84:391–395
Becker T, Carter M, Naeve J (2005) Experts playing the traveler's dilemma. Discussion paper 252/2005, Universitt Hohenheim
Bernheim BD (1984) Rationalizable strategic behavior. Econometrica 52:1007–1028
Brandenburger A, Dekel E (1987) Rationalizability and correlated equilibria. Econometrica 55:1391–1402
Goeree JK, Holt CA (2001) Ten little treasures of game theory and ten intuitive contradictions. Am Econ Rev 91:1402–1422
Harsanyi JC (1967–1968) Games of incomplete information played by "Bayesian Players". Part I, II, III. Manag Sci 14:159–182, 320–334, 486–502
Kreps DM (1995) Nash equilibrium. In: The new palgrave: game theory. Norton, New York, pp. 176–177

McKelvey RD, Palfrey TR (1995) Quantal response equilibria for normal form games. Games Econ Behav 10:6–38

Pearce D (1984) Rationalizable strategic behavior and the problem of perfection. Econometrica 52:1029–1050

Perea A (2007a) A one-person doxastic characterization of nash strategies. Synthese 158:1251–1271

Perea A (2007b) Epistemic conditions for backward induction: an overview. In: Interactive logic proceedings of the 7th Augustus de Morgan Workshop, London. Texts in logic and games 1. Amsterdam University Press. pp. 159–193

Perea A (2011) An algorithm for proper rationalizability. Games Econ Behav 72:510–525

Perea A (2012) Epistemic game theory: reasoning and choice. Cambridge University Press, Cambridge

Rosenthal RW (1989) A bounded-rationality approach to the study of noncooperative games. Int J Game Theory 18:273–292

Schuhmacher F (1999) Proper rationalizability and backward induction. Int J Game Theory 28:599–615

Tan TCC, Werlang SRC (1988) The Bayesian foundation of solution concepts of games. J Econ Theory 45:370–391