Epistemic Game Theory
Part 2: Lexicographic Beliefs in Static Games

Andrés Perea

Maastricht University

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Outline

- Yesterday, we investigated standard beliefs: probability distributions over the opponents' choices.
- Today, we concentrate on cautious reasoning:
- You never discard any opponent's choice from consideration,
- yet you may deem some opponent's choices much more likely – in fact, infinitely more likely – than other choices.
- This can be modelled by lexicographic beliefs.

Outline

- We present, formalize, and compare, three different ways of reasoning:
- Primary belief in the opponent's rationality
- Respecting the opponent's preferences
- Assuming the opponent's rationality
- We discuss recursive procedures that characterize the choices induced by these concepts.

Example: Should I call or not?

Story

- This evening, Barbara will go to the cinema.
- You can join if you wish, but Barbara decides on the movie.
- There is the choice between The Godfather and Casablanca.
- You prefer The Godfather (utility 1) to Casablanca (utility 0).
- For Barbara it is the other way around.
- Staying at home gives you utility 0.
- Question: Should you call Barbara or not?

		Barbara	
		The Godfather	Casablanca
You	Call	$1, 0$	$0, 1$
	Don't call	$0, 0$	$0, 1$

- Intuitively, your unique best choice is to call.
- However, if you hold a standard belief, and believe that Barbara chooses rationally, then you must assign probability 0 to Barbara choosing The Godfather.
- But then, both call and don't call would be optimal for you.
- We want to model a state of mind in which you
- deem Casablanca much more likely (in fact, infinitely more likely) than The Godfather, but
- do not completely rule out the possibility that Barbara will choose The Godfather.
- This can be modeled by a lexicographic belief.

		Barbara	
		The Godfather	Casablanca
You	Call	$1, 0$	$0, 1$
	Don't call	$0, 0$	$0, 1$

- Consider the following lexicographic belief about Barbara's choice:
- Your primary belief is that Barbara will choose Casablanca.
- Your secondary belief is that Barbara will choose The Godfather.
- Interpretation: You deem Casablanca infinitely more likely than The Godfather, but you still deem The Godfather possible.
- In your primary belief, you believe that Barbara chooses rationally: You primarily believe in Barbara's rationality.
- Under this lexicographic belief, your unique optimal choice is to call.

Example: Where to read my book?

Story

- You want to go to a pub to read your book.
- Barbara told you that she will also go to a pub, but you forgot to ask which one.
- Your only objective is to avoid Barbara, since you want to read your book in silence.
- Barbara prefers Pub a to Pub b, and Pub b to Pub c.
- Question: To which pub should you go?

		Barbara		
		Pub a	Pub b	Pub c
You	Pub a	$0,3$	$1,2$	$1,1$
	Pub b	$1,3$	$0,2$	$1,1$
	Pub c	$1,3$	$1,2$	$0,1$

- If you primarily believe in Barbara's rationality, then your primary belief should assign probability 1 to Barbara visiting Pub a.
- Hence, you must deem Pub a infinitely more likely than Pub b and Pub c, but you can rank Pub b and Pub c in any way you wish.
- Since you can deem Pub b or Pub c least likely for Barbara, it can be optimal for you to go to Pub b or Pub c.
- Conclusion: If you primarily believe in Barbara's rationality, you can rationally visit Pub b or Pub c.
- Problem: Intuitively, Pub c is the "least likely choice" for Barbara, and hence you should go to Pub c, and not to Pub b.

		Barbara		
		Pub a	Pub b	Pub c
You	Pub a	$0,3$	$1,2$	$1,1$
	Pub b	$1,3$	$0,2$	$1,1$
	Pub c	$1,3$	$1,2$	$0,1$

- Pub b is better for Barbara than Pub c, and hence it seems natural to deem her better choice Pub b infinitely more likely than her inferior choice Pub c.
- In general, if choice c_j is better for opponent j than choice c_j', then you must deem c_j infinitely more likely than c_j'.
- In that case, you respect the opponent's preferences.
- If you respect Barbara's preferences, you deem her choice Pub a infinitely more likely than her choice Pub b, and you deem her choice Pub b infinitely more likely than her choice Pub c.
- Hence, your unique optimal choice would be to visit Pub c.

