Mini Course on Epistemic Game Theory
Toulouse, June 30 - July 3, 2015
Exercises

Part I: Common Belief in Rationality

Exercise 1. A game of cards

Barbara, Chris and you are sitting in a bar, having a drink before the movie starts. You have brought a pack of playing cards with you, and tell your friends about a new cards game you invented last night. The rules are easy: There are three decks of cards on the table with their faces down. One deck contains 2, 5, 8 and jack of hearts, another deck contains 3, 6, 9 and queen of hearts, and the last deck contains 4, 7, 10 and king of hearts, and everybody knows this. The jack is worth 11 points, the queen is worth 12 points and the king 13 points. Each of the three players receives one of these decks, and everybody knows the decks that are given to the other two players. Then, all players simultaneously choose one card from their deck, and put in on the table. The player putting the card whose value is the middle value on the table wins the game. Every loosing player pays the value of his own card in euros to the winning player. Suppose that when you start playing the game, you hold the deck with the 3, 6, 9 and the queen.

(a) Which cards are rational for you? For every rational card, find a belief about your friends' choices for which this card is optimal. For every irrational card, find another card, or randomization over cards, that strictly dominates it.

(b) Which cards are rational for your two friends? For every rational card, find a belief for which this card is optimal. For every irrational card, find another card, or randomization over cards, that strictly dominates it.

(c) Based on your findings in (a) and (b), construct an epistemic model for this game.

(d) Consider your type that supports playing the queen, and your type that supports playing the 6. For both types, state the first three levels of the belief hierarchy they have.

(e) Use the algorithm of iterated elimination of strictly dominated choices to find those cards you and your friends can rationally play under common belief in rationality. After how many rounds does the algorithm stop?

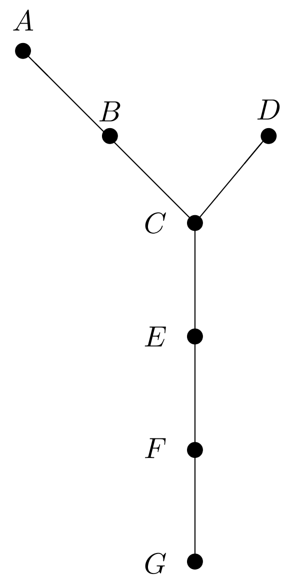

Figure 1: Houses for sale in "The mother-in-law"

(f) Construct an epistemic model such that, for each of the choices c_i found in (e), there is a type $t_i^{c_i}$ such that

- choice c_i is optimal for $t_i^{c_i}$, and
- type $t_i^{c_i}$ expresses common belief in rationality.

(g) Which player, or players, do you expect to be able to win the game under common belief in rationality? How much money do you expect the winner, or possible winners, to earn?

Exercise 2. The mother-in-law

Suppose that you and your partner are planning to move to another village, which consists of only three streets. There are seven houses for sale in that village, and their locations are depicted in Figure 1. The distance between two houses on this map is 100 metres. So, the distance between house A and house B is 100 metres, the distance between house B and house C is 100 metres, and so on.

When your mother-in-law learned about your plans, she decided to move to the same village as well, and she can choose from the same seven houses. Tomorrow, you and your mother-in-law have to sign up for one of the seven houses. If you both choose different houses, you will both get the house of your choice. If you both choose the same house, the house will go to a third party, and you will both not be able to move.

Suppose that your relationship with her is not quite optimal, and that you attempt to maximize the distance to her house. Your utility is equal to the distance between the houses if you both get a house in the village, and is equal to 0 if you cannot move to the village. The mother-in-law, on the other hand, wishes to minimize the distance to your house, as she likes to visit her child every day, and check whether you have cleaned the house properly. More precisely, the utility for your mother-in-law is equal to

$$600 - \text{ distance between the houses}$$

if you both get a house in the village, and is equal to 0 if she cannot move to the village.