Example: Spy game

Story

- Story is largely the same as in "Where to read my book?"
- However, now Barbara suspects that you are having an affair. She therefore would like to spy on you.
- Spying gives Barbara an additional utility of 3.
- Spying is only possible if you are in Pub a and she is in Pub c, or vice versa.

	Pub a	Pub b	Pub c
Pub a	$0, 3$	$1, 2$	$1, 4$
Pub b	$1, 3$	$0, 2$	$1, 1$
Pub c	$1, 6$	$1, 2$	$0, 1$

- Barbara prefers Pub a to Pub b. So, if you respect Barbara's preferences, then you must deem her choice a infinitely more likely than her choice b.
- Then, you will prefer Pub b to Pub a. Hence, if you believe that Barbara respects your preferences as well, you believe that Barbara deems your choice b infinitely more likely than your choice a.
- Hence, Barbara will prefer Pub b to Pub c. So, you must deem her choice b infinitely more likely than her choice c.
- But then, you must visit Pub c.
- Hence, reasoning in line with respect of the opponent's preferences uniquely leads you to Pub c.

	Pub a	Pub b	Pub c
Pub a	$0,3$	$1,2$	$1,4$
Pub b	$1,3$	$0,2$	$1,1$
Pub c	$1,6$	$1,2$	$0,1$

- Alternative way of reasoning:
- For Barbara, visiting Pub a and Pub c can both be optimal, but Pub b can never be optimal.
- Therefore, deem Barbara's choices a and c infinitely more likely than her choice b. We say that you assume Barbara's rationality.
- In general, if the opponent's choice c_j can be optimal for some cautious lexicographic belief, but c_j' cannot, then you must deem c_j infinitely more likely than c_j'.
- Assume the opponent's rationality.
- If you assume Barbara's rationality, you must visit Pub b, and not Pub c.

Lexicographic beliefs

- We want to model a state of mind in which you deem all opponent's choices possible, yet may deem some choice infinitely more likely than another choice.

Definition (Lexicographic belief)

A lexicographic belief for player i about player j's choice is a sequence of probability distributions

$$b_i = (b_i^1;\ b_i^2;\ \ldots ; b_i^K),$$

where $b_i^1, \ldots, b_i^K$ are probability distributions on the set of j's choices.

Here, b_i^1 is the primary belief, b_i^2 is the secondary belief, ..., b_i^K is the level K belief.

- Based on Blume, Brandenburger and Dekel (1991a,b).
- The lexicographic belief b_i is cautious if all opponent's choices receive positive probability somewhere in b_i.

	Pub a	Pub b	Pub c
Pub a	$0,3$	$1,2$	$1,4$
Pub b	$1,3$	$0,2$	$1,1$
Pub c	$1,6$	$1,2$	$0,1$

- Some examples of cautious lexicographic beliefs about Barbara's choice:
- $(a; b; c)$,
- $(a; c; b)$,
- $(a; \frac{1}{3}b + \frac{2}{3}c)$.

Lexicographic belief hierarchies

- To formalize reasoning concepts à la common belief in rationality, we need
- your lexicographic belief about the opponent's choice (first-order belief),
- your lexicographic belief about the opponent's lexicographic belief about your choice (second-order belief),
- and so on.
- Lexicographic belief hierarchy.
- Again, these cannot be written down explicitly, because they contain infinitely many orders.
- How can we encode lexicographic belief hierarchies in an easy way?

Types

- In a lexicographic belief hierarchy, you hold a lexicographic belief about
- the opponents' choices,
- the opponents' first-order beliefs,
- the opponents' second-order beliefs,
- and so on.

- Hence, in a lexicographic belief hierarchy, you hold a lexicographic belief about
- the opponents' choices, and the opponents' lexicographic belief hierarchies.

- Like before, call a lexicographic belief hierarchy a type.
- Then, a type holds a lexicographic belief about the opponents' choices and the opponents' types.

Definition (Epistemic model)

A finite epistemic model with lexicographic beliefs specifies for every player i a finite set T_i of possible types.