(a) Show that location C is strictly dominated for you by a randomized choice in which you randomize over the locations A, D and G. That is, find probabilities α, β and γ with $\alpha + \beta + \gamma = 1$ such that location C is strictly dominated by the randomized choice in which you choose location A with probability α, location D with probability β and location G with probability γ.

(b) Which are the rational locations for you? For every rational location, find a belief about your mother-in-law's choice for which this location is optimal. For every irrational location, find a randomized choice that strictly dominates it.

Hint to (b): Every irrational location is strictly dominated by a randomized choice in which you randomize over the locations A, D and G.

(c) Use the algorithm of iterated elimination of strictly dominated choices to find those locations you and the mother-in-law can rationally choose under common belief in rationality. After how many steps does the algorithm stop? Do you expect that both you and your mother-in-law could choose the same house?

(d) Construct an epistemic model such that, for each of the choices c_i found in (c), there is a type $t_i^{c_i}$ such that

- choice c_i is optimal for $t_i^{c_i}$, and

- type $t_i^{c_i}$ expresses common belief in rationality.

Part II: Nash equilibrium

Exercise 3: Black or white?

This evening there will be a party in the village for which you and Barbara are invited. The problem is that you don't know whether to go or not, and if you go, which color to wear. Assume that you only have white and black suits in your cupboard, and the same holds for Barbara. You and Barbara have conflicting interests when it comes to wearing clothes: You strongly dislike it when Barbara wears the same color as you do, whereas Barbara prefers to wear the same color as you do. At the same time, you know that you will only have a good time at the party if Barbara goes, and similarly for Barbara.

More precisely, your utilities are as follows: Staying at home gives you a utility of 2. If you go to the party, and Barbara shows up with a different color than you, your utility will be 3. In all other cases, your utility is 0.

For Barbara the utilities are similar. The only difference is that she gets a utility of 3 if she goes to the party and you show up with the *same* color as she wears.

(a) Model this situation as a game between you and Barbara. That is, make a table in which you list the choices and utilities for you and Barbara.

(b) Make a beliefs diagram and translate it into an epistemic model. Which choices can you rationally make under common belief in rationality?

(c) Which types t_i in your model believe that the opponent j has correct beliefs? Which of these types t_i believe that the opponent j believes that i has correct beliefs too?

(d) Compute all Nash equilibria in this game. For every Nash equilibrium, state the belief hierarchy it induces for you.

(e) Compute all choices you can rationally make under common belief in rationality if you believe that Barbara is correct about your beliefs, and you believe that Barbara believes you are correct about her beliefs.

Exercise 4: A high-school reunion

Tomorrow there will be a reunion of your former class mates from high-school. Let us number the class mates $1, 2, 3, ..., 30$. Every person i had exactly one favorite class mate, namely $i+1$, and one least preferred class mate, namely $i-1$. If $i=1$, then i's least preferred class mate is 30, and if $i=30$, then i's favorite class mate is 1. For every class mate, the presence of his favorite class mate would increase his utility by 3, whereas the presence of his least preferred class mate would decrease his utility by 3. The presence of other class mates would not affect his utility. Every class mate must decide whether or not to join the reunion. Assume that staying at home would yield a utility of 2.

(a) Explain why under common belief in rationality both joining the reunion and staying at home can be optimal. What would change if one of the class mates would *not* have a favorite class mate?

(b) For any number n between 0 and 29, construct a belief hierarchy expressing common belief in rationality in which you believe that exactly n fellow class mates will show up at the reunion.

(c) What choice, or choices, can you rationally make under a belief hierarchy that is induced by a Nash equilibrium?

(d) How many class mates do you think will show up under a belief hierarchy that is induced by a Nash equilibrium?