Moreover, for every type t_i it specifies a lexicographic belief $b_i(t_i)$ over the set $C_{-i} \times T_{-i}$ of opponents' choice-type combinations.

- Implicit epistemic model: For every type, we can derive the lexicographic belief hierarchy induced by it.
- Based on Brandenburger (1992).

	Pub a	Pub b	Pub c
Pub a	$0,3$	$1,2$	$1,1$
Pub b	$1,3$	$0,2$	$1,1$
Pub c	$1,3$	$1,2$	$0,1$

$$b_1(t_1) = ((a, t_2); \tfrac{2}{3}(b, t_2) + \tfrac{1}{3}(c, t_2))$$

$$b_2(t_2) = ((c, t_1); \tfrac{1}{2}(a, t_1) + \tfrac{1}{2}(b, t_1))$$

- Optimal choice for type t_1?
- Under primary belief, choice a gives 0, while b and c give 1. To break the tie between b and c, go to the secondary belief.
- Under the secondary belief, choice b gives $\frac{1}{3}$ and c gives $\frac{2}{3}$.
- Optimal choice for t_1 is c. In fact, type t_1 prefers c to b, and b to a.

Optimal choice

- Consider a type t_i with lexicographic belief $b_i(t_i) = (b_i^1; \, b_i^2; \, ... \, ; b_i^K)$ about j's choice-type pairs.
- Type t_i prefers choice c_i to choice c_i' if there is some level k such that
- choice c_i yields a higher expected utility than c_i' under b_i^k, and
- choices c_i and c_i' yield the same expected utility under the beliefs $b_i^1, ..., b_i^{k-1}$.
- Choice c_i is optimal for type t_i if t_i does not prefer any other choice to c_i.

Cautious types

- Consider a type t_i with lexicographic belief $b_i(t_i) = (b_i^1; b_i^2; \ldots ; b_i^K)$ about j's choice-type pairs.
- Type t_i is cautious if, for every type t_j that is deemed possible by $b_i(t)$, and every choice c_j, the choice-type pair (c_j, t_j) is deemed possible by $b_i(t_i)$.
- $$\begin{array}{rcl} b_1(t_1) & = & ((a, t_2);\ \frac{2}{3}(b, t_2') + \frac{1}{3}(c, t_2)) \\ b_2(t_2) & = & ((c, t_1);\ \frac{1}{2}(a, t_1) + \frac{1}{2}(b, t_1)) \\ b_2(t_2') & = & ((a, t_1);\ (b, t_1);\ (c, t_1)) \end{array}$$
- Type t_1 is not cautious, but type t_2 is.

Primary belief in rationality, and respect of preferences

- Consider a cautious type t_i with lexicographic belief $b_i(t_i)$ on the opponent's choice-type pairs.
- Type t_i primarily believes in the opponent's rationality if t_i's primary belief only assigns positive probability to choice-type pairs (c_j, t_j) where c_j is optimal for t_j.
- Type t_i respects the opponent's preferences if for every type t_j deemed possible by t_i, and every two choices c_j, c_j':

  if t_j prefers c_j to c_j', then t_i deems (c_j, t_j) infinitely more likely than (c_j', t_j).
- Observation: If t_i respects the opponent's preferences, then t_i primarily believes in the opponent's rationality.

	Pub a	Pub b	Pub c
Pub a	$0, 3$	$1, 2$	$1, 1$
Pub b	$1, 3$	$0, 2$	$1, 1$
Pub c	$1, 3$	$1, 2$	$0, 1$

$$\begin{array}{rcl} b_1(t_1) & = & ((a, t_2);\ (b, t_2);\ (c, t_2)) \\ b_1(t_1') & = & ((a, t_2');\ \frac{1}{3}(b, t_2') + \frac{2}{3}(c, t_2') \\ b_2(t_2) & = & ((c, t_1);\ (b, t_1);\ (a, t_1)) \\ b_2(t_2') & = & ((b, t_1');\ \frac{2}{3}(a, t_1') + \frac{1}{3}(c, t_1')) \end{array}$$

- All types primarily believe in the opponent's rationality.
- Only types t_1 and t_2 respect the opponent's preferences.

Iterating "primary belief in rationality"

Definition

(Induction start) Type t_i expresses 1-fold full belief in "caution and primary belief in rationality" if t_i is cautious and primarily believes in the opponents' rationality.

(Inductive step) For every $k \geq 2$, type t_i expresses k-fold full belief in "caution and primary belief in rationality" if t_i only deems possible opponents' types that express $(k-1)$-fold full belief in "caution and primary belief in rationality".

Type t_i expresses common full belief in "caution and primary belief in rationality" if t_i expresses k-fold full belief in "caution and primary belief in rationality" for all k.

- Also known as permissibility (Brandenburger (1992), Börgers (1994)).
- Equilibrium counterpart is trembling-hand perfect equilibrium (Selten (1975)).

Iterating "respect of preferences"

Definition

(Induction start) Type t_i expresses 1-fold full belief in "caution and respect of preferences" if t_i is cautious and respects the opponent's preferences.

(Inductive step) For every $k \geq 2$, type t_i expresses k-fold full belief in "caution and prespect of preferences" if t_i only deems possible opponents' types that express $(k-1)$-fold full belief in "caution and respect of preferences".

Type t_i expresses common full belief in "caution and respect of preferences" if t_i expresses k-fold full belief in "caution and respect of preferences" for all k.

- Also known as proper rationalizability (Schuhmacher (1999), Asheim (2001)).
- Equilibrium counterpart is proper equilibrium (Myerson (1978)).

	Pub a	Pub b	Pub c
Pub a	$0,3$	$1,2$	$1,1$
Pub b	$1,3$	$0,2$	$1,1$
Pub c	$1,3$	$1,2$	$0,1$

$$\begin{aligned} b_1(t_1) &= ((a,t_2);\ (b,t_2);\ (c,t_2)) \\ b_1(t_1') &= ((a,t_2');\ \tfrac{1}{3}(b,t_2') + \tfrac{2}{3}(c,t_2') \end{aligned}$$

$$\begin{aligned} b_2(t_2) &= ((c,t_1);\ (b,t_1);\ (a,t_1)) \\ b_2(t_2') &= ((b,t_1');\ \tfrac{2}{3}(a,t_1') + \tfrac{1}{3}(c,t_1')) \end{aligned}$$

- All types express common full belief in "caution and primary belief in rationality".
- Only types t_1 and t_2 express common full belief in "caution and respect of preferences".

Assuming the opponent's rationality

- Consider an epistemic model M and a cautious type t_i within M.
- Type t_i assumes the opponent's rationality if:
- (richness condition) for every opponent's choice c_j that is optimal for some cautious type in some epistemic model, the model M contains at least one cautious type t_j for which c_j is optimal, and
- (optimality condition) type t_i deems all choice-type pairs (c_j, t_j), where c_j is optimal for t_j and t_j is cautious, infinitely more likely than all other choice-type pairs.
- Observation: If t_i assumes the opponent's rationality, then t_i primarily believes in the opponent's rationality.
- Iterating this condition leads to common assumption of rationality.
- Based on Brandenburger, Friedenberg and Keisler (2008).
- There is no equilibrium analogue to common assumption of rationality.
- Details in Chapter 7 of the book.

Recursive Procedures

- We wish to find recursive procedures that characterize the choices induced by the three concepts.

Lemma (Based on Pearce (1984))

A choice c_i is optimal for some cautious lexicographic belief about the opponents' choices, if and only if, c_i is not weakly dominated by any randomized choice.

- Here, a randomized choice r_i for player i is a probability distribution on i's choices.
- Choice c_i is weakly dominated by the randomized choice r_i if

$$u_i(c_i, c_{-i}) \leq u_i(r_i, c_{-i})$$

for every opponents' choice-combination $c_{-i} \in C_{-i}$, and

$$u_i(c_i, c_{-i}) < u_i(r_i, c_{-i})$$

for at least one c_{-i}.

Definition (Dekel-Fudenberg procedure)

Consider a finite static game Γ.

(Round 0) Let $\Gamma^0 := \Gamma$ be the original game.