Part III: Backward induction reasoning

Exercise 5: Two parties in a row

After ten consecutive attempts Chris finally passed the driving test. In order to celebrate this memorable event, he organizes two parties in a row – one on Friday and one on Saturday. You and Barbara are both invited for the first party on Friday. The problem, as usual, is to decide which color to wear for that evening. You can both choose between *blue, green, red* and *yellow.* The utilities that you and Barbara derive from wearing these colors are given by Table 1. As before, you both feel unhappy when you wear the same color as the friend. In that case, the utility for you and Barbara would only be 1.

In order to decide which people to invite for the second party, Chris applies a very strange selection criterion: Only those people dressed in yellow will be invited for the party on Saturday. Moreover, you will only go to the party on Saturday if Barbara is invited as well, and similarly for Barbara.

	blue	green	red	yellow	same color as friend
you	6	4	3	2	1
Barbara	6	2	4	3	1

Table 1: The utilities for you and Barbara in "Two parties in a row"

In case you are both invited for the party on Saturday, you face the same decision problem as on Friday, namely which color to wear for that evening.

Suppose that the total utility for you and Barbara is simply the sum of the utilities you derive from the parties you visit.

(a) Model this situation as a dynamic game between you and Barbara. How many strategies do you have in this game?

(b) Is this a game with observed past choices? Is this a game with perfect information?

(c) Which strategies can you rationally choose under common belief in future rationality? Which colors can you rationally wear on Friday under common belief in future rationality? And on Saturday, provided you go to the party on that day? What about Barbara?

(d) Construct an epistemic model such that, for every strategy found in (c), there is some type that expresses common belief in future rationality, and for which that strategy is rational.

Exercise 6: The love letter

During a beautiful holiday on Mallorca you have met Charlie, to whom you are writing love letters ever since. This morning Barbara has told you that she found out, and you can feel that she is extremely jealous about all this. She is even threatening to burn your letter if you write to Charlie again! Now, you are sitting behind your desk, and must decide whether or not to write a letter to Charlie. If you write the letter then Barbara will certainly find out, after which she can then either search for the letter and burn it, or let it go. Charlie, both upon receiving a letter and upon observing that no letter has arrived, must choose between writing you or not. In case no letter has arrived, Charlie does not know whether this is because you did not send one, or because Barbara has burned your letter.

Assume that the utilities for you, Barbara and Charlie are as follows: Writing a letter requires a considerable effort from you, and decreases your utility by 2. On the other hand, receiving a letter from Charlie would increase your utility by 5. However, if Barbara were to burn your letter, then this would bring you in a state of shock and would decrease your utility by 5.

For Barbara, searching for the letter and burning it is a difficult task and decreases her utility by 2. However, if Charlie does not write you, then Barbara would have succeeded in her mission and her utility would increase by 10.

Charlie only wants to write to you if you have written a letter too. More precisely, Charlie's utility decreases by 5 if Charlie writes a letter whereas you have not, or if Charlie does not write a letter whereas you have written one.

(a) Model this situation as a dynamic game between you, Barbara and Charlie. Is this a game with observed past choices? Is this a game with perfect information?

(b) Which strategy, or strategies, can you, Barbara and Charlie rationally choose under common belief in future rationality?

(c) Construct an epistemic model such that, for every strategy found in (b), there is some type that expresses common belief in future rationality, and for which that strategy is rational.

(d) Suppose now that you do not only express common belief in future rationality, but that you also believe that Barbara holds the same belief about Charlie's strategy choice as you do. Which strategy, or strategies, can you then rationally choose?

Part IV: Forward Induction Reasoning

Exercise 7: Never let a lady wait

It is Saturday afternoon, and Barbara and you want to have dinner this evening at 8.00 pm. In the village where you live there are only two restaurants – an Italian restaurant and a Chinese restaurant. The problem is that you prefer the Italian restaurant whereas Barbara prefers the Chinese restaurant. More precisely, having dinner in the Italian restaurant gives you a utility of 10, and eating in the Chinese restaurant yields you a utility of 7, but for Barbara it is the other way around. Even after a long discussion this morning you could not reach an agreement about the restaurant. For this reason you will both go to one of the restaurants this evening, without knowing to which restaurant the other person goes, and hope that you are lucky enough to find your friend there.

In the past you have built up a reputation of coming late, and Barbara knows that you have often used it as a strategic weapon. Also this evening you will strategically choose between arriving on time, or arriving one hour late. That is, at 7.45 pm you decide between walking to one of the restaurants and be there at 8.00 pm precisely, or wait until 8.45 pm to leave for one of the restaurants. In the latter case you would not know, however, in which restaurant Barbara is. Barbara, on the other hand, is always on time, and so it will be this evening. So, at 7.45 pm she decides to which restaurant she will go, and she will be there exactly at 8.00 pm. If you are both in the same restaurant at 8.00 pm then you will have dinner together, and the utilities will be as described above.

The other possibility is that Barbara, at 8.45 pm, is still waiting in front of an empty chair in her restaurant of choice. In that case, Barbara does not know whether you are momentarily at the other restaurant, or that you are waiting until 8.45 pm to leave your house. She then has two options – to stay in the same restaurant and hope that you will come at 9.00 pm, or to go to the other restaurant and hope that you will be there at 9.00 pm. On the other hand, you will never switch to the other restaurant if you left your house at 7.45 pm and are still alone in the restaurant at 8.45 pm – and Barbara knows this! If you are both in the same restaurant at 9.00 pm then you will have dinner together at 9.00 pm, but the utilities for both of you will be decreased by 2 units because of the one hour delay. Assume, moreover, that walking to the other restaurant would decrease Barbara's utility by 1 extra unit because it is raining outside. If you are both at different restaurants at 9.00 pm, then you will both be disappointed, go home and have a sandwich, yielding both of you a utility of 0 in total.

(a) Model this situation as a dynamic game between you and Barbara. Be careful how to model the information sets!

(b) Find the strategies that you and Barbara can rationally choose under common strong belief in rationality. Do you let Barbara wait?

(c) Describe verbally the reasoning that leads to these strategy choices. Who has an advantage under common strong belief in rationality, and why?

Exercise 8: Read my mind

You and Barbara participate in a TV show that is called "Read my mind". The rules are very simple: On a table there are six different objects, which are numbered from 1 to 6. You must take one of the six objects, but Barbara cannot see this since she is blindfolded. However, the showmaster tells Barbara whether the number you chose is odd or even. Afterwards, Barbara must try to guess the number of the object you chose. If you choose object number k and Barbara guesses it correctly, then both you and Barbara get $1000k^2$ euros. If Barbara is wrong about the object you chose, you both get nothing.

(a) Model this situation as a game between you and Barbara.

(b) Find the strategies that you and Barbara can rationally choose under common strong belief in rationality. What outcomes do you deem possible under common strong belief in rationality?

Hint for (b): The full decision problems in this game are fairly large, and therefore explicitly writing down these full decision problems is not a very good idea. Instead, use the graphical representation of the game you constructed in (a), and find a way to "eliminate strategies directly from the graphical representation". This will save you a lot of writing.

(c) Describe verbally the reasoning that leads to the strategy choices in (b).

(d) Construct an epistemic model such that, for every strategy found in (b), there is a type that expresses common strong belief in rationality, and for which this strategy is optimal.

Hint for (d): Build an epistemic model with the following properties: For every $k \in \{0, 1, 2, ...\}$ and every strategy s_i that can rationally be chosen under expressing up to k-fold strong belief in rationality, construct a type $t_i^{s_i}$ such that

- strategy s_i is optimal for type $t_i^{s_i}$, and
- type $t_i^{s_i}$ expresses up to k-fold strong belief in rationality.

(e) Find the strategies that you and Barbara can rationally choose under common belief in future rationality. What outcomes do you deem possible under common belief in future rationality? Compare this to your answers in (b).