(Round 1) Let Γ^1 be the game which results if we eliminate from Γ^0 all choices that are weakly dominated within Γ^0.

(Further rounds) For every $k \geq 2$ let Γ^k be the game which results if we eliminate from Γ^{k-1} all choices that are strictly dominated within Γ^{k-1}.

- Procedure taken from Dekel and Fudenberg (1990).
- This procedure characterizes exactly those choices that can rationally be made under common full belief in "caution and primary belief in rationality".
- Result based on Brandenburger (1992).

Example: Stealing an apple

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Story

- You have stolen an apple, and since then you are being followed by an angry farmer.
- You decide to hide in the castle above. But in what chamber?
- Famer must decide in what chamber to look for you.
- He will find you whenever his chamber is the same as your chamber, or horizontally, vertically, or diagonally adjacent to your chamber.
- If he finds you, your utility is 0 and the farmer's utility is 1. Otherwise, your utility is 1 and the farmer's utility is 0.

You				
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Farmer				
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

- Apply Dekel-Fudenberg procedure.
- Round 1: For you, 2, 6 and 7 weakly dominated by 1, 8 weakly dominated by 3. Similarly for other chambers.
- For farmer, 1, 2 and 6 weakly dominated by 7, 3 weakly dominated by 8. Similarly for other chambers.

You				
1		3		5
11		13		15
21		23		25

Farmer				
	7	8	9	
	12	13	14	
	17	18	19	

- Round 2: For you, 13 is strictly dominated by $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 25$.

You				
1		3		5
11				15
21		23		25

Farmer				
	7	8	9	
	12	13	14	
	17	18	19	

- Round 3: For farmer, 13 is strictly dominated by $\frac{1}{4} \cdot 7 + \frac{1}{4} \cdot 9 + \frac{1}{4} \cdot 17 + \frac{1}{4} \cdot 19.$

You				
1		3		5
11				15
21		23		25

Farmer				
	7	8	9	
	12		14	
	17	18	19	

- Procedure terminates.

Definition (Iterated elimination of weakly dominated choices)

Consider a finite static game Γ.

(Round 0) Let $\Gamma^0 := \Gamma$ be the original game.

(Further rounds) For every $k \geq 1$, let Γ^k be the game which results if we eliminate from Γ^{k-1} all choices that are weakly dominated within Γ^{k-1}.

- Is a refinement of the Dekel-Fudenberg procedure.
- This procedure characterizes exactly those choices that can rationally be made under common assumption of rationality.
- Result based on Brandenburger, Friedenberg and Keisler (2008).

You				
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Farmer				
1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

- Apply iterated elimination of weakly dominated choices.
- Round 1: For you, 2, 6 and 7 weakly dominated by 1, 8 weakly dominated by 3. Similarly for other chambers.
- For farmer, 1, 2 and 6 weakly dominated by 7, 3 weakly dominated by 8. Similarly for other chambers.

You				
1		3		5
11		13		15
21		23		25

Farmer				
	7	8	9	
	12	13	14	
	17	18	19	

- Round 2: For you, 13 is weakly dominated by 1 , 3 and 11 weakly dominated by 1. Similarly for other chambers.
- For farmer, 8, 12 and 13 weakly dominated by 7. Similarly for other chambers.

You				
1				5
21				25

Farmer				
	7		9	
	17		19	

- Procedure terminates.

Procedure for respect of preferences

- Is there a recursive procedure for common full belief in "caution and respect of preferences"?
- Yes, as shown in Perea (2011).
- But it cannot be an elimination procedure.

Example: Spy game

	Pub a	Pub b	Pub c
Pub a	$0,3$	$1,2$	$1,4$
Pub b	$1,3$	$0,2$	$1,1$
Pub c	$1,6$	$1,2$	$0,1$

- We have seen: Common full belief in "caution and respect of preferences" uniquely leads you to Pub c.
- The only choice that can be eliminated is Barbara's choice b.
- But then, your choice b could never be eliminated afterwards.
- Hence, elimination of choices cannot work for common full belief in "caution and respect of preferences".
- Details in Chapter 6 of the book.

